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Revisiting Ćirić-Type Contraction with Caristi's Approach

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Received: 22 April 2019; Accepted: 24 May 2019; Published: 28 May 2019

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Abstract: The essential goal of the main result is to merge two celebrated fixed-point results that belong to Ćirić and Caristi. The obtained result not only combines but also extends these two results in the context of complete metric spaces. An example is presented to indicate the validity and genuineness of the main result.

Keywords: Ćirić-type contraction; Caristi fixed-point theorem; fixed point; contraction

MSC: 46T99; 47H10; 54H25

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1. Introduction and Preliminaries

Metric fixed-point theory was emergent with the abstraction of successive approximations that was reported by Banach as a contraction mapping principle. Since then, this topic has been studied and extended by several authors. It would not be wrong to say that, in the last decades, one of the hottest research topics has been fixed-point theory. The main reason behind this fact is the application potential of the observed results. It is possible to find several distinct applications of fixed-point theory in almost all quantitative sciences. Apart from the several branches of mathematics, economics, and computer science, there are very well-known crucial and interesting disciplines for applications of fixed-point theory.

We start by recalling the pioneer results in metric fixed-point theory:

Theorem 1. *Banach [1] Let (M,d) be a complete metric space and $f : M \to M$ be mapping. Suppose that there is $q \in [0,1)$, such that*

$$d(fx, fy) \leq qd(x,y), \tag{1}$$

for all $x, y \in M$. Then, f has a unique fixed point in M.

Since then, this result has been extended in several aspects (see, e.g., References [2–10] and the references therein).

In this paper, we restricted ourselves to merge two interesting fixed-point results that belong to Caristi [11] and Ćirić [2,3]. Indeed, the nature of these results is quite different from each other. Roughly speaking, Ćirić [2,3] involved all distances $d(fx,fy), d(x,y), d(x,fx), d(y,fy), d(x,fy), d(y,fx)$ in his contraction in a linear way, while Banach [1] used only the first two distances, where f is assumed as self-mapping on a metric space (M, d) with $x, y \in M$. More precisely, the renowned Ćirić [2,3] for a single-valued map is the following:

Theorem 2. *Ćirić [2,3] Let (M, d) be a complete metric space and $f : M \to M$ be mapping. Suppose that there is $q \in [0, 1)$ such that*

$$d(fx, fy) \leq qN(x, y), \tag{2}$$

for all $x, y \in M$, where,

$$N(x, y) = \max \{d(x, y), d(x, fx), d(y, fy), d(y, fx), d(x, fy)\}$$

Then, f has a unique fixed point in M.

Note that this theorem covers almost all fixed-point theorems via linear contraction $N(x, y)$, in particular, Banach's fixed-point theorem. Indeed, since $d(x, y) \leq N(x, y)$, Theorem 1, Banach contraction mapping turns to be a corollary of Ćirić's fixed-point theorem. In particular, for positive real numbers α_i, $i = 1, 2, ..., 5$, we have

- (*i*) (Kannan [6]): $\alpha_1 d(x, fx) + \alpha_1 d(y, fy) \leq N(x, y)$, where $\alpha_1 + \alpha_2 \leq 1$;
- (*ii*) (Chatterjea [7]): $\alpha_1 d(x, fy) + \alpha_1 d(y, fx) \leq N(x, y)$, where $\alpha_1 + \alpha_2 \leq 1$;
- (*iii*) Reich [8]): $\alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) \leq N(x, y)$, where $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$
- (*iv*) (Anonymous): $\alpha_1 d(x, y) + \alpha_2 d(x, fy) + \alpha_3 d(y, fx) \leq N(x, y)$, where $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$
- (*v*) (Hardy-Rogers [9]): $\alpha_1 d(x, y) + \alpha_2 d(x, fx) + \alpha_3 d(y, fy) + \alpha_4 \frac{d(x,fy)+d(y,fx)}{2} \leq N(x, y)$, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq 1$.

From Observations (*i*–*v*) above, one can conclude that all famous fixed-point theorems of a linear type, for instance, Kannan [6], Chatterjea [7], Reich [8], Hardy-Rogers [9], are a consequence of Ćirić's fixed-point theorem.

On the other hand, Caristi [11] also only considered distance $d(fx, fy)$ that was dominated by the difference of the image of x and fx under a suitable lower semicontinuous function. For the sake of completeness, we recollect Caristi's fixed-point theorem as follows:

Theorem 3. *Caristi [11] Let (M, d) be a complete metric space, and $\varphi : M \to [0, \infty)$ be a lower semicontinuous and bounded below function. Suppose that f is Caristi-type mapping on M dominated by φ; that is, f satisfies*

$$d(x, fx) \leq \varphi(x) - \varphi(fx)$$

for each $x \in M$. Then, f has a fixed point in M.

In this short note, we aimed to merge these two significant fixed-point theorems and extend them. This note can be thought as a continuation of Reference [10].

2. Main Result

The main result of this note is the following:

Theorem 4. *Suppose that f is self-mapping on complete metric (M, d). If there is a $\varphi : M \to [0, \infty)$, such that*

$$d(x, fx) > 0 \textit{ implies } d(fx, fy) \leq (\varphi(x) - \varphi(fx))N(x, y), \tag{3}$$

in which

$$N(x, y) = \max \{d(x, y), d(x, fx), d(y, fy), d(y, fx), d(x, fy)\}$$

for all $x, y \in M$. Then, f has a fixed point.

Proof. Let $x_0 \in M$. If $fx_0 = x_0$, the proof is completed. Herewith, we assume that $d(x_0, fx_0) > 0$. Without loss of generality, keeping the same argument in mind, we assume that $x_{n+1} = fx_n$ and, hence,

$$d(x_n, x_{n+1}) = d(x_n, fx_n) > 0. \tag{4}$$

For that sake of convenience, suppose that $b_n = d(x_n, x_{n-1})$. From Equation (3), we derive that

$$\begin{aligned} b_{n+1} &= d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \\ &\leq (\varphi(x_{n-1}) - \varphi(fx_{n-1}))N(x_{n-1}, x_n) \\ &= (\varphi(x_{n-1}) - \varphi(x_n)) \max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\}. \end{aligned} \tag{5}$$

We divide our proof into three steps:

Step 1: There exist $\gamma \in [0, 1)$, such that

$$d(x_n, x_{n+1}) \leq \gamma d(x_n, x_{n-1}).$$

To reach the goal, we consider three cases:

Case 1: If $\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} = d(x_n, x_{n-1})$, taking into account Equation (5), we have

$$\begin{aligned} &b_{n+1} = d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) = (\varphi(x_{n-1}) - \varphi(x_n))d(x_n, x_{n-1}) \\ &= (\varphi(x_{n-1}) - \varphi(x_n))b_n. \end{aligned}$$

So, we get

$$0 < \frac{b_{n+1}}{b_n} \leq \varphi(x_{n-1}) - \varphi(x_n) \text{ for each } n \in \mathbb{N}.$$

Thus, sequence $\{\varphi(x_n)\}$ is necessarily positive and nonincreasing. Hence, it converges to some $r \geq 0$. On the other hand, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{b_{k+1}}{b_k} &\leq \sum_{k=1}^{n} \varphi(x_{k-1}) - \varphi(x_k) \\ &= (\varphi(x_0) - \varphi(x_1)) + (\varphi(x_1) - \varphi(x_2)) + ... + (\varphi(x_{n-1}) - \varphi(x_n)) \\ &= \varphi(x_0) - \varphi(x_n) \to \varphi(x_0) - r < \infty, \text{ as } n \to \infty. \end{aligned}$$

This means that

$$\sum_{n=1}^{\infty} \frac{b_{n+1}}{b_n} < \infty.$$

Accordingly, we have

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 0. \tag{6}$$

On account of Equation (10), for $\gamma \in (0, 1)$, there exists $n_0 \in \mathbb{N}$, such that

$$\frac{b_{n+1}}{b_n} \leq \gamma, \tag{7}$$

for all $n \geq n_0$. It yields that

$$d(x_{n+1}, x_n) \leq \gamma d(x_n, x_{n-1}), \tag{8}$$

for all $n \geq n_0$.

Case 2: If $\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} = d(x_n, x_{n+1})$, then Equation (5) implies that

$$d(x_{n+1}, x_n) \le (\varphi(x_{n-1}) - \varphi(x_n))d(x_{n+1}, x_n).$$

So $\{\varphi(x_n)\}$ is a nonincreasing and positive sequence, and so converges to some $r \ge 0$. Since $d(x_n, x_{n+1}) \ne 0$, we have $1 \le (\varphi(x_{n-1}) - \varphi(x_n)) \to 0$ as $n \to \infty$, which is a contradiction.

Case 3: Suppose that $\max\{d(x_n, x_{n-1}), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} = d(x_{n-1}, x_{n+1})$, revisiting Equation (5) and taking $b_n = d(a_{n-1}, a_n)$ for the sake of convenience, we have

$$\begin{aligned} b_{n+1} &= d(x_{n+1}, x_n) \\ &\le (\varphi(a_{n-1}) - \varphi(a_n))d(a_{n-1}, a_{n+1}) \\ &\le (\varphi(x_{n-1}) - \varphi(x_n))(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &= (\varphi(x_{n-1}) - \varphi(x_n))(b_n + b_{n+1}). \end{aligned}$$

Therefore, we have

$$\frac{b_{n+1}}{b_n + b_{n+1}} \le \varphi(x_{n-1}) - \varphi(x_n). \tag{9}$$

So $\{\varphi(x_n)\}$ is positive and nonincreasing. Hence, it converges to some $r \ge 0$. On the other hand, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{b_{k+1}}{b_k + b_{k+1}} &\le \sum_{k=1}^{n} \varphi(x_{k-1}) - \varphi(x_k) \\ &= (\varphi(x_0) - \varphi(x_1)) + (\varphi(x_1) - \varphi(x_2)) + ... + (\varphi(x_{n-1}) - \varphi(x_n)) \\ &= \varphi(x_0) - \varphi(x_n) \to \varphi(x_0) - r < \infty, \text{ as } n \to \infty. \end{aligned}$$

It means that

$$\sum_{n=1}^{\infty} \frac{b_{n+1}}{b_n + b_{n+1}} < \infty.$$

Accordingly, we have

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n + b_{n+1}} = 0. \tag{10}$$

On account of Equation (10), for $\beta \in (0, \frac{1}{2})$, there exists $n_0 \in \mathbb{N}$, such that

$$\frac{b_{n+1}}{b_n + b_{n+1}} \le \beta, \tag{11}$$

for all $n \ge n_0$. It yields that

$$b_{n+1} \le \frac{\beta}{1-\beta} b_n, \tag{12}$$

Since $\beta < \frac{1}{2}$, we have $\frac{\beta}{1-\beta} < 1$. By taking $\gamma = \frac{\beta}{1-\beta}$, then we have

$$d(x_{n+1}, x_n) \le \gamma d(x_n, x_{n-1}), \tag{13}$$

for all $n \ge n_0$.

Step 2: Sequence $\{x_n\}$ converges to some $\omega \in M$.

Note that Step 1 shows that sequence $\{d(x_{n+1}, x_n)\}$ is nonincreasing and bounded below. So, it is convergent to some $q \geq 0$. Since $\gamma < 1$, it is easily verified that $q = 0$. For each $m, n \in \mathbb{N}$ with $m > n$, we also have

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} d(x_i, x_{i+1}) \leq \frac{\gamma^n}{1-\gamma} d(x_0, x_1).$$

This means that $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$. Therefore, $\{x_n\}$ is a Cauchy sequence and, since M is complete, there exists $\omega \in M$, such that $\{x_n\}$ converges to ω.

Step 3: ω is a fixed point of f.

By employing Equation (3), if $d(\omega, f\omega) > 0$, then we find that

$$\begin{aligned}
& d(\omega, f\omega) \\
\leq\ & d(\omega, x_{n+1}) + d(x_{n+1}, f\omega) \\
=\ & d(\omega, x_{n+1}) + d(fx_n, f\omega) \\
=\ & d(\omega, x_{n+1}) + \\
& (\varphi(x_n) - \varphi(fx_n))N(x_n, \omega) \\
=\ & d(\omega, x_{n+1}) + \\
& (\varphi(x_n) - \varphi(x_{n+1})) \max\{d(x_n, \omega), d(x_n, fx_n), d(\omega, f\omega), d(x_n, f\omega), d(fx_n, \omega)\} \\
=\ & d(\omega, x_{n+1}) + \\
& (\varphi(x_n) - \varphi(x_{n+1})) \max\{d(x_n, \omega), d(x_n, fx_n), d(\omega, f\omega), d(x_n, f\omega), d(x_{n+1}, \omega)\}
\end{aligned} \tag{14}$$

Since sequences $\{\varphi(x_n)\}$ tend to $r \geq 0$, for sufficiently large $n \in \mathbb{N}$, we have

$$d(\omega, f\omega) \leq \lim_{n\to\infty} (d(\omega, x_{n+1}) + (\varphi(x_n) - \varphi(x_{n+1})))d(\omega, f\omega) = 0.$$

Consequently, we obtain $d(\omega, f\omega) = 0$, that is, $T\omega = \omega$.

□

From Theorem 4, we obtain the corresponding result for complete metric spaces. The following example shows that Theorem 4 is not a consequence of Banach's contraction principle.

Example 1. *Let $M = \{0, 1, 2\}$, endowed with the following metric:*

$$d(0,1) = 1, d(2,0) = 1, d(1,2) = \frac{3}{2} \ \textit{and}\ \ d(x,x) = 0,\ \forall\, x \in M\ \ d(x,y) = d(y,x)\ \ \forall\, x, y \in M.$$

Define $f : M \to M$ by $f0 = 0, f1 = 2, f2 = 0$ and $\varphi : M \to [0, \infty)$ by $\varphi(2) = 2, \varphi(0) = 0, \varphi(1) = 4$. If $x \in M$ and $d(x, fx) > 0$, then $x \neq 0$. So, we have

$$\begin{aligned}
d(f1, f2) &\leq (\varphi(1) - \varphi(f1))N(2,1), \\
d(f2, f1) &\leq (\varphi(2) - \varphi(f2))N(2,1), \\
d(f1, f0) &\leq (\varphi(1) - \varphi(f1))N(1,0), \\
d(f2, f0) &\leq (\varphi(2) - \varphi(f2))N(2,0).
\end{aligned}$$

Thus, f satisfies the condition of our theorem, so f has a fixed point. Note that $d(f1, f0) = d(1, 0)$. So, f does not satisfy the Banach contraction principle. Moreover, it is clear that Reich's fixed-point theorem, Hardy-Rogers's fixed-point theorem, and Ćirić's fixed-point theorem are not applicable in this example for the same reason.

Again, by Observations (*i*–*v*) above, we deduced the following corollaries:

Corollary 1. *Suppose that f is self-mapping on complete metric (M, d). If there is a $\varphi : M \to [0, \infty)$, such that*

$$d(x, fx) > 0 \text{ implies } d(fx, fy) \leq (\varphi(x) - \varphi(fx))d(x, y),$$

for all $x, y \in M$. Then, f has at least one fixed point.

Corollary 2. *Suppose that f is self-mapping on complete metric (B, d). If there exists $\varphi : B \to [0, \infty)$ and $\alpha_1, \alpha_2 \in [0, 1]$, such that*

$$d(x, fx) > 0 \text{ implies } d(fx, fy) \leq (\varphi(x) - \varphi(fx))[\alpha_1 d(x, fx) + \alpha_2 d(y, fy)],$$

for all $x, y \in M$, where $\alpha_1 + \alpha_2 \leq 1$ Then, f has at least one fixed point.

Corollary 3. *Suppose that f is self-mapping on complete metric (M, d). If there exists $\varphi : M \to [0, \infty)$ and $\alpha_1, \alpha_2 \in [0, 1]$, such that*

$$d(x, fx) > 0 \text{ implies } d(fx, fy) \leq (\varphi(x) - \varphi(fx))[\alpha_1 d(x, fy) + \alpha_2 d(y, fx)],$$

for all $x, y \in M$, where $\alpha_1 + \alpha_2 \leq 1$ Then, f has at least one fixed point.

Corollary 4. *Suppose that f is self-mapping on complete metric (M, d). If there exists $\varphi : B \to [0, \infty)$ and $\alpha_0, \alpha_1, \alpha_2 \in [0, 1]$, such that*

$$d(x, fx) > 0 \text{ implies } d(fx, fy) \leq (\varphi(x) - \varphi(fx))[\alpha_0 d(x, y) + \alpha_1 d(x, fx) + \alpha_2 d(y, fy)],$$

for all $x, y \in M$, where $\alpha_0 + \alpha_1 + \alpha_2 \leq 1$ Then, f has at least one fixed point.

Corollary 5. *Suppose that f is self-mapping on complete metric (M, d). If there exists $\varphi : M \to [0, \infty)$ and $\alpha_0, \alpha_1, \alpha_2 \in [0, 1]$, such that*

$$d(x, fx) > 0 \text{ implies } d(fax, fy) \leq (\varphi(x) - \varphi(fx))[\alpha_0 d(x, y) + \alpha_1 d(x, fx) + \alpha_2 d(y, fy) + \alpha_3 \frac{d(x, fy) + d(y, fx)}{2}],$$

for all $x, y \in M$, where $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 \leq 1$ Then, f has at least one fixed point.

Remark 1. *Consider $q := \sup\{\varphi(x) - \varphi(fx) : d(x, fx) > 0\}$. If $q < 1$, then Theorem 4 covered Theorem 2. Moreover, if $q < 1$, then Corollaries 1–5 covered the corresponding famous fixed-point results [1,6–9], respectively. For instance, Corollary 1 is covered by the renowned Banach's fixed-point theorem.*

Author Contributions: All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Funding: This work is funded by Prince Sultan University through research group NAMAM, group number RG-DES-2017-01-17.

Acknowledgments: The authors thank the anonymous referees for their remarkable comments, suggestions, and ideas that helped to improve this paper. The third author also thanks Prince Sultan University for funding this work through research-group number RG-DES-2017-01-17.

Conflicts of Interest: The authors declare no conflict of interest.

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