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Free Vibration Analysis of Functionally Graded Shells Using an Edge-Based Smoothed Finite Element Method

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Abstract: An edge-based smoothed finite element method (ES-FEM) combined with the mixed interpolation of tensorial components technique for triangular shell element (MITC3), called ES-MITC3, for free vibration analysis of functionally graded shells is investigated in this work. In the formulation of the ES-MITC3, the stiffness matrices are obtained by using the strain-smoothing technique over the smoothing domains that are formed by two adjacent MITC3 triangular shell elements sharing an edge. The strain-smoothing technique can improve significantly the accuracy and convergence of the original MITC3. The material properties of functionally graded shells are assumed to vary through the thickness direction by a power-rule distribution of volume fractions of the constituents. The numerical examples demonstrated that the present ES-MITC3 method is free of shear locking and achieves the high accuracy compared to the reference solutions in the literature.

Keywords: FGMshells; edge-based smoothed finite element method (ES-FEM); mixed interpolation of tensorial components (MITC)

1. Introduction

Functionally graded materials (FGM) are usually made from a mixture of metals and ceramics, whose material properties vary smoothly and continuously from one surface to the other of the structure according to volume fraction power-law distribution. It is well known that the ceramics are capable of resisting high temperature, while the metals provide structural strength and fracture toughness. They are therefore suitable to apply for aerospace structures, nuclear plants, and other applications. With the advantageous features of the FGM in many practical applications, the problem of static and free vibration behaviors of FGM shell structures are attractive to many researchers over the world. Woo and Meguid [1] used an analytical solution based on the von Karman theory to investigate nonlinear response of FGM plates and shallow shells. Matsunaga [2] carried out the power series expansion of displacement component approach, which relied on higher-order shear deformation theory (HSDT) to analyze free vibration and buckling of FGM shells. Nguyen et al. [3] proposed an analytical solution using Reddy's HSDT to solve nonlinear dynamic and free vibration of piezoelectric FGM double curved shallow shells subjected to electrical, thermal, mechanical, and damping loads. Dao et al. [4] presented nonlinear vibration of stiffened functionally graded double curved shells on an

elastic foundation using the first order shear deformation theory (FSDT) and stress function. Due to the complication of mathematics, it is generally difficult to use analytical methods for all problems. Thus, numerical methods have been devised to study the behavior of FGM structural components. Among these numerical approaches, the finite element method has become the most powerful, reliable, and simply tool to solve FGM shells. Arciniega and Reddy [5] presented a finite element formulation for nonlinear analysis of FGM shell based on the FSDT, consisting of seven parameters. Pradyumna and Bandyopadhyay [6] used the shell element, consisting of nine degrees of freedom per node, to investigate free vibration of FGM shells. Kordkheili and Naghdabadi [7] proposed a finite element model for geometrically nonlinear thermos-elastic analysis of FGM plates and shells.

In addition, in the recent trend of development of numerical methods, the edge-based smoothed finite element method (ES-FEM) combined with the mixed interpolation of tensorial components using triangular element (MITC3), named ES-MITC3, has been proposed to investigate plate and shell structures. For instance, Chau-Dinh et al. [8] proposed an ES-MITC3 to analyze static and free vibration of plates. Nguyen et al. [9] developed the ES-MITC3 for static and vibration analysis of isotropic and functionally graded plates. Pham et al. [10] examined the static and free vibration of composite shells using ES-MITC3 shell element. Pham et al. [11] used ES-MITC3 shell element to study geometrically nonlinear analysis of FGM shells based on FSDT. Pham-Tien et al. [12] investigated the dynamic response of composite shells based on the FSDT and ES-MITC3 element. Hoang-Nam Nguyen et al. [13] used FSDT to investigate dynamic composite shell with shear connectors. For shell structures, especially the shell with two curvatures, the employing of quadrilateral elements will not accurately describe the model due to the distorting during the bending process. In this case, the using of triangle elements is suitable because they can rotate freely around their three edges. However, the using of these elements can meet the shear locking phenomenon; thus, we propose the new method, in which the triangle element is combined with an edge-based smoothed finite element method (ES-MITC3) to analyze the shell structures. Its accuracy in comparison with other methods is shown in the numerical exploration.

This paper now further extends the ES-MITC3 method for free vibration analysis of functionally graded shell structures. The material properties of functionally graded shells are assumed to vary continuously and smoothly through the thickness based on a simple power-law of the volume fractions exponents. The formulation is based on the FSDT and flat shell theory due to the simplicity and computational efficiency. The accuracy and reliability of the present method are verified by comparing with those of others available numerical results.

2. Theoretical Formulation

2.1. Functionally Grade Material

FGM is formed by a mixture of ceramic and metal, as shown in Figure 1. The material properties change continuously from a surface to the other surface according to a power-law of volume fraction

$$P(z) = (P_c - P_m)V_c + P_m \quad (1)$$

$$V_c(z) = \left(\frac{1}{2} + \frac{z}{h} \right)^n \quad (2)$$

where $P(z)$ represents the effective material properties: Young's modulus E , density ρ and Poisson ratio ν ; P_c and P_m denote the properties of the ceramic and metal, respectively; V_c is the volume fractions of the ceramic; h the thickness of structure; $n \geq 0$ the volume fraction exponent; $z \in [-h/2, h/2]$ is the thickness coordinate of the structure. Figure 2 illustrates the variation of the volume fraction of ceramic and metal through the thickness via the volume fraction exponent n .

Figure 1. A functionally graded shell.

Figure 2. Variation of the volume fraction versus the non-dimensional thickness.

2.2. The FGM Shell Model

Consider an FGM shell element subjected to both in-plane forces and bending moments as shown in Figure 3. The middle (neutral) surface of the shell is chosen as the reference plane that occupies a domain $\Omega \in \mathbb{R}^3$. Let u_0 , v_0 , and w_0 be the displacements of the middle plane in the x , y , and z directions; β_x , β_y , and β_z be the rotations of the middle surface of the shell around the y -axis, x -axis, and z -axis, respectively, as indicated in Figure 3. The unknown vector of an FGM shell including six independent variables at any point in the problem domain can be written as

$$\mathbf{u} = \begin{bmatrix} u_0 & v_0 & w_0 & \beta_x & \beta_y & \beta_z \end{bmatrix} \quad (3)$$

The linear strain–displacement relationship can be given as

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \boldsymbol{\varepsilon}_m + z\boldsymbol{\kappa} \quad (4)$$

$$\varepsilon_m = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{Bmatrix}, \kappa = \begin{Bmatrix} \frac{\partial \beta_x}{\partial x} \\ \frac{\partial \beta_y}{\partial y} \\ \frac{\partial \beta_x}{\partial y} + \frac{\partial \beta_y}{\partial x} \end{Bmatrix}; \quad (5)$$

$$\gamma = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_0}{\partial x} + \beta_x \\ \frac{\partial w_0}{\partial y} + \beta_y \end{Bmatrix} \quad (6)$$

From Hooke's law, the constitutive relations of FGM shells are expressed as:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (7)$$

where

$$Q_{11}(z) = \frac{E(z)}{1-\nu(z)^2}, Q_{12}(z) = \nu(z)Q_{11}(z), Q_{22}(z) = Q_{11}(z), Q_{44}(z) = Q_{55}(z) = Q_{66}(z) = \frac{E(z)}{2(1+\nu(z))} \quad (8)$$

A weak form of the free vibration analysis for FGM shells can be briefly given as:

$$\int_{\Omega} \delta \bar{\varepsilon}^T \mathbf{D} \bar{\varepsilon} d\Omega + \int_{\Omega} \delta \gamma^T \mathbf{C} \gamma d\Omega = \int_{\Omega} \delta \mathbf{u}^T \mathbf{m} \ddot{\mathbf{u}} d\Omega \quad (9)$$

where

$$\bar{\varepsilon} = \begin{bmatrix} \varepsilon_m \\ \kappa \end{bmatrix}, \mathbf{D} = \begin{bmatrix} \mathbf{D}^m & \mathbf{B} \\ \mathbf{B} & \mathbf{D}^b \end{bmatrix} \quad (10)$$

Figure 3. Three-node triangular element.

In which

$$\mathbf{D}^m = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}, \mathbf{D}^b = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \quad (11)$$

$$, \mathbf{C} = \begin{bmatrix} C_{55} & C_{45} \\ C_{45} & C_{44} \end{bmatrix}$$

In Equation (11) A_{ij} , B_{ij} , D_{ij} , and C_{ij} are given by:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2) dz, \quad i, j = 1, 2, 6 \quad (12)$$

$$C_{ij} = k \int_{-h/2}^{h/2} Q_{ij} dz, \quad i, j = 4, 5 \quad (13)$$

where $k = 5/6$ is transverse shear correction coefficient and $\mathbf{m}$ is the mass matrix containing ρ calculated as

$$\mathbf{m} = \begin{bmatrix} I_0 & 0 & 0 & I_1 & 0 & 0 \\ 0 & I_0 & 0 & 0 & I_1 & 0 \\ 0 & 0 & I_0 & 0 & 0 & 0 \\ I_1 & 0 & 0 & I_2 & 0 & 0 \\ 0 & I_1 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}; \quad (I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho(1, z, z^2) dz \quad (14)$$

2.3. Finite Element Formulation for Shell Analysis

Now, we discretize the bounded domain Ω into n^e finite three-node triangular elements with n^n nodes such that $\Omega \approx \sum_{e=1}^{n^e} \Omega_e$ and $\Omega_i \cap \Omega_j = \emptyset, i \neq j$. The displacement field $\mathbf{u}^e = \{u^e \ v^e \ w^e \ \beta_x^e \ \beta_y^e \ \beta_z^e\}^T$ of the finite element solution can be expressed as

$$\mathbf{u}^e = \sum_{i=1}^{n^n} \begin{bmatrix} N_i(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & N_i(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & N_i(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & N_i(x) & 0 & 0 \\ 0 & 0 & 0 & 0 & N_i(x) & 0 \\ 0 & 0 & 0 & 0 & 0 & N_i(x) \end{bmatrix} \mathbf{d}_i^e = \sum_{i=1}^{n^n} \mathbf{N}_i \mathbf{d}_i^e \quad (15)$$

where $\mathbf{d}_i^e = \{u_{0i}^e \ v_{0i}^e \ w_{0i}^e \ \beta_{xi}^e \ \beta_{yi}^e \ \beta_{zi}^e\}^T$ is the nodal displacement at the i th node; $N_i(\mathbf{x})$ is shape function for the i th node.

The approximation of the membrane, the bending and the shear strains of the triangular element can be written in matrix forms as follows

$$\boldsymbol{\epsilon}^e = \begin{bmatrix} \mathbf{B}_{m1}^e & \mathbf{B}_{m2}^e & \mathbf{B}_{m3}^e \end{bmatrix} \mathbf{d}^e = \mathbf{B}_m^e \mathbf{d}^e \quad (16)$$

$$\boldsymbol{\kappa}^e = \begin{bmatrix} \mathbf{B}_{b1}^e & \mathbf{B}_{b2}^e & \mathbf{B}_{b3}^e \end{bmatrix} \mathbf{d}^e = \mathbf{B}_b^e \mathbf{d}^e \quad (17)$$

$$\boldsymbol{\gamma}^e = \begin{bmatrix} \mathbf{B}_{s1}^e & \mathbf{B}_{s2}^e & \mathbf{B}_{s3}^e \end{bmatrix} \mathbf{d}^e = \mathbf{B}_s^e \mathbf{d}^e \quad (18)$$

where

$$\mathbf{B}_{mi}^e = \begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 & 0 & 0 \\ N_{i,y} & N_{i,x} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (19)$$

$$\mathbf{B}_{bi}^e = \begin{bmatrix} 0 & 0 & 0 & N_{i,x} & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{i,y} & 0 \\ 0 & 0 & 0 & N_{i,y} & N_{i,x} & 0 \end{bmatrix} \quad (20)$$

$$\mathbf{B}_{si}^e = \begin{bmatrix} 0 & 0 & N_{i,x} & N_i & 0 & 0 \\ 0 & 0 & N_{i,y} & 0 & N_i & 0 \end{bmatrix} \quad (21)$$

By substituting the discrete displacement field into Equation (9) the discretized equation for free vibration analysis can be written into matrix form such as

$$(\mathbf{K} - \omega^2 \mathbf{M}) \hat{\mathbf{d}} = 0, \quad (22)$$

where ω is the natural frequency, $\mathbf{K}$ and $\mathbf{M}$ are the global stiffness and mass matrices, respectively,

$$\mathbf{K} = \sum_{e=1}^{n^e} \mathbf{T}^T \mathbf{K}^e \mathbf{T} \quad (23)$$

with

$$\mathbf{K}^e = \int_{\Omega_e} (\mathbf{B}^e)^T \hat{\mathbf{D}} \mathbf{B}^e d\Omega_e \quad (24)$$

and

$$\mathbf{B}^e = \begin{bmatrix} \mathbf{B}_m^e \\ \mathbf{B}_b^e \\ \mathbf{B}_s^e \end{bmatrix}, \quad \hat{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^m & \mathbf{B} & 0 \\ \mathbf{B} & \mathbf{D}^b & 0 \\ 0 & 0 & \mathbf{C} \end{bmatrix} \quad (25)$$

$$\mathbf{M} = \sum_{e=1}^{n^e} \mathbf{T}^T \mathbf{M}^e \mathbf{T}, \quad (26)$$

$$\mathbf{M}^e = \int_{\Omega_e} \mathbf{N}^T \mathbf{m} \mathbf{N} d\Omega_e \quad (27)$$

in which $\mathbf{T}$ is the transformation matrix between the local coordinate system $Oxyz$ and the global coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ [14].

The problem of zero stiffness that appears with using the drilling degree of freedom β_z , which can cause a singularity in the global stiffness matrix when all the elements meeting at a node are coplanar. To deal with this issue, a simple modification coefficient is chosen to be 10^{-3} times the maximum diagonal value of the element stiffness matrix at the zero drilling degree of freedom to avoid the drill rotation locking [15].

3. Formulation of ES-MITC3 Finite Element Method for FGM Shells

3.1. Brief on the MITC3 Formulation

In the linear triangular MITC3, the approximated displacement field $\mathbf{u}$ is simply interpolated using the linear basic functions for membrane, deflection, and rotation without adding any new variables. Herein, the membrane and bending strains of the standard finite elements are unchanged, while the transverse shear strains, which are modified by the mixed interpolation of tensorial components [16].

As a result, the transverse shearstrain field [8,10] is being obtained as

$$\gamma_{\text{MITC3}}^e = \begin{bmatrix} \mathbf{B}_{s1}^e & \mathbf{B}_{s2}^e & \mathbf{B}_{s3}^e \end{bmatrix} \mathbf{d}^e = \mathbf{B}_s^e \mathbf{d}^e \quad (28)$$

where

$$\mathbf{B}_{s1}^e = J^{-1} \begin{bmatrix} 0 & 0 & -1 & \frac{a}{3} + \frac{d}{6} & \frac{b}{3} + \frac{c}{6} & 0 \\ 0 & 0 & -1 & \frac{d}{3} + \frac{a}{6} & \frac{c}{3} + \frac{b}{6} & 0 \end{bmatrix}, \quad (29)$$

$$\hat{\mathbf{B}}_{s2}^e = J^{-1} \begin{bmatrix} 0 & 0 & 1 & \frac{a}{2} - \frac{d}{6} & \frac{b}{2} - \frac{c}{6} & 0 \\ 0 & 0 & 0 & \frac{d}{6} & \frac{c}{6} & 0 \end{bmatrix}, \quad (30)$$

$$\hat{\mathbf{B}}_{s3}^e = J^{-1} \begin{bmatrix} 0 & 0 & 0 & \frac{a}{6} & \frac{b}{6} & 0 \\ 0 & 0 & 1 & \frac{d}{2} - \frac{a}{6} & \frac{c}{2} - \frac{b}{6} & 0 \end{bmatrix} \quad (31)$$

with

$$J^{-1} = \frac{1}{2A^e} \begin{bmatrix} c & -b \\ -d & a \end{bmatrix} \quad (32)$$

in which $a = x_2 - x_1$, $b = y_2 - y_1$, $c = y_3 - y_1$, and $d = x_3 - x_1$, as pointed out in Figure 4 and A_e is the area of the triangular element.

Figure 4. Three-node triangular element and local coordinates.

3.2. The ES-MITC3 Formulation

In the ES-FEM, the strains are smoothed over local smoothing domains Ω^k , the computation for stiffness matrix is no longer based on elements, but on these smoothing domains. These smoothing domains are formed based on edge of elements such as $\Omega = \cup_{k=1}^{n^k} \Omega^k$ and $\Omega^i \cap \Omega^j = \emptyset$ for $i \neq j$, in which n^k is the total number of edges of all the elements. On a curved geometry of shell models, an edge-based smoothing domain Ω^k associated with the inner edge k is created two sub-domains of two non-planar adjacent MITC3 triangular elements as shown in Figure 5. These triangular elements are defined by two local coordinate systems $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$. In order to compute the edge-based smoothing strain Ω^k for two non-planar adjacent elements, the virtual coordinate system $\tilde{O}\tilde{x}\tilde{y}\tilde{z}$ is proposed as shown in Figure 6, whereas the $\tilde{x}$ -axis coinciding with the edge k , the $\tilde{z}$ -axis with the average direction between the $\hat{z}_1$ -axis and $\hat{z}_2$ -axis, and the $\tilde{y}$ -axis is given by the cross-product of the unit vectors in the $\tilde{x}$ -axis and $\tilde{z}$ -axis.

Figure 5. The smoothing domain; Ω^k is formed by triangular elements.

Hence, a smoothed membrane strain $\bar{\epsilon}^k$, a smoothed bending strain $\bar{\kappa}^k$, a smoothed shearstrain $\bar{\gamma}^k$ of the smoothing domain Ω^k in the global coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ can be derived as

$$\bar{\epsilon}^k = \sum_{j=1}^{n^{nk}} \bar{\mathbf{B}}_{mj}^k \mathbf{d}_j^k, \bar{\kappa}^k = \sum_{j=1}^{n^{nk}} \bar{\mathbf{B}}_{bj}^k \mathbf{d}_j^k, \bar{\gamma}^k = \sum_{j=1}^{n^{nk}} \bar{\mathbf{B}}_{sj}^k \mathbf{d}_j^k \quad (33)$$

where n^{nk} is the number of the neighboring nodes of edge k . $\mathbf{d}_j^k$ is the nodal degrees of freedom at the j th node of the smoothing domain Ω^k in $\hat{O}\hat{x}\hat{y}\hat{z}$. $\bar{\mathbf{B}}_{mj}^k$, $\bar{\mathbf{B}}_{bj}^k$, and $\bar{\mathbf{B}}_{sj}^k$ are the membrane, the bending and the MITC3 shear smoothed gradient matrices at the j th node of the smoothing domain Ω^k in the global coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$, respectively. The $\bar{\mathbf{B}}_{mj}^k$, $\bar{\mathbf{B}}_{bj}^k$ and $\bar{\mathbf{B}}_{sj}^k$ can be computed by

$$\bar{\mathbf{B}}_{mj}^k = \frac{1}{A^k} \sum_{i=1}^{n^{ek}} \frac{1}{3} A^i \Lambda_{m1}^k \Lambda_{m2}^i \mathbf{B}_{mj}^i \mathbf{T}_j^i \quad (34)$$

$$\bar{\mathbf{B}}_{bj}^k = \frac{1}{A^k} \sum_{i=1}^{n^{ek}} \frac{1}{3} A^i \Lambda_{b1}^k \Lambda_{b2}^i \mathbf{B}_{bj}^i \mathbf{T}_j^i \quad (35)$$

$$\bar{\mathbf{B}}_{sj}^k = \frac{1}{A^k} \sum_{i=1}^{n^{ek}} \frac{1}{3} A^i \Lambda_{s1}^k \Lambda_{s2}^i \mathbf{B}_{sj}^i \mathbf{T}_j^i \quad (36)$$

in which Λ_{m1}^k , Λ_{b1}^k , and Λ_{s1}^k are strain transformation matrices between the global coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ and the virtual coordinate system $\tilde{O}\tilde{x}\tilde{y}\tilde{z}$, respectively; Λ_{m2}^i , Λ_{b2}^i and Λ_{s2}^i are the strain transformation matrices between the local coordinate system $Oxyz$ of i th adjacent triangular elements and the virtual coordinate system $\tilde{O}\tilde{x}\tilde{y}\tilde{z}$, respectively; $\mathbf{T}_j^i$ is the transformation matrix between the local coordinate system $Oxyz$ at the j th node of the i th adjacent triangular element and the global coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$. More detailed information about these strain transformation matrices can be found in [14]. The area A^k of the smoothing domain Ω^k is computed by

$$A^k = \int_{\Omega^k} d\Omega = \frac{1}{3} \sum_{i=1}^{n^{ek}} A^i \quad (37)$$

where n^{ek} is the number of the adjacent triangular elements in the smoothing domain Ω^k , and A^i is the area of the i th triangular element around the edge k .

Figure 6. Global, local, and virtual coordinates.

As a result, the global stiffness matrix of the FGM shell in Equation (22) is rewritten as

$$(\tilde{\mathbf{K}} - \omega^2 \mathbf{M}) \hat{\mathbf{d}} = 0, \quad (38)$$

$$\tilde{\mathbf{K}} = \sum_{i=1}^{n^n} \tilde{\mathbf{K}}^k \quad (39)$$

where

$$\tilde{\mathbf{K}}^k = \int_{\Omega_k} (\tilde{\mathbf{B}}^k)^T \hat{\mathbf{D}} \tilde{\mathbf{B}}^k d\Omega_k \quad (40)$$

with

$$\tilde{\mathbf{B}}^k = \begin{bmatrix} \tilde{\mathbf{B}}_m^k & \tilde{\mathbf{B}}_b^k & \tilde{\mathbf{B}}_s^k \end{bmatrix}^T \quad (41)$$

4. Numerical Results

In the section, several numerical examples are provided to show the performance of the ES-MITC3 element for free vibration analysis of FGM shell and results obtained are compared to those published [6, 16–19]. For convenience to the numerical comparison, the non-dimensional frequency parameters ω^* are expressed to the following equation as

$$\omega^* = \omega a^2 \sqrt{\rho_m h / D_m^*}, \quad D_m^* = E_m h^3 / 12(1 - \nu_m^2) \quad (42)$$

First, let us consider free vibration for analysis of clamped functionally graded cylindrical shell ($R_x = R$, $R_y = \infty$) with radius-to-length $R/a = 100$, $a/h = 10$. The functionally graded shell is made from Silicon nitride (Si_3N_4) and Stainless steel (SUS304), which material properties are $E_c = 322.2715$ GPa, $\nu_c = 0.24$, $\rho_c = 2370$ kg/m³, $E_m = 207.7877$ GPa, $\nu_m = 0.31776$, and $\rho_m = 8166$ kg/m³. The first four non-dimensional frequency of the present method list in Table 1 are compared with MITC3 [16], a higher-order theory based on radial basis functions collocation including transverse normal deformation (HSDT RBFC-1) and discarding transverse normal deformation (HSDT RBFC-2) by Neves et al. [17], a higher-order theory and finite element formalation (HSDT FEM) by Pradyumna and Bandyopadhyay [6], a higher-order theory and semi-analytical method relied on Galerkin (HSDT SAG) by Yang and Shen [18], and Quasi-3D Ritz model (ED₅₅₅) by Fazzolari and Carrera [19]. From Table 1 we can see that this proposed method (ES-MITC3) is more accurate than other methods, such as MITC3, HSDT RBFC-1, HSDT RBFC-2, HSDT FEM and HSDT SAG. The errors are less than 3% in comparison with the exact solution ED₅₅₅ [19]. Figure 7 shows non-dimensional frequency parameter for four first modes of clamped functionally graded cylindrical shell using various methods.

Table 1. Non-dimensional frequency parameter for clamped cylindrical functionally graded materials (FGM) shell with $R/a = 100$, and relative error between methods (ED₅₅₅ [19] is fixed). Error (%) = $\frac{100 \times |\text{Method} - \text{ED}_{555}[19]|}{\text{ED}_{555}[19]}$.

Mode	Method	n				
		0	0.2	2	10	∞
1	ES-MITC3	75.4587	61.3587	40.9880	35.3951	33.0594
	%	0.2776	0.0298	0.3963	0.7275	0.5532
	MITC3 [16]	72.7508	59.0689	39.4771	34.0744	31.8220
	%	3.3209	3.7031	4.0679	4.4317	4.2754
	HSDT FEM [6]	72.9613	60.0269	39.1457	33.3666	32.0274
	%	3.0412	2.1413	4.8733	6.4169	3.6576
	HSDT RBFC-1 [17]	74.2634	60.0061	40.5259	35.1663	32.6108
	%	1.3108	2.1752	1.5193	1.3693	1.9026
	HSDT RBFC-2 [17]	74.5821	60.3431	40.8262	35.4229	32.8593
	%	0.8873	1.6258	0.7895	0.6496	1.1551
	HSDT SAG [18]	74.5180	57.4790	40.7500	35.8520	32.7610
	%	0.9725	6.2950	0.9747	0.5539	1.4508
2	ED ₅₅₅ [19]	75.2498	61.3404	41.1511	35.6545	33.2433
	ES-MITC3	144.4760	117.6462	78.5402	67.7320	63.3473
	%	0.6724	0.6147	0.5174	0.3138	0.4088
	MITC3 [16]	140.8063	114.5113	76.4785	65.9309	61.6559
	%	1.8847	2.0664	2.1212	2.3537	2.2722
	HSDT FEM [6]	138.5552	113.8806	74.2915	63.2869	60.5546
	%	3.4533	2.6058	4.9201	6.2695	4.0178
	HSDT RBFC-1 [17]	141.6779	114.3788	76.9725	66.6482	61.9329
	%	1.2773	2.1797	1.4889	1.2913	1.8331
	HSDT RBFC-2 [17]	142.4281	115.2134	77.6639	67.1883	62.4886
	%	0.7546	1.4660	0.6041	0.4914	0.9523
	HSDT SAG [18]	144.6630	111.7170	78.8170	69.0750	63.3140
3	%	0.8027	4.4562	0.8717	2.3029	0.3560
	ED ₅₅₅ [19]	143.5110	116.9275	78.1359	67.5201	63.0894
	ES-MITC3	145.1510	118.1985	78.9069	68.0474	63.6440
	%	1.0284	0.9602	0.8727	2.2434	0.7658
	MITC3 [16]	141.7861	115.3112	77.0122	66.3906	62.0864
	%	1.3137	1.5061	1.5494	4.6235	1.7003
	HSDT FEM [6]	138.5552	114.0266	74.3868	63.3668	60.6302
	%	3.5625	2.6033	4.9056	8.9675	4.0058
	HSDT RBFC-1 [17]	141.8485	114.5495	77.0818	66.7332	62.0082
	%	1.2702	2.1567	1.4604	4.1314	1.8241
	HSDT RBFC-2 [6]	142.6024	115.3665	77.7541	67.2689	62.5668
	%	0.7455	1.4588	0.6010	3.3618	0.9397
4	HSDT SAG [18]	145.7400	112.5310	79.4070	67.5946	63.8060
	%	1.4383	3.8808	1.5121	2.8939	1.0223
	ED ₅₅₅ [19]	143.6735	117.0744	78.2242	69.6090	63.1603
	ES-MITC3	204.0647	166.3177	111.0461	95.6539	89.5229
	%	1.1780	1.2302	1.3863	1.2447	1.2996
	MITC3 [16]	195.3261	158.8135	106.1329	91.3802	85.4901
	%	3.1547	3.3373	3.0995	3.2788	3.2637
	HSDT FEM [6]	195.5366	160.6235	104.7687	89.1970	85.1788
	%	3.0503	2.2357	4.3450	5.5896	3.6160
	HSDT RBFC-1 [17]	199.1566	160.7355	107.9484	93.3350	86.8160
	%	1.2555	2.1675	1.4419	1.2097	1.7634
	HSDT RBFC-2 [17]	200.3158	162.0337	108.9677	94.0923	87.6341
5	%	0.6808	1.3773	0.5113	0.4081	0.8377
	HSDT SAG [18]	206.9920	159.8550	112.4570	98.3860	90.3700
	%	2.6294	2.7034	2.6745	4.1365	2.2581
	ED ₅₅₅ [19]	201.6888	164.2966	109.5277	94.4779	88.3744

Figure 7. Non-dimensional frequency parameter for four first modes. (a) Mode 1; (b) mode 2; (c) mode 3; (d) mode 4.

Next, we investigate the first non-dimensional frequencies ω^* of functionally graded spherical ($R_x = R_y = R$) and cylindrical shells ($R_x = R, R_y = \infty$) with geometric data: radius to edge R/a and a/h are varied from 5 to 50 and 10, respectively. The functionally graded shells in these studies are made from aluminum, and alumina whose material properties are $E_m = 70$, GPa, $\nu_m = 0.3$, $\rho_m = 2707$ kg/m³, $E_c = 380$ GPa, $\nu_c = 0.3$, and $\rho_c = 3000$ kg/m³. Again, it is seen from Tables 2–5 that the results of the present approach are very close to an HSDT RBFC-1, HSDT RBFC-2 [17], and ED555 [19]. Figures 8–11 show non-dimensional frequency parameter for the first mode of cylindrical FGM shell and spherical FGM shell with different n , respectively. The first six mode shapes of simply supported spherical FGM shell are illustrated in Figure 12.

Figure 8. Non-dimensional frequency parameter for the first mode of clamped cylindrical FGM shell with different n . (a) $R/a = 5$; (b) $R/a = 10$; (c) $R/a = 50$.

Table 2. Non-dimensional frequency parameter for clamped cylindrical FGM shell with $a/h = 10$ and different R/a ratios.

$\frac{R}{a}$	Method	n						
		0	0.2	0.5	1	2	10	∞
5	ES-MITC3	73.4741	67.2928	60.6591	53.9842	48.1650	41.4718	33.4122
	HSDT FEM [6]	71.8861	68.1152	63.1896	56.5546	36.2487	33.6611	32.4802
	HSDT RBFC-1 [17]	73.1640	66.6620	60.2477	53.5430	47.5205	40.8099	33.0576
	HSDT RBFC-2 [17]	73.6436	67.1004	60.6568	53.9340	47.9060	41.0985	33.2743
10	ES-MITC3	72.6253	65.5578	60.0417	53.4874	47.7863	41.1837	33.0311
	HSDT FEM [6]	71.0394	67.3320	62.4687	55.8911	35.6633	31.1474	32.0976
	HSDT RBFC-1 [17]	72.3304	65.8808	59.5215	52.8800	46.9447	40.4145	32.6810
	HSDT RBFC-2 [17]	72.8141	66.3235	59.9353	53.2759	47.3343	40.7046	32.8995
50	ES-MITC3	72.3439	66.3519	59.9114	53.4282	47.7802	41.1529	32.9058
	HSDT FEM [6]	70.7660	67.0801	62.2380	55.6799	35.4745	32.9812	31.9741
	HSDT RBFC-1 [17]	72.0614	65.6371	59.3022	52.6864	46.7820	40.3028	32.5594
	HSDT RBFC-2 [17]	72.5465	66.0814	59.7178	53.0841	47.1726	40.5923	32.7786

Table 3. Non-dimensional frequency parameter for simply supported cylindrical FGM shell with $a/h = 10$ and different R/a ratios.

$\frac{R}{a}$	Method	n						
		0	0.2	0.5	1	2	10	∞
5	ES-MITC3	42.9913	39.3028	35.4690	31.7485	28.6106	24.7564	19.5592
	HSDT FEM [6]	42.2543	40.1621	37.2870	33.2268	27.4449	19.3892	19.0917
	HSDT RBFC-1 [17]	42.6701	38.7168	34.8768	30.9306	27.5362	24.2472	19.2796
	HSDT RBFC-2 [17]	42.7172	38.7646	34.9273	30.9865	27.5977	24.2839	19.3008
	ED ₅₅₅ [19]	42.7160	39.0642	35.0811	31.0414	27.5634	24.1245	19.3003
10	ES-MITC3	42.5231	38.9004	35.1357	31.4868	28.4168	24.6061	19.3492
	HSDT FEM [6]	41.9080	39.8472	36.9995	32.9585	27.1879	19.1562	18.9352
	HSDT RBFC-1 [17]	42.3153	38.3840	34.5672	30.6485	27.2979	24.1063	19.1193
	HSDT RBFC-2 [17]	42.3684	38.4368	34.6219	30.7077	27.3616	24.1444	19.1433
	ED ₅₅₅ [19]	42.3677	38.7377	34.7661	30.7621	27.3258	23.9848	19.1429
50	ES-MITC3	42.3669	38.7889	35.0696	31.4631	28.4233	24.5937	19.2798
	HSDT FEM [6]	41.7963	39.7465	36.9088	32.8750	27.0961	19.0809	18.8848
	HSDT RBFC-1 [17]	42.2008	38.2842	34.4809	30.5759	27.2423	24.0762	19.0675
	HSDT RBFC-2 [17]	42.2560	38.3384	34.5365	30.6355	27.3055	24.1125	19.0924
	ED ₅₅₅ [19]	42.2553	38.6391	34.6904	30.6890	27.2682	23.9515	19.0922

Figure 9. Non-dimensional frequency parameter for the first mode of simple supported cylindrical FGM shell with different n . (a) $R/a = 5$; (b) $R/a = 10$; (c) $R/a = 50$.

Table 4. Non-dimensional frequency parameter for simply supported spherical FGM shell with $a/h = 10$ and different R/a ratios.

$\frac{R}{a}$	Method	n						
		0	0.2	0.5	1	2	10	∞
5	ES-MITC3	44.4405	40.6238	36.6449	32.7529	29.4124	25.2893	20.2096
	HSDT FEM [6]	44.0073	41.7782	38.7731	34.6004	28.7459	20.4691	19.8838
	HSDT RBFC-1 [17]	44.4555	40.3936	36.4453	32.3691	28.7833	25.0772	20.0818
	HSDT RBFC-2 [17]	44.4697	40.4211	36.6004	32.4101	28.8329	25.1038	20.0927
	ED ₅₅₅ [19]	44.4671	40.7166	36.6297	32.4645	28.7996	24.9403	20.0915
10	ES-MITC3	42.9198	39.2373	35.4098	31.6957	28.5633	24.7153	19.5267
	HSDT FEM [6]	42.3579	40.2608	37.3785	33.3080	27.5110	19.4357	19.1385
	HSDT RBFC-1 [17]	42.7709	38.8074	34.9574	31.0012	27.5984	24.3034	19.3251
	HSDT RBFC-2 [17]	42.8180	38.8551	35.0080	31.0572	27.6602	24.3401	19.3464
	ED ₅₅₅ [19]	42.8169	39.1556	35.1622	31.1122	27.6258	24.1803	19.3459
50	ES-MITC3	42.4046	38.8147	35.0835	31.4662	28.4197	24.5977	19.2966
	HSDT FEM [6]	41.8145	39.7629	36.9234	32.8881	27.1085	19.0922	18.8930
	HSDT RBFC-1 [17]	42.2192	38.2988	34.4922	30.5840	27.2474	24.0791	19.0759
	HSDT RBFC-2 [17]	42.2741	38.3528	34.5478	30.6437	27.3109	24.1168	19.1006
	ED ₅₅₅ [19]	42.2735	38.6538	34.7018	30.6975	27.2741	23.9567	19.1004

(a)

(b)

(c)

Figure 10. Non-dimensional frequency parameter for the first mode of simply supported spherical FGM shell with different n . (a) $R/a = 5$; (b) $R/a = 10$; (c) $R/a = 50$.

Table 5. Non-dimensional frequency parameter for clamped spherical FGM shell with different R/a ratios.

R/a	Method	n						
		0	0.2	0.5	1	2	10	∞
5	ES-MITC3	74.3416	68.0034	61.2122	54.3761	48.3922	41.5911	33.7972
	HSDT FEM [6]	73.5550	69.6597	64.6114	57.8619	37.3914	34.6658	33.2343
	HSDT RBFC-1 [17]	74.8207	68.2142	61.6902	54.8597	48.6656	41.6016	33.8061
	HSDT RBFC-2 [17]	75.2810	68.6329	62.0789	55.2302	49.0328	41.8796	34.0141
10	ES-MITC3	72.8831	66.7331	60.1352	53.5040	47.7428	41.1652	33.1447
	HSDT FEM [6]	71.4659	67.7257	62.8299	56.2222	35.9568	33.4057	32.2904
	HSDT RBFC-1 [17]	72.7536	66.2686	59.8745	53.1956	47.2135	40.5990	32.8722
	HSDT RBFC-2 [17]	73.2322	66.7063	60.2831	53.5864	47.5990	40.8883	33.0884
50	ES-MITC3	72.3889	66.3780	59.9190	53.4192	47.7612	41.1495	32.9258
	HSDT FEM [6]	70.7832	67.0956	62.2519	55.6923	35.4861	32.9916	31.9819
	HSDT RBFC-1 [17]	72.0784	65.6498	59.3112	52.6921	46.7849	40.3049	32.5671
	HSDT RBFC-2 [17]	72.5633	66.0938	59.7265	53.0895	47.1574	40.5946	32.7862

(a)

(b)

(c)

Figure 11. Non-dimensional frequency parameter for the first mode of clamped spherical FGM shell with different n . (a) $R/a = 5$; (b) $R/a = 10$; (c) $R/a = 50$.

Figure 12. First six mode shapes of simply supported spherical FGM shell ($R/a = 10$, $a/h = 10$, $n = 0.2$).

For the fully clamped spherical shell, the second vibration mode shape and the third vibration mode shape are similar to each other (their natural frequencies are equal), they just have different views. Adding, the value of non-dimensional frequency of the fifth and the sixth vibration mode shape are approximatively each other. This thing is consistent with the actual symmetrical shell structures with the same boundary conditions.

5. Conclusions

In this paper, the free vibration analysis of functionally graded shells is studied by using the ES-MITC3. Herein, the stiffness matrices obtained based on the strain-smoothing technique over the smoothing domains associated with edges of MITC3 shell elements. The present approach uses a triangular element and hence much easily generated automatically, even for complicated geometries. The numerical results showed that the ES-MITC3 is a good agreement with the reference solutions, which are a requirement of high computational costs, such as ED₅₅₅ [19] and a higher-order theory based on radial basis functions [17]. The ES-MITC3 presented herein is promising to be a simple and effective finite element method for analysis of functionally graded shells in practice.

The combination of finite element method (FEM) with an edge-based smoothed finite element method (ES-FEM) is very suitable for analyzing shell structures, especially for the complicated structures such as shell with reinforced stiffeners, reinforced nano graphene, micro shell structures, nano shell structures, and so on. This combination allows us to calculate exactly plate and shell structures with thin thicknesses ($h = a/10^8$) due to overcoming the shear locking phenomenon.

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References

1. Woo, J.; Meguid, S. Nonlinear analysis of functionally graded plates and shallow shells. *Int. J. Solids Struct.* **2001**, *38*, 7409–7421. [\[CrossRef\]](#)
2. Matsunaga, H. Free vibration and stability of functionally graded shallow shells according to a 2D higher-order deformation theory. *Compos. Struct.* **2008**, *84*, 132–146. [\[CrossRef\]](#)
3. Duc, N.D.; Quan, T.Q.; Luat, V.D. Nonlinear dynamic analysis and vibration of shear deformable piezoelectric FGM double curved shallow shells under damping-thermo-electro-mechanical loads. *Compos. Struct.* **2015**, *125*, 29–40. [\[CrossRef\]](#)
4. Bich, D.H.; Duc, N.D.; Quan, T.Q. Nonlinear vibration of imperfect eccentrically stiffened functionally graded double curved shallow shells resting on elastic foundation using the first order shear deformation theory. *Int. J. Mech. Sci.* **2014**, *80*, 16–28. [\[CrossRef\]](#)
5. Arciniega, R.A.; Reddy, J.N. Large deformation analysis of functionally graded shells. *Int. J. Solids Struct.* **2007**, *44*, 2036–2052. [\[CrossRef\]](#)
6. Pradyumna, S.; Bandyopadhyay, J.N. Free vibration analysis of functionally graded curved panels using a higher-order finite element formulation. *J. Sound Vib.* **2008**, *318*, 176–192. [\[CrossRef\]](#)
7. Kordkheili, S.A.H.; Naghdabadi, R. Geometrically non-linear thermoelastic analysis of functionally graded shells using finite element method. *Int. J. Numer. Methods Eng.* **2007**, *72*, 964–986. [\[CrossRef\]](#)
8. Chau-Dinh, T.; Nguyen-Duy, Q.; Nguyen-Xuan, H. Improvement on MITC3 plate finite element using edge-based strain smoothing enhancement for plate analysis. *Acta Mech.* **2017**, *228*, 2141–2163. [\[CrossRef\]](#)
9. Nguyen, T.-K.; Nguyen, V.-H.; Chau-Dinh, T.; Vo, T.P.; Nguyen-Xuan, H. Static and vibration analysis of isotropic and functionally graded sandwich plates using an edge-based MITC3 finite elements. *Compos. Part B Eng.* **2016**, *107*, 162–173. [\[CrossRef\]](#)
10. Pham, Q.-H.; Tran, T.-V.; Pham, T.-D.; Phan, D.-H. An Edge-Based Smoothed MITC3 (ES-MITC3) Shell Finite Element in Laminated Composite Shell Structures Analysis. *Int. J. Comput. Methods* **2017**, *15*, 1850060. [\[CrossRef\]](#)
11. Pham, Q.-H.; Pham, T.-D.; Trinh, Q.V.; Phan, D.-H. Geometrically nonlinear analysis of functionally graded shells using an edge-based smoothed MITC3 (ES-MITC3) finite elements. *Eng. Comput.* **2019**, 1–14. [\[CrossRef\]](#)
12. Pham-Tien, D.; Pham-Quoc, H.; Vu-Khac, T.; Nguyen-Van, N. Transient Analysis of Laminated Composite Shells Using an Edge-Based Smoothed Finite Element Method. In Proceedings of the International Conference on Advances in Computational Mechanics, Phu Quoc Island, Vietnam, 2–4 August 2017; pp. 1075–1094.

13. Nguyen, H.N.; Canh, T.N.; Thanh, T.T.; Ke, T.V.; Phan, V.D.; Thom, D.V. Finite Element Modelling of a Composite Shell with Shear Connectors. *Symmetry* **2019**, *11*, 527. [[CrossRef](#)]
14. Nguyen-Hoang, S.; Phung-Van, P.; Natarajan, S.; Kim, H.-G. A combined scheme of edge-based and node-based smoothed finite element methods for Reissner–Mindlin flat shells. *Eng. Comput.* **2016**, *32*, 267–284. [[CrossRef](#)]
15. Nguyen-Thoi, T.; Phung-Van, P.; Thai-Hoang, C.; Nguyen-Xuan, H. A cell-based smoothed discrete shear gap method (CS-DSG3) using triangular elements for static and free vibration analyses of shell structures. *Int. J. Mech. Sci.* **2013**, *74*, 32–45. [[CrossRef](#)]
16. Lee, P.-S.; Bathe, K.-J. Development of MITC isotropic triangular shell finite elements. *Comput. Struct.* **2004**, *82*, 945–962. [[CrossRef](#)]
17. Neves, A.; Ferreira, A.; Carrera, E.; Cinefra, M.; Roque, C.; Jorge, R.; Soares, C.; Ferreira, A.; Jorge, R.N.; Soares, C.M.M. Free vibration analysis of functionally graded shells by a higher-order shear deformation theory and radial basis functions collocation, accounting for through-the-thickness deformations. *Eur. J. Mech. A/Solids* **2013**, *37*, 24–34. [[CrossRef](#)]
18. Yang, J.; Shen, H.-S. Free vibration and parametric resonance of shear deformable functionally graded cylindrical panels. *J. Sound Vib.* **2003**, *261*, 871–893. [[CrossRef](#)]
19. Fazzolari, F.A.; Carrera, E. Refined hierarchical kinematics quasi-3D Ritz models for free vibration analysis of doubly curved FGM shells and sandwich shells with FGM core. *J. Sound Vib.* **2014**, *333*, 1485–1508. [[CrossRef](#)]

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