symmetry

Article

Local and Semilocal Convergence of Wang-Zheng's Method for Simultaneous Finding Polynomial Zeros

Slav I. Cholakov

Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, 24 Tzar Asen, 4000 Plovdiv, Bulgaria; cholakovs@uni-plovdiv.bg

Received: 9 May 2019; Accepted: 28 May 2019; Published: 29 May 2019

Abstract: In 1984, Wang and Zheng (J. Comput. Math. 1984, 1, 70–76) introduced a new fourth order iterative method for the simultaneous computation of all zeros of a polynomial. In this paper, we present new local and semilocal convergence theorems with error estimates for Wang–Zheng's method. Our results improve the earlier ones due to Wang and Wu (Computing 1987, 38, 75–87) and Petković, Petković, and Rančić (J. Comput. Appl. Math. 2007, 205, 32–52).

Keywords: iteration methods; simultaneous methods; polynomial zeros; Wang–Zheng method; local convergence; semilocal convergence; error estimates

1. Introduction

Over the last few decades, we have observed rapid development of the theory of iterative methods for simultaneously finding all roots of a polynomial (see, e.g., [1–34] and the references given therein). The present paper deals with a thorough local and semilocal convergence analysis of a well-known iterative method, which was introduced in [34].

In the paper, $(\mathbb{K}, |\cdot|)$ stands for a valued field with absolute value $|\cdot|$ and $\mathbb{K}[z]$ stands for the ring of polynomials over $\mathbb{K}$. We endow the vector space $\mathbb{K}^n$ with the norm $\|\cdot\|_p \colon \mathbb{K}^n \to \mathbb{R}$ defined by $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ $(1 \le p \le \infty)$.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$. A vector $\xi \in \mathbb{K}^n$ is said to be a *root vector* of polynomial f if: $f(z) = a_0 \prod_{i=1}^n (z - \xi_i)$ for all $z \in \mathbb{K}$, where $a_0 \in \mathbb{K}$.

In 1984, Wang and Zheng [34] introduced the following simultaneous method:

$$x^{(k+1)} = T(x^{(k)}), \qquad k = 0, 1, 2, \ldots, \tag{1}$$

where $T \colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by $T(x) = (T_1(x), \ldots, T_n(x))$ with:

$$T_i(x) = \begin{cases} x_i - Z_i(x)^{-1} & \text{if } f(x_i) \neq 0, \\ x_i & \text{if } f(x_i) = 0, \end{cases} \tag{2}$$

where:

$$Z_i(x) = \frac{f'(x_i)}{f(x_i)} - \frac{f''(x_i)}{2f'(x_i)} - \frac{f(x_i)}{2f'(x_i)} \left(\left(\sum_{j \neq i} \frac{1}{x_i - x_j} \right)^2 + \sum_{j \neq i} \frac{1}{(x_i - x_j)^2} \right).$$

Apparently, the domain $\mathscr{D}$ of Wang–Zheng's iteration function T is:

$$\mathscr{D} = \{x \in \mathcal{D} \colon \ f'(x_i) \neq 0 \ \text{and} \ Z_i(x) \neq 0 \ \text{whenever} \ f(x_i) \neq 0\}. \tag{3}$$

In the paper, we denote by $\mathcal{D}$ the set of the vectors in $\mathbb{K}^n$ with pairwise distinct coordinates, i.e.,

$$\mathcal{D} = \{x \in \mathbb{K}^n \colon x_i \neq x_j \text{ for all } i \neq j\}. \tag{4}$$

In 1987, Wang and Wu [32] gave the following local convergence theorem for Wang–Zheng's method:

Theorem 1 ([32])**.** *Suppose $f \in \mathbb{C}[z]$ is a polynomial of degree $n \geq 2$ with simple zeros. Let $x^{(0)} \in \mathbb{C}^n$ be an initial approximation with pairwise distinct coordinates satisfying:*

$$\|x^{(0)} - \xi\|_\infty < \min\left\{\frac{\delta(\xi)}{2n}, \frac{\delta(\xi)^2}{(n+2)\Delta(\xi)}\right\}, \tag{5}$$

where $\delta(\xi) = \min_{i \neq j} |\xi_i - \xi_j|$ and $\Delta(\xi) = \max_{i \neq j} |\xi_i - \xi_j|$. Then, Wang–Zheng's iteration (1) is convergent to a root vector of f with convergence order four and with error estimate:

$$\|x^{(k+1)} - \xi\|_\infty \leq 13.6\, n^2\, \Delta(\xi)\, \delta(\xi)^{-4}\, \|x^{(k)} - \xi\|_\infty^4 \qquad \textit{for all } k \geq 0. \tag{6}$$

In order to formulate the semilocal convergence results, we need the function $W_f \colon \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ (Weierstrass correction) defined by:

$$W_f(x) = (W_1(x), \ldots, W_n(x)) \quad \text{with} \quad W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)}, \tag{7}$$

where a_0 is the leading coefficient of polynomial f with degree n.

In 2007, Petković, Petković, and Rančić [13] gave a semilocal convergence result for Wang–Zheng's method (1), improving the previous result due to Petković and Herceg [12].

Theorem 2 ([13])**.** *Suppose $f \in \mathbb{C}[z]$ is a polynomial of degree $n \geq 2$ with simple zeros. Let $x^{(0)} \in \mathbb{C}^n$ be an initial approximation with pairwise distinct coordinates satisfying:*

$$\|W_f(x^{(0)})\|_\infty < c_n\, \delta(x^{(0)}), \quad \textit{where} \quad c_n = \begin{cases} 1/(3n + 2.4), & 3 \leq n \leq 20, \\ 1/(3n), & n \geq 21, \end{cases} \tag{8}$$

where $\delta(x^{(0)}) = \min_{i \neq j} |x_i^{(0)} - x_j^{(0)}|$, then Wang–Zheng's iteration (1) is convergent to a root vector of f with convergence order four.

The purpose of our study is to give a comprehensive convergence analysis for Wang–Zheng's method (1). We present two local convergence theorems (Theorem 5 and Theorem 6) and a semilocal convergence theorem (Theorem 9). Our first local convergence theorem improves the result of Wang and Wu (Theorem 1), and our semilocal convergence result improves the result of Petković, Petković, and Rančić (Theorem 2).

2. Preliminaries

Recently, Proinov [16–18,35,36] developed a general convergence theory for iterative methods of the type (1), where $T \colon D \subset X \to X$ is an iteration function in metric, cone metric, or vector space. On the basis of this theory lays the notion *function of initial conditions* of T since the convergence of any iterative method of the type (1) is studied with respect to some function of initial conditions (see [35,36]). Some applications of this theory can be found in [1,2,5,7,8,14–29,35–38].

Let $\mathbb{R}^n$ be equipped with coordinate-wise ordering $\preceq$ defined by:

$$x \preceq y \quad \text{if and only if} \quad x_i \leq y_i \ \text{ for } \ i = 1, \ldots, n.$$

Furthermore, $\mathbb{K}^n$ is equipped with a vector norm $\|\cdot\|\colon \mathbb{K}^n \to \mathbb{R}^n$ defined by:

$$\|x\| = (|x_1|, \ldots, |x_n|).$$

For a given p ($1 \leq p \leq \infty$), we always define a number q by:

$$1 \leq q \leq \infty \quad \text{with} \quad 1/p + 1/q = 1.$$

Furthermore, for given n and p, we use the denotations:

$$a = (n-1)^{1/q}, \quad b = 2^{1/q}. \tag{9}$$

We observe that $1 \leq a \leq n-1$ and $1 \leq b \leq 2$. Henceforth, for two vectors $x \in \mathbb{K}^n$ and $y \in \mathbb{R}^n$, we shall use the denotation x/y for a vector in $\mathbb{R}^n$ defined by:

$$\frac{x}{y} = \left(\frac{|x_1|}{y_1}, \cdots, \frac{|x_n|}{y_n} \right)$$

provided that y has only nonzero coordinates. Furthermore, we define $0^0 = 1$, and we denote by $S_k(r)$ the sum of the first k terms of geometric sequence $1, r, r^2, \cdots$, i.e.,

$$S_k(r) = 1 + r + \cdots + r^{k-1}. \tag{10}$$

Definition 1 ([36])**.** *A function $\varphi\colon J \subset \mathbb{R} \to \mathbb{R}_+$ is called quasi-homogeneous of degree $r \geq 0$ if it satisfies $\varphi(\alpha t) \leq \alpha^r \varphi(t)$ for $\alpha \in [0,1]$ and $t \in J$.*

We define the function $d\colon \mathbb{K}^n \to \mathbb{R}^n$ by:

$$d(x) = (d_1(x), \ldots, d_n(x)) \quad \text{with} \quad d_i(x) = \min_{j \neq i} |x_i - x_j| \qquad (i = 1, \ldots, n). \tag{11}$$

The following theorem of Proinov [18] deals with local convergence of the Picard iterative sequence (1) regarding the function $E\colon \mathbb{K}^n \to \mathbb{R}_+$ defined as follows:

$$E(x) = \left\| \frac{x - \xi}{d(\xi)} \right\|_p \qquad (1 \leq p \leq \infty). \tag{12}$$

Theorem 3 ([18])**.** *Let $T\colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ be an iteration function, $\xi \in \mathbb{K}^n$ be a vector with pairwise distinct coordinates, and $E\colon \mathbb{K}^n \to \mathbb{R}_+$ be defined by* (12)*. Suppose there exists a quasi-homogeneous function $\phi\colon J \to \mathbb{R}_+$ of degree $m \geq 0$ such that for each vector $x \in \mathbb{K}^n$ with $E(x) \in J$, the following conditions hold:*

$$x \in D \quad \text{and} \quad \|T(x) - \xi\| \preceq \phi(E(x)) \|x - \xi\|. \tag{13}$$

Let $x^{(0)} \in \mathbb{K}^n$ be an initial approximation such that:

$$E(x^{(0)}) \in J \quad \text{and} \quad \phi(E(x^{(0)})) < 1. \tag{14}$$

Then, the Picard iteration (1) *is well defined and converges to* ξ *with order* $r = m + 1$ *and with error estimates:*

$$\|x^{(k+1)} - \xi\| \preceq \lambda^{r^k} \|x^{(k)} - \xi\| \quad \textit{and} \quad \|x^{(k)} - \xi\| \preceq \lambda^{S_k(r)} \|x^{(0)} - \xi\| \quad \textit{for all } k \geq 0,$$

where $\lambda = \phi(E(x^{(0)}))$ *and* $S_k(r)$ *is defined by* (10).

The next theorem of Proinov [18] deals with local convergence of the Picard iteration (1) with respect to the function of initial conditions $E \colon \mathcal{D} \to \mathbb{R}_+$ defined by:

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p \qquad (1 \leq p \leq \infty). \tag{15}$$

Theorem 4 ([18]). *Let* $T \colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ *be an iteration function,* $\xi \in \mathbb{K}^n$, *and* $E \colon \mathcal{D} \subset \mathbb{K}^n \to \mathbb{R}_+$ *is defined by* (15). *Suppose there is a nonzero quasi-homogeneous function* $\beta \colon J \to \mathbb{R}_+$ *of degree* $m \geq 0$ *such that for any* $x \in \mathcal{D}$ *with* $E(x) \in J$, *the conditions:*

$$x \in D \quad \textit{and} \quad \|T(x) - \xi\| \preceq \beta(E(x)) \|x - \xi\| \tag{16}$$

are fulfilled. Let also $x^{(0)} \in \mathcal{D}$ *be an initial approximation such that:*

$$E(x^{(0)}) \in J \quad \textit{and} \quad \Psi(E(x^{(0)})) \geq 0, \tag{17}$$

where $\Psi(t) = 1 - bt - \beta(t)(1 + bt)$. *Then, the Picard iteration* (1) *is well defined and converges to* ξ *with error estimates:*

$$\|x^{(k+1)} - \xi\| \preceq \theta \lambda^{r^k} \|x^{(k)} - \xi\| \quad \textit{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k \lambda^{S_k(r)} \|x^{(0)} - \xi\| \textit{ for all } k \geq 0,$$

where $r = m + 1$, $\theta = \psi(E(x^{(0)}))$, $\lambda = \phi(E(x^{(0)}))$, $\psi(t) = 1 - bt(1 + \beta(t))$, $\phi(t) = \beta(t)/\psi(t)$, *and* $S_k(r)$ *is defined by* (10).

To prove our auxiliary results, we use the following technical lemmas.

Lemma 1 ([17]). *Let* $x, \xi \in \mathbb{K}^n$, *vector* ξ *be with distinct coordinates, and* $1 \leq p \leq \infty$. *Then, for* $i \neq j$,

$$|x_i - x_j| \geq (1 - bE(x)) d_j(\xi) \quad \textit{and} \quad |x_i - \xi_j| \geq (1 - E(x)) d_i(\xi),$$

where b *is defined by* (9) *and* $E \colon \mathbb{K}^n \to \mathbb{R}_+$ *is defined by* (12).

Lemma 2 ([17]). *Let* $x, \xi \in \mathbb{K}^n$, *vector* x *be with distinct coordinates, and* $1 \leq p \leq \infty$. *Then, for* $i \neq j$,

$$|x_i - x_j| \geq d_j(x) \quad \textit{and} \quad |x_i - \xi_j| \geq (1 - E(x)) d_i(x),$$

where $E \colon \mathcal{D} \to \mathbb{R}_+$ *is defined by* (15).

Lemma 3 ([22]). *Let* $f \in \mathbb{K}[z]$ *be a polynomial of degree* $n \geq 2$, *which splits in* $\mathbb{K}$. *Suppose* $\xi \in \mathbb{K}^n$ *is a root vector of* f *and* $x \in \mathbb{K}^n$ *is a vector with a coordinate* x_i, *which is not a zero of* f *and* f'. *Then:*

$$\frac{f'(x_i)}{f(x_i)} = \frac{1 + \sigma_i}{x_i - \xi_i} \quad \textit{and} \quad \frac{f'(x_i)}{f(x_i)} - \frac{f''(x_i)}{2f'(x_i)} = \frac{2 + 2\sigma_i + \sigma_i^2 + \tau_i}{2(1 + \sigma_i)(x_i - \xi_i)}, \tag{18}$$

where σ_i and τ_i are defined by:

$$\sigma_i = (x_i - \xi_i) \sum_{j \neq i} \frac{1}{x_i - \xi_j} \quad \text{and} \quad \tau_i = (x_i - \xi_i)^2 \sum_{j \neq i} \frac{1}{(x_i - \xi_j)^2}\,. \tag{19}$$

3. Local Convergence Analysis of the First Type

In 2016, Proinov [16] categorized into three types the most commonly-used initial conditions in the convergence analysis of simultaneous methods. The objective of this section is to provide a local convergence theorem of the first type for Wang–Zheng's method.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$, and let $\xi \in \mathbb{K}^n$ be a root vector of f.

In the present section, we investigate the convergence of Wang–Zheng's iteration (1) with respect to the function of initial conditions $E\colon \mathbb{K}^n \to \mathbb{R}_+$ defined by (12).

For the sake of simplicity, we introduce the following denotations:

$$A_i = (x_i - \xi_i) \sum_{j \neq i} \frac{x_j - \xi_j}{(x_i - \xi_j)(x_i - x_j)}\,, \tag{20}$$

$$B_i = (x_i - \xi_i) \sum_{j \neq i} \left(\frac{1}{x_i - \xi_j} + \frac{1}{x_i - x_j} \right), \tag{21}$$

$$C_i = (x_i - \xi_i)^2 \sum_{j \neq i} \left(\frac{x_j - \xi_j}{(x_i - \xi_j)^2 (x_i - x_j)} + \frac{x_j - \xi_j}{(x_i - \xi_j)(x_i - x_j)^2} \right). \tag{22}$$

Lemma 4. *Suppose $f \in \mathbb{K}[z]$ is a polynomial of degree $n \geq 2$, which splits in $\mathbb{K}$ and $\xi \in \mathbb{K}^n$ is a root vector of f. If $x \in \mathscr{D}$ is a vector with $f(x_i) \neq 0$ for some i, then:*

$$T_i(x) - \xi_i = - \frac{A_i B_i + C_i}{2(1 + \sigma_i) - A_i B_i - C_i} (x_i - \xi_i), \tag{23}$$

where σ_i, A_i, B_i, and C_i are defined by (19), (20), (21), and (22), respectively.

Proof. From (2) and Lemma 3, taking into account that $f(x_i) \neq 0$ and that ξ is a root vector of f, we obtain:

$$\begin{aligned}
& T_i(x) - \xi_i = \\
& = x_i - \xi_i - \left(\frac{2 + 2\sigma_i + \sigma_i^2 + \tau_i}{2(1 + \sigma_i)(x_i - \xi_i)} - \frac{x_i - \xi_i}{2(1 + \sigma_i)} (S_i^2 + G_i) \right)^{-1} \\
& = \left(1 - \frac{2(1 + \sigma_i)}{2(1 + \sigma_i)^2 - (\sigma_i^2 + 2\sigma_i - \tau_i) - \mu_i^2 - \nu_i} \right) (x_i - \xi_i) \\
& = \left(1 - \frac{2(1 + \sigma_i)}{2(1 + \sigma_i) + (\sigma_i - \mu_i)(\sigma_i + \mu_i) + \tau_i - \nu_i} \right) (x_i - \xi_i),
\end{aligned}$$

where τ_i is defined by (19),

$$S_i = \sum_{j \neq i} \frac{1}{x_i - x_j}, \quad G_i = \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}, \quad \mu_i = (x_i - \xi_i) S_i \quad \text{and} \quad \nu_i = (x_i - \xi_i)^2 G_i\,.$$

Now, taking into account that $\sigma_i - \mu_i = -A_i$, $\sigma_i + \mu_i = B_i$ and $\tau_i - \nu_i = -C_i$, we get (23). □

Define the real functions ϕ as follows:

$$\phi(t) = \frac{\gamma(t)t^3}{g(t)}, \tag{24}$$

where the functions γ and g are defined by:

$$\gamma(t) = an(2-(b+1)t) \quad \text{and} \quad g(t) = 2(1-t)(1-nt)(1-bt)^2 - \gamma(t)t^3 \tag{25}$$

and a, b are defined by (9). We observe that the function g is continuous and decreasing on $[0, 1/n]$, and $g(0) > 0$ and $g(1/n) < 0$. It follows from this that there exists a unique zero μ of g in $(0, 1/n)$. It is easy to show that the function $\omega(t) = \gamma(t)/g(t)$ is increasing on $[0, \mu)$. This yields that ϕ is a quasi-homogeneous function of degree $m = 3$ on $[0, \mu)$. In the following lemma, we prove that the function ϕ satisfies the assumption (13) of Theorem 3 for the iteration function $T\colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ defined by (2).

Lemma 5. *Suppose $f \in \mathbb{K}[z]$ is a polynomial of degree $n \geq 2$, which splits in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ is a root vector of f, and $1 \leq p \leq \infty$. Let a vector $x \in \mathbb{K}^n$ be such that:*

$$E(x) < \mu, \tag{26}$$

where E is defined by (12) and the positive number μ is the unique zero in $(0, 1/n)$ of the function g defined in: (25). Then:

(i) *$x \in \mathscr{D}$, where $\mathscr{D}$ is defined by (3);*

(ii) *$\|T(x) - \xi\| \preceq \phi(E(x))\, \|x - \xi\|$, where ϕ is defined by (24).*

Proof. (i) First, we note that Lemma 1 and Condition (26) show that $x \in \mathcal{D}$. Let $f(x_i) \neq 0$ for some i. It follows from Lemma 3 that $f'(x_i) \neq 0$ is equivalent to $\sigma_i \neq -1$. From (19), the triangle inequality, Lemma 1, and (26), we obtain:

$$|\sigma_i| \leq |x_i - \xi_i| \sum_{j \neq i} \frac{1}{|x_i - \xi_j|} \leq \frac{|x_i - \xi_i|}{(1 - E(x))d_i(\xi)} \sum_{j \neq i} 1 \leq \frac{(n-1)E(x)}{1 - E(x)} < 1, \tag{27}$$

which yields $\sigma_i \neq -1$. Taking into account the definition of $\mathscr{D}$ in (3), it remains to prove that $Z_i(x) \neq 0$. From Lemma 3, we get:

$$Z_i(x) = \frac{2(1+\sigma_i) - A_i B_i - C_i}{2(1+\sigma_i)(x_i - \xi_i)}, \tag{28}$$

where σ_i, A_i, B_i, and C_i are defined by (19) (20), (21), and (22). It follows from (28) that $Z_i(x) \neq 0$ is equivalent to $2(1+\sigma_i) \neq A_i B_i + C_i$. By (20), the triangle inequality, Lemma 1, and Hölder's inequality, we obtain:

$$\begin{aligned} |A_i| &\leq |x_i - \xi_i| \sum_{j \neq i} \frac{|x_j - \xi_j|}{|x_i - \xi_j||x_i - x_j|} \\ &\leq \frac{|x_i - \xi_i|}{(1 - E(x))(1 - bE(x))d_i(\xi)} \sum_{j \neq i} \frac{|x_j - \xi_j|}{d_j(\xi)} \\ &\leq \frac{aE(x)^2}{(1 - E(x))(1 - bE(x))}. \end{aligned} \tag{29}$$

From (21) and the triangle inequality, it follows that:

$$|B_i| \le |x_i - \xi_i| \left(\sum_{j \neq i} \frac{1}{|x_i - \xi_j|} + \sum_{j \neq i} \frac{1}{|x_i - x_j|} \right). \tag{30}$$

From (30) and Lemma 1, we obtain the following estimate:

$$|B_i| \le \frac{(n-1)E(x)}{1 - E(x)} + \frac{(n-1)E(x)}{1 - bE(x)} = \frac{(n-1)\,(2 - (b+1)E(x))\,E(x)}{(1 - E(x))\,(1 - bE(x))}. \tag{31}$$

From (22) and the triangle inequality, it follows that:

$$|C_i| \le |x_i - \xi_i|^2 \left(\sum_{j \neq i} \frac{|x_j - \xi_j|}{|x_i - \xi_j|^2\,|x_i - x_j|} + \sum_{j \neq i} \frac{|x_j - \xi_j|}{|x_i - \xi_j|\,|x_i - x_j|^2} \right). \tag{32}$$

From (32), Lemma 1, and Hölder's inequality, we get the bound:

$$\begin{aligned} |C_i| &\le \frac{aE(x)^3}{(1 - E(x))^2\,(1 - bE(x))} + \frac{aE(x)^3}{(1 - E(x))(1 - bE(x))^2} \\ &= \frac{a(2 - (b+1)E(x))\,E(x)^3}{(1 - E(x))^2\,(1 - bE(x))^2}. \end{aligned} \tag{33}$$

It follows from Condition (26) and the monotonicity of g that $g(E(x)) > g(\mu) = 0$. Now, using the triangle inequality and the inequalities (27), (29), (31), and (33), we get:

$$\begin{aligned} &|2(1 + \sigma_i) - A_i B_i - C_i| \\ &\ge 2(1 - |\sigma_i|) - |A_i||B_i| - |C_i| \\ &\ge \frac{2(1 - E(x))(1 - nE(x))(1 - bE(x))^2 - \gamma(E(x))E(x)^3}{(1 - E(x))^2\,(1 - bE(x))^2} \\ &= \frac{g(E(x))}{(1 - E(x))^2\,(1 - bE(x))^2} > 0. \end{aligned} \tag{34}$$

This means that $2(1 + \sigma_i) \neq A_i B_i + C_i$. Therefore, $x \in \mathscr{D}$.

(ii) We have to prove that:

$$|T_i(x) - \xi_i| \le \phi(E(x))\,|x_i - \xi_i| \quad \text{for all} \;\; i = 1, \ldots, n. \tag{35}$$

If $x_i = \xi_i$, then $T_i(x) = \xi_i$, and so, (35) becomes an equality. Suppose $x_i \neq \xi_i$. By Lemma 4 and the triangle inequality, we get:

$$|T_i(x) - \xi_i| \le \frac{|A_i|\,|B_i| + |C_i|}{|2(1 + \sigma_i) - A_i B_i - C_i|}\,|x_i - \xi_i|. \tag{36}$$

Combining (36) and the estimates (29), (31), (33), and (34), we get (35). □

Now, we are ready to state the first main theorem of this paper. First, we define a real function Φ as follows:

$$\Phi(t) = (1 - nt)(1 - t)(1 - bt)^2 - a\,n(2 - (b+1)t)t^3, \tag{37}$$

where a, b are defined by (9).

Theorem 5. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ with n simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root vector of f, and $1 \leq p \leq \infty$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation satisfying the following conditions:*

$$E(x^{(0)}) < 1/n \qquad \text{and} \qquad \Phi(E(x^{(0)})) > 0, \tag{38}$$

where E and Φ are the functions defined by (12) and (37), respectively. Then, Wang–Zheng's iteration (1) is well defined and convergent to ξ with fourth order and with error estimates:

$$\|x^{(k)} - \xi\| \preceq \lambda^{(4^k-1)/3}\|x^{(0)} - \xi\|, \quad \|x^{(k+1)} - \xi\| \preceq \lambda^{4^k}\|x^{(k)} - \xi\| \quad \text{for all } k \geq 0, \tag{39}$$

$$\|x^{(k+1)} - \xi\|_p \leq \omega(E(x^{(0)}))\delta(\xi)^{-3}\|x^{(k)} - \xi\|_p^4 \quad \text{for all } k \geq 0, \tag{40}$$

where $\lambda = \phi(E(x^{(0)}))$, $\omega(t) = \gamma(t)/g(t)$ and the functions ϕ, γ, g are defined by (24) and (25).

Proof. We will apply Proinov's Theorem 3 to the function $T\colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ defined by (2). First, we prove that the initial Condition (39) implies condition:

$$E(x^{(0)}) < \mu \qquad \text{and} \qquad \phi(E(x^{(0)})) < 1, \tag{41}$$

where μ is the unique zero of the function g in $(0, 1/n)$. It is easy to see that Φ is decreasing on $[0, 1/n]$ and $\Phi(t) = \frac{1}{2}(g(t) - \gamma(t))$. This implies $\Phi(\mu) = -\frac{1}{2}\gamma(\mu) < 0$. Hence, if $0 \leq t < 1/n$ and $\Phi(t) > 0$, then $t < \mu$. On the other hand, if $0 \leq t < \mu$, then $\phi(t) < 1$ if and only if $\Phi(t) > 0$. Consequently, (38) implies (41). Now, Theorem 3 and Lemma 5 lead to the conclusion that the iterative sequence (1) is well defined and convergent to ξ with error bounds (39). It remains to prove the estimate (40).

It follows from the first estimate in (39) that $E(x^{(k)}) \leq E(x^{(0)})$ for $k \geq 0$. We also note the following simple facts: $\phi(t) = \omega(t)\, t^3$ for $t \in [0, \mu)$; the function ω is increasing on $[0, \mu)$; $E(x) \leq \|x - \xi\|_p / \delta(\xi)$ for $x \in \mathbb{K}^n$. Now, applying Lemma 5 (ii) with $x = x^{(k)}$ and taking into account these facts, we obtain:

$$\|x^{(k+1)} - \xi\|_p \leq \phi(E(x^{(k)}))\|x^{(k)} - \xi\|_p = \omega(E(x^{(k)}))E(x^{(k)})^3\|x^{(k)} - \xi\|_p \leq \omega(E(x^{(0)}))\delta(\xi)^{-3}\|x^{(k)} - \xi\|_p^4$$

which proves (40). □

In the case $p = \infty$, we obtain the following consequence of Theorem 5:

Corollary 1. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ possessing n simple zeros in $\mathbb{K}$, and let $\xi \in \mathbb{K}^n$ be a root vector of f. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation satisfying the following condition:*

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(\xi)} \right\|_\infty \leq \frac{4}{7n}\,. \tag{42}$$

Then, the iterative sequence (1) is well defined and convergent to ξ with error bounds (39) and (40) with $p = \infty$.

Proof. We prove that $x^{(0)}$ satisfies the conditions (38) of Theorem 5 with the function Φ defined by (37) with $p = \infty$. Since Φ is decreasing on $[0, 1/n]$, it is sufficient to prove that $\Phi(4/(7n)) > 0$. The last inequality is equivalent to $1029n^3 - 3836n^2 + 4352n - 1536 > 0$, which holds for all $n \geq 2$. This completes the proof. □

The following corollary is an improvement of Wang–Wu's result (Theorem 1).

Corollary 2. *Suppose $f \in \mathbb{K}[z]$ is a polynomial of degree $n \geq 2$ possessing n simple zeros in $\mathbb{K}$ and $\xi \in \mathbb{K}^n$ is a root vector of f. Let a vector $x \in \mathbb{K}^n$ be with pairwise distinct coordinates and let it satisfy:*

$$\|x^{(0)} - \xi\|_\infty \leq \frac{\delta(\xi)}{2n}\,, \tag{43}$$

where $\delta(\xi) = \min_{i \neq j} |\xi_i - \xi_j|$. Then, the iterative sequence (1) is well defined and convergent to ξ with error bounds:

$$\|x^{(k+1)} - \xi\|_\infty \leq 5\, n^2\, \delta(\xi)^{-3}\, \|x^{(k)} - \xi\|_\infty^4 \quad \textit{for all } \ k \geq 0. \tag{44}$$

Proof. From (43), we get:

$$\left\| \frac{x^{(0)} - \xi}{d(\xi)} \right\|_\infty \leq \frac{\|x^{(0)} - \xi\|_\infty}{\delta(\xi)} \leq \frac{1}{2n} < \frac{4}{7n}\,.$$

Consequently, Corollary 1 guarantees the convergence of the iterative sequence (1) to the root vector ξ with the error bound (40) for $p = \infty$. From (43), we get:

$$\omega(E(x^{(k)}) \leq \omega\left(\frac{1}{2n}\right) = \frac{8n^3(4n-3)}{16n^2 - 28n + 11} < 5n^2.$$

From this and (40), we obtain the estimate (44), which ends the proof. □

4. Local Convergence Analysis of the Second Type

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$, and let $\xi \in \mathbb{K}^n$ be a root vector of f. The objective of this section is to provide a local convergence theorem of the second type. More precisely, we study the convergence of Wang–Zheng's method (1) with respect to the function of initial conditions $E \colon \mathcal{D} \to \mathbb{R}_+$ defined by (15).

The main role in this section is played by a real function β defined by:

$$\beta(t) = \frac{a\, n(2-t)t^3}{2(1-nt)(1-t) - a\, n(2-t)t^3}, \tag{45}$$

where a is defined by (9). Let us denote by ν the unique zero of the function:

$$\Lambda(t) = 2(1-nt)(1-t) - a\, n(2-t)t^3 \tag{46}$$

in the interval $(0, 1/n)$. It easy to see that the function Λ is decreasing on $[0, 1/n)$ and the function β is quasi-homogeneous of degree $m = 3$ on $[0, \nu)$. In the next lemma, we prove that the function β satisfies the assumptions (16) of Theorem 4 for $T \colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ defined by (2).

Lemma 6. *Suppose $f \in \mathbb{K}[z]$ is a polynomial of degree $n \geq 2$ with n simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ is a root vector of f, and $1 \leq p \leq \infty$. Let a vector $x \in \mathbb{K}^n$ be with pairwise distinct coordinates and let it satisfy:*

$$E(x) < \nu, \tag{47}$$

where the functions E is defined by (15) and ν is the unique zero in $(0, 1/n)$ of the function Λ defined by (46). Then:

(i) *$x \in \mathscr{D}$, where $\mathscr{D}$ is defined by (3);*

(ii) *$\|T(x) - \xi\| \preceq \beta(E(x))\, \|x - \xi\|$, where β is defined by (45).*

Proof. The proof is similar to the proof of Lemma 5. We again define the quantities σ_i, A_i, B_i, and C_i by (19), (20), (21), and (22). Using Lemma 2, we obtain the following estimates:

$$|\sigma_i| \leq \frac{(n-1)E(x)}{1-E(x)}, \quad |A_i| \leq \frac{aE(x)^2}{1-E(x)}, \quad |B_i| \leq \frac{(n-1)(E(x)-2)E(x)}{1-E(x)}, \quad |C_i| \leq \frac{a(E(x)-2)\,E(x)^3}{(1-E(x))^2}\,.$$

From these estimates, we get:

$$|2(1+\sigma_i) - A_iB_i - C_i| \geq \frac{\Lambda(E(x))}{(1-E(x))^2} > 0.$$

The rest of the proof is the same as in Lemma 5. □

Now, we are ready to state the second main result of this paper. In the formulation of the theorem we use the following real functions:

$$\Psi(t) = 1 - bt - \beta(t)(1+bt), \tag{48}$$

$$\psi(t) = 1 - bt(1+\beta(t)) \quad \text{and} \quad \phi(t) = \beta(t)/\psi(t), \tag{49}$$

where β is defined by (45) and b is defined in (9).

Theorem 6. *Let a polynomial $f \in \mathbb{K}[z]$ be of degree $n \geq 2$ with n simple zeros in $\mathbb{K}$, $1 \leq p \leq \infty$ and $\xi \in \mathbb{K}^n$ be a root vector of f. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation with distinct coordinates satisfying:*

$$E(x^{(0)}) < 1/n, \quad \Lambda(E(x^{(0)})) > 0 \quad \text{and} \quad \Psi(E(x^{(0)})) \geq 0, \tag{50}$$

where E, Λ, and Ψ are defined by (15), (46) and (48), respectively. Then, the Wang–Zheng's iterative sequence (1) is well defined and convergent to ξ with error estimates:

$$\|x^{(k+1)} - \xi\| \preceq \theta\lambda^{4^k}\|x^{(k)} - \xi\| \quad \text{and} \quad \|x^{(k)} - \xi\| \preceq \theta^k\lambda^{(4^k-1)/3}\|x^{(0)} - \xi\| \quad \text{for all } k \geq 0, \tag{51}$$

where $\theta = \psi(E(x^{(0)}))$, $\lambda = \phi(E(x^{(0)}))$, and ψ, ϕ are defined by (49). Besides, if $\Psi(E(x^{(0)})) > 0$, then the order of convergence is at least four.

Proof. Let $T\colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ be defined by (2), and let the real function β be defined by (45). It follows from the first two inequalities of (50) that $E(x^{(0)}) < \nu$, where ν is the unique zero of the function Λ in $(0, 1/n)$. Hence, the initial conditions (50) can be written in the form:

$$E(x^{(0)}) < \nu \quad \text{and} \quad \Psi(E(x^{(0)})) \geq 0.$$

It follows from Lemma 6 that the initial condition (16) holds with $J = [0, \nu]$. Applying Theorem 4 to the iteration function T, we conclude that the iterative sequence (1) is well defined and convergent to ξ with order four and with error bounds (51). □

Applying Theorem 6 with $p = \infty$, we get the next result.

Corollary 3. *Let a polynomial $f \in \mathbb{K}[z]$ be of degree $n \geq 2$ with n simple zeros in $\mathbb{K}$ and $\xi \in \mathbb{K}^n$ be a root vector of f. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation with distinct coordinates satisfying:*

$$E(x^{(0)}) = \left\| \frac{x^{(0)} - \xi}{d(x^{(0)})} \right\|_\infty \leq \frac{20}{33n}\,. \tag{52}$$

Then, the iteration (1) is well defined and convergent to ξ with order four and with error estimates (51), where the functions ψ and ϕ are defined by (49) with $p = \infty$.

Proof. According to Theorem 6, it is sufficient to prove the following two inequalities $\Lambda(20/(33n)) > 0$ and $\Psi(20/(33n)) > 0$, where the real functions Λ and Ψ are defined by (46) and (48) with $p = \infty$. We prove only the second inequality since the first one is trivial. It is easy to show that the second inequality can be written in the form $a_n < b_n$, where $a_n = \beta(20/(33n))$ and $b_n = (33n - 40)/(33n + 40)$. Taking into account that $(a_n)_{n=2}^{\infty}$ is a decreasing sequence and $(b_n)_{n=2}^{\infty}$ is an increasing sequence, we get $a_n \le a_2 < b_2 \le b_n$. This completes the proof. □

5. Semilocal Convergence Analysis

Let f be a polynomial $\mathbb{K}[z]$ of degree $n \ge 2$. In the present section, we provide two semilocal convergence theorems for Wang–Zheng's method (1), which improve the result of Petković, Petković, and Rančić (Theorem 2). Here, we study the convergence of (1) with respect to the function of initial conditions $E_f \colon \mathcal{D} \to \mathbb{R}_+$ given by:

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p \qquad (1 \le p \le \infty), \tag{53}$$

where the function $W_f \colon \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by (7).

Henceforth, we use a metric ρ on $\mathbb{K}^n$, which was introduced in [23]. First, we define an equivalence relation $\equiv$ on $\mathbb{K}^n$ as follows: $x \equiv y$ if there exists a permutation $(i_1, \ldots, i_n)$ of the indexes $(1, \ldots, n)$ such that $(x_1, \ldots, x_n) = (y_{i_1}, \ldots, y_{i_n})$. Then, the distance $\rho(x, y)$ between two vectors $x, y \in \mathbb{K}^n$ is defined by:

$$\rho(x, y) = \min_{u \equiv y} \|x - u\|_p \,. \tag{54}$$

To prove the results in this section, we need Theorem 6 and Corollary 3, as well as the following two theorems of Proinov [16].

Theorem 7 ([16])**.** *Let $(\mathbb{K}, |\cdot|)$ be an algebraically-closed valued field and $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$. Suppose there exists a vector $x \in \mathbb{K}^n$ with pairwise distinct coordinates such that:*

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p < \tau = \frac{1}{(1 + \sqrt{a})^2} \tag{55}$$

for some $1 \le p \le \infty$, where a is defined by (9). Then, f possesses only simple zeros, and there is a root vector $\xi \in \mathbb{K}^n$ of f such that:

$$\rho(x, \xi) \le \alpha(E_f(x)) \|W_f(x)\|_p \quad \textit{and} \quad \left\| \frac{x - \xi}{d(x)} \right\|_p < h(E_f(x)), \tag{56}$$

where $\alpha, h \colon [0, \tau] \to \mathbb{R}_+$ are defined by:

$$\alpha(t) = \frac{2}{1 - (a-1)t + \sqrt{(1 - (a-1)t)^2 - 4t}} \quad \textit{and} \quad h(t) = t\alpha(t). \tag{57}$$

Theorem 8 ([16])**.** *Let $(\mathbb{K}, |\cdot|)$ be an algebraically-closed valued field and $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose there exists a vector $x \in \mathbb{K}^n$ with pairwise distinct coordinates such that:*

$$\left\| \frac{W_f(x)}{d(x)} \right\|_p < \frac{R(1-R)}{1+(a-1)R} \tag{58}$$

for some $1 \leq p \leq \infty$ and $0 \leq R \leq 1/(1+\sqrt{a})$, where a is defined by (9). Then, polynomial f possesses only simple zeros, and there exists a root vector $\xi \in \mathbb{K}^n$ of f such that:

$$\left\| \frac{x-\xi}{d(x)} \right\|_p < R, \tag{59}$$

where the function α is defined by (57).

Now, we are ready to state the third main theorem of this paper.

Theorem 9. *Let $(\mathbb{K}, |\cdot|)$ be an algebraically-closed valued field and $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ and $1 \leq p \leq \infty$. Suppose $x^{(0)} \in \mathbb{K}^n$ is an initial approximation with pairwise distinct coordinates satisfying:*

$$E_f(x^{(0)}) \leq \frac{n-1}{n(a+n-1)}, \quad \Lambda(h(E_f(x^{(0)}))) > 0 \quad \textit{and} \quad \Psi(h(E_f(x^{(0)}))) \geq 0, \tag{60}$$

where a is defined in (9), and E_f, Λ, Ψ, and h are defined by: (53), (46), (48), and (57), respectively. Then, f possesses only simple zeroes, and Wang–Zheng's iteration (1) is well defined and convergent to a root vector ξ of f with order four and with error estimate:

$$\rho(x^{(k)}, \xi) \leq \alpha(E_f(x^{(k)}))\|W_f(x^{(k)})\|_p \quad \textit{for all } k \geq 0 \textit{ with } E_f(x^{(k)}) < \tau, \tag{61}$$

where the function α is defined by (57) and τ is defined in (55).

Proof. Let $\tau_n = (n-1)/(an+n^2-n)$. It can be proven that $\tau_n < \tau$ and $h(\tau_n) = 1/n$. It follows from $\tau_n < \tau$ and the first inequality in (60) that $E_f(x^{(0)}) < \tau$. By Theorem 7, f possesses only simple zeros, and there is a root vector $\xi \in \mathbb{K}^n$ of f such that:

$$E(x^{(0)}) < h(E_f(x^{(0)})), \tag{62}$$

where the function E is defined by (15). On the other hand, it follows from $E_f(x^{(0)}) < \tau_n$ that:

$$h(E_f(x^{(0)})) < h(\tau_n) = 1/n. \tag{63}$$

Combining (62) and (63), we get $E(x^{(0)}) < 1/n$. From (63) and the second inequality in (60), we deduce that $h(E_f(x^{(0)})) < \nu$, where ν is the unique zero of the function Λ in $(0, 1/n)$. From (62) and the second inequality in (60), taking into account that Λ is decreasing on $[0, \nu)$, we obtain:

$$\Lambda(E(x^{(0)})) > \Lambda(h(E_f(x^{(0)}))) > 0.$$

Analogously, from (62) and the third inequality in (60), taking into account that Ψ is decreasing on $[0, \nu)$, we get:

$$\Psi(E(x^{(0)})) > \Psi(h(E_f(x^{(0)}))) \geq 0.$$

Thus, $x^{(0)}$ satisfies the initial conditions (50). Then, it follows from Theorem 6 that the iterative sequence (1) is well defined and convergent to ξ with order four. The estimate (61) follows from Theorem 7. □

From Corollary 3 and Theorem 8, we obtain the next theorem, which improves and complements the result of Petković, Petković, and Rančić (Theorem 2).

Theorem 10. *Let $(\mathbb{K}, |\cdot|)$ be an algebraically-closed valued field. Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ and $x^{(0)} \in \mathbb{K}^n$ be an initial approximation with pairwise distinct coordinates satisfying:*

$$E_f(x^{(0)}) = \left\| \frac{W_f(x^{(0)})}{d(x^{(0)})} \right\|_\infty \leq R'_n = \frac{20(33n-20)}{33n(53n-40)}, \tag{64}$$

where E_f is defined by (53). Then, f has only simple zeros, and Wang–Zheng's iteration (1) is well defined and convergent to a root vector ξ of f with order four and with error estimate (61) with $p = \infty$.

Proof. If we take $R = 20/(33n)$, then we can write (64) in the form:

$$\left\| \frac{W_f(x^{(0)})}{d(x^{(0)})} \right\|_\infty < \frac{R(1-R)}{1+(a-1)R}$$

with $a = n-1$. Then, by Theorem 8, f possesses only simple zeros, and there is a root vector $\xi \in \mathbb{K}^n$ of f such that:

$$E(x^{(0)}) < R,$$

where E is defined by (15). Then, it follows from Corollary 3 that the iterative sequence (1) is convergent to ξ with order four. The estimate (61) follows Theorem 7. □

6. Conclusions

In 1984, Wang and Zheng [34] derived a family of iterative methods for simultaneously finding all zeros $\xi_1, \ldots, \xi_n$ of a polynomial f of degree $n \geq 2$. The present paper deals with the convergence of the method (1), which is a well-known member of the Wang–Zheng family. We have presented three types of convergence theorems for Wang–Zheng's method (1).

In 1987, Wang and Wu [32] established a local convergence result for Wang–Zheng's method (Theorem 1), which gives a lower bound for the convergence radius and an a posteriori error estimate. They proved that the method (1) is convergent under the initial condition of the type:

$$\|x^{(0)} - \xi\|_\infty < R_n,$$

where the radius of convergence R_n depends on n and the parameters $\delta(\xi) = \min_{i \neq j} |\xi_i - \xi_j|$ and $\Delta(\xi) = \max_{i \neq j} |\xi_i - \xi_j|$. Their error estimate has the form:

$$\|x^{(k+1)} - \xi\|_\infty \leq C_n \, \|x^{(k)} - \xi\|_\infty^4 \quad \text{for all } k \geq 0. \tag{65}$$

where C_n depends on n, $\delta(\xi)$, and $\Delta(\xi)$. In Section 3, we have obtained a local convergence result (Theorem 5), which improves the result of Wang and Wu [32]. The advantages of this result are:

- a larger convergence domain of the method (1);
- a greater convergence radius R_n, which does not depend on $\Delta(\xi)$;

- an error estimate of the form (65) with smaller C_n, which does not depend on $\Delta(\xi)$;
- two new error estimates of the form:

$$\|x^{(k)} - \xi\| \preceq \lambda^{(4^k-1)/3} \|x^{(0)} - \xi\| \quad \text{and} \quad \|x^{(k+1)} - \xi\| \preceq \lambda^{4^k} \|x^{(k)} - \xi\| \quad \text{for all } k \geq 0,$$

where $0 \leq \lambda < 1$.

In Section 4, we have established a local convergence result of the second type (Theorem 6) for Wang–Zheng's method (1). The second type of convergence results are closer to the semilocal convergence results. The first convergence result of the second type were obtained by Wang and Zhao [33] for the Weierstrass method. Our local convergence result of the second type is the first result of this type for Wang–Zheng's method (1). The convergence results of the second type for other simultaneous methods can be found in [17,22–24,28,29].

In Section 5, we have obtained two semilocal convergence theorems (Theorem 9 and Theorem 10) for Wang–Zheng's method (1). In these theorems, we prove the convergence of the method under the initial conditions of the form:

$$\left\| \frac{W_f(x^{(0)})}{d(x^{(0)})} \right\|_\infty \leq R_n \,, \tag{66}$$

where R_n depends on n and the functions W_f and d are defined by (7) and (11), respectively. Initial conditions of the type (66) were considered for the first time by Proinov [14,15]. Our semilocal convergence results improved the result of Petković, Petković and Rančić [13] (Theorem 2). The advantages of this result are:

- weaker convergence conditions of the method (1);
- a verifiable a posteriori error estimate, which can be used as a stop criterion when applying Wang–Zheng's method;
- we did not assume either the simplicity, or existence of the zeros of f.

Finally, we refer the reader to some recent papers [2,17,20,22–25,28,29], which investigate initial conditions of the type (66).

Funding: This research was supported by the National Science Fund of the Bulgarian Ministry of Education and Science under Grant DN 12/12.

Acknowledgments: The author wishes to express gratitude to Professor Petko D. Proinov for his valuable suggestions and comments on this research.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Cholakov, S.I. Local convergence of Chebyshev-like method for simultaneous finding polynomial zeros. *Compt. Rend. Acad. Bulg. Sci.* **2013**, *66*, 1081–1090.
2. Cholakov, S.I.; Vasileva, M.T. A convergence analysis of a fourth-order method for computing all zeros of a polynomial simultaneously. *J. Comp. Appl. Math.* **2017**, *321*, 270–283. [CrossRef]
3. Iliev, A.I. Generalization of Ehrlich-Kjurkchiev method for multiple roots of algebraic equations. *Serdica Math. J.* **1998**, *24*, 482–488.
4. Iliev, A.I.; Kyurkchiev, N.V. On a generalization of Obreshkoff-Ehrlich method for simultaneous inclusion of only part of all multiple roots of algebraic polynomials. *Comput. Math. Math. Phys.* **2003**, *43*, 482–488.
5. Ivanov, S.I. A unified semilocal convergence analysis of a family of iterative algorithms for computing all zeros of a polynomial simultaneously. *Numer. Algorithms* **2017**, *75*, 1193–1204. [CrossRef]
6. Kjurkchiev, N.V. *Initial Approximations and Root Finding Methods*; Wiley: Berlin, Germany, 1998.

7. Kyncheva, V.K.; Yotov, V.V.; Ivanov, S.I. Convergence of Newton, Halley and Chebyshev iterative methods as methods for simultaneous determination of multiple polynomial zeros. *Appl. Numer. Math.* **2017**, *112*, 146–154. [CrossRef]
8. Kyncheva, V.K.; Yotov, V.V.; Ivanov, S.I. On the convergence of Schröder's method for the simultaneous computation of polynomial zeros of unknown multiplicity. *Calcolo* **2017**, *54*, 1199–1212. [CrossRef]
9. McNamee, J.M. *Numerical Methods for Roots of Polynomials, Part I. Studies in Computational Mathematics*; Elsevier: Amsterdam, The Netherlands, 2007.
10. Petković, M. *Point Estimation of Root Finding Methods. Lecture Notes in Mathematics*; Springer: Berlin, Germany, 2008.
11. Nedzhibov, G.H.; Petkov, M.G. On a family of iterative methods for simultaneous extraction of all roots of algebraic polynomial. *Appl. Math. Comput.* **2005**, *162*, 427–433. [CrossRef]
12. Petković, M. Herceg, D. On the convergence of Wang–Zheng's method. *J. Comput. Appl. Math* **1998**, *91*, 123–135. [CrossRef]
13. Petković, M.; Petković, L.; Rančić, L. Point estimation of simultaneous methods for solving polynomial equations: A survey (II). *J. Comput. Appl. Math* **2007**, *205*, 32–52. [CrossRef]
14. Proinov, P.D. A new semilocal convergence theorem for the Weierstrass method from data at one point. *Compt. Rend. Acad. Bulg. Sci.* **2006**, *59*, 131–136.
15. Proinov, P.D. Semilocal convergence of two iterative methods for simultaneous computation of polynomial zeros. *Compt. Rend. Acad. Bulg. Sci.* **2006**, *59*, 705–712.
16. Proinov, P.D. Relationships between different types of initial conditions for simultaneous root finding methods. *Appl. Math. Lett.* **2016**, *52*, 102–111. [CrossRef]
17. Proinov, P.D. General convergence theorems for iterative processes and applications to the Weierstrass root-finding method. *J. Complex.* **2016**, *33*, 118–144. [CrossRef]
18. Proinov, P.D. Unified convergence analysis for Picard iteration in n-dimensional vector spaces. *Calcolo* **2018**, *55*, 6. [CrossRef]
19. Proinov, P.D. On the local convergence of Gargantini-Farmer-Loizou method for simultaneous approximation of multiple polynomial zeros. *J. Nonlinear Sci. Appl.* **2018**, *11*, 1045–1055. [CrossRef]
20. Proinov, P.D.; Cholakov, S.I. Semilocal convergence of Chebyshev-like root-finding method for simultaneous approximation of polynomial zeros. *Appl. Math. Comput.* **2014**, *236*, 669–682. [CrossRef]
21. Proinov, P.D.; Ivanov, S.I. Convergence of Schröder's method for polynomial zeros of unknown multiplicity. *Compt. Rend. Acad. Bulg. Sci.* **2013**, *66*, 1073–1080.
22. Proinov, P.D.; Ivanov, S.I. On the convergence of Halley's method for simultaneous computation of polynomial zeros. *J. Numer. Math.* **2015**, *23*, 379–394. [CrossRef]
23. Proinov, P.D.; Ivanov, S.I. Convergence analysis of Sakurai–Torii–Sugiura iterative method for simultaneous approximation of polynomial zeros. *J. Comp. Appl. Math.* **2019**, *357*, 56–70. [CrossRef]
24. Proinov, P.D.; Ivanov, S.I.; Petković, M.S. On the convergence of Gander's type family of iterative methods for simultaneous approximation of polynomial zeros. *Appl. Math. Comput.* **2019**, *349*, 168–183. [CrossRef]
25. Proinov, P.D.; Petkova, M.D. Convergence of the Weierstrass method for simultaneous approximation of polynomial zeros. *Compt. Rend. Acad. Bulg. Sci.* **2013**, *66*, 809–818. [CrossRef]
26. Proinov, P.D.; Petkova, M.D. A new semilocal convergence theorem for the Weierstrass method for finding zeros of a polynomial simultaneously. *J. Complexity* **2014**, *30*, 366–380. [CrossRef]
27. Proinov, P.D.; Petkova, M.D. Convergence of the two-point Weierstrass root-finding method. *Japan J. Industr. Appl. Math.* **2014**, *31*, 279–292. [CrossRef]
28. Proinov, P.D.; Vasileva, M.T. On the convergence of high-order Ehrlich-type iterative methods for approximating all zeros of a polynomial simultaneously. *J. Inequal. Appl.* **2015**, *2015*, 336. [CrossRef]
29. Proinov, P.D.; Vasileva, M.T. On a family of Weierstrass-type root-finding methods with accelerated convergence. *Appl. Math. Comput.* **2016**, *273*, 957–968. [CrossRef]
30. Sendov Bl.; Andreev, A.; Kjurkchiev, N. Numerical Solution of Polynomial Equations. *Handb. Numer. Anal.* **1994**, *3*, 1625–1778.

31. Sakurai, T.; Petković, M.S. On some simultaneous methods based on Weierstrass correction. *J. Comput. Appl. Math.* **1996**, *72*, 275–291. [CrossRef]
32. Wang, D.; Wu, Y. Some modifications of the parallel Halley iteration method and their convergence. *Computing* **1987**, *38*, 75–87. [CrossRef]
33. Wang, D.; Zhao, F. On the determination of a safe initial approximation for the Durand-Kerner algorithm. *J. Comput. Appl. Math.* **1991**, *38*, 447–456.
34. Wang, X.; Zheng, S. A family of parallel ind interval iterations for finding all roots of a polynomial simultaneously with rapid convergence I. *J. Comput. Math.* **1984**, *1*, 70–76.
35. Proinov, P.D. General local convergence theory for a class of iterative processes and its applications to Newton's method. *J. Complex.* **2009**, *25*, 38–62. [CrossRef]
36. Proinov, P.D. New general convergence theory for iterative processes and its applications to Newton Kantorovich type theorems. *J. Complex.* **2010**, *26*, 3–42. [CrossRef]
37. Ivanov, S.I. On the convergence of Chebyshev's method for multiple polynomial zeros. *Results Math.* **2016**, *69*, 93–103. [CrossRef]
38. Proinov, P.D.; Ivanov, S.I. On the convergence of Halley's method for multiple polynomial zeros. *Mediterr. J. Math.* **2015**, *12*, 555–572. [CrossRef]