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New Fixed Point Results for Modified Contractions and Applications

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Abstract: In this study, we introduce a new type of contractive mapping to establish the existence and uniqueness of fixed points for this type of contraction. Some related examples are built demonstrating the superiority of our results compared to the existing onesin the literature. As applications of the results obtained, some new fixed point theorems are presented for graph-type contractions. Furthermore, sufficient conditions are discussed to ensure the existence underlying various approaches of a solution for a functional equation originating in dynamic programming.

Keywords: admissible mapping; α-ψ-contraction; graph; functional equation

1. Introduction and Preliminaries

Let $\mathcal{T}$ be a self-mapping on a nonempty set M, and denote the set of all real numbers, the set of all non-negative real numbers, and the set of all natural numbers by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$, respectively. By $Fix(\mathcal{T}) = \{u \in M : \mathcal{T}u = u\}$, we denote the set of all fixed points of $\mathcal{T}$. We denote by Σ the set of functions $\sigma \colon (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(σ1) σ is non-decreasing;
(σ2) for each sequence $\{u_n\} \subset (0,\infty)$, we have $\lim_{n\to\infty} \sigma(u_n) = 1$ iff $\lim_{n\to\infty} u_n = 0$;
(σ3) there exist $r \in (0,1)$ and $\lambda \in (0,\infty]$ such that $\lim_{t\to 0^+} \dfrac{\sigma(t)-1}{t^r} = \lambda$.

Jleli and Samet introduced in [1] a new type of contraction by using the function σ and established the following fixed point theorem.

Theorem 1. *Let (M,d) be a complete metric space and $\mathcal{T} : M \to M$ be a given mapping. Suppose that there exist $k \in (0,1)$ and $\sigma \in \Sigma$ such that:*

$$u,v \in M, \quad d(\mathcal{T}u,\mathcal{T}v) > 0 \quad \Longrightarrow \quad \sigma(d(\mathcal{T}u,\mathcal{T}v)) \le [\sigma(d(u,v))]^k.$$

Then, $\mathcal{T}$ has a unique fixed point.

Later, Isik and Shatanawi [2] stated that the condition $(\sigma 2)$ is not required in the proof of Theorem 1 with the help of the following lemma.

Lemma 1. *[2] Let $\sigma\colon (0,\infty) \to (1,\infty)$ be a non-decreasing function and $\{t_n\} \subset (0,\infty)$ a decreasing sequence such that $\lim_{n\to\infty} \sigma(t_n) = 1$. Then, we have $\lim_{n\to\infty} t_n = 0$.*

We now denote by Ξ the set of all functions $\sigma\colon (0,\infty) \to (1,\infty)$ satisfying the conditions $(\sigma 1)$ and $(\sigma 3)$. In the following examples, we see some functions that belong to the set Ξ, but do not belong to Σ.

Example 1. *[2] Define $\sigma\colon (0,\infty) \to (1,\infty)$ with $\sigma(t) = e^{\sqrt{t+1}}$. Evidently, σ satisfies $(\sigma 1)$, and since $\lim_{t\to 0^+} (e^{\sqrt{t+1}} - 1)/t^r = \infty$ for $r \in (0,1)$, also $(\sigma 3)$. However, σ does not satisfy the condition $(\sigma 2)$. Indeed, consider $t_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} \sigma(t_n) = e \neq 1$. Consequently, $\sigma \in \Xi$ while $\sigma \notin \Sigma$.*

Example 2. *[2] Let $a > 1$ and $\sigma(t) = a + \ln(\sqrt{t+1})$. It can easily be seen that σ satisfies the conditions $(\sigma 1)$ and $(\sigma 3)$. However, if we take $t_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty} \sigma(t_n) = a > 1$. Hence, $\sigma \in \Xi$ and $\sigma \notin \Sigma$.*

In 2012, Samet et al. [3] adopted the notion of α-ψ-contractive mappings and confirmed the existence and uniqueness of a fixed point for such mappings. Let Ψ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ in order that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$. If $\psi \in \Psi$, then it is easy to see that $\psi(t) < t$ for all $t > 0$.

Let (M,d) be a metric space. A self-map $\mathcal{T}$ on M is stated to be an α-ψ-contraction, if:

$$\alpha(u,v)d(\mathcal{T}u,\mathcal{T}v) \leq \psi(d(u,v)), \quad \text{for all } u,v \in M, \tag{1}$$

where $\psi \in \Psi$ and $\alpha : M \times M \to [0,\infty)$.

A self-map $\mathcal{T}$ on M is stated to be α-admissible, if there exists $\alpha : M \times M \to [0,+\infty)$ in order that:

$$u,v \in M, \quad \alpha(u,v) \geq 1 \Longrightarrow \alpha(\mathcal{T}u,\mathcal{T}v) \geq 1.$$

Using this concept, many fixed point results appeared; see [4–15]. The results presented in [3] can be abstracted as follows.

Theorem 2 ([3])**.** *Given a complete metric space (M,d), let $\mathcal{T} : M \to M$ be a mapping such that it is α-admissible and an α-ψ-contraction. Assume that the following conditions are satisfied:*

(i) there exists $\xi_0 \in M$ in order that $\alpha(\xi_0, \mathcal{T}\xi_0) \geq 1$;
(ii) the mapping $\mathcal{T}$ is continuous or;
(iii) for each sequence $\{\xi_n\}$ in M in order that $\xi_n \to u \in M$ and $\alpha(\xi_n, \xi_{n+1}) \geq 1$, then $\alpha(\xi_n, u) \geq 1$ for all $n \in \mathbb{N}$.

Then, the mapping $\mathcal{T}$ possesses a fixed point. If in addition, we assume that for every $(\xi,u) \in M^2$, there exists $z \in M$ in order that $\alpha(\xi,z) \geq 1$ and $\alpha(u,z) \geq 1$, then such a fixed point is unique.

In the indicated study, we introduce a new type of contractive mapping and establish the existence and uniqueness results for fixed points of this new type of contraction. Our results generalize and improve Theorems 1 and 2 and many others in the literature. Several examples are constructed in order to illustrate the generality of our results. As applications of the obtained results, some new fixed point theorems are presented for graph-type contractions. Moreover, sufficient conditions are discussed to ensure the existence underlying various approaches of a solution for a functional equation originating in various dynamic programming.

2. Main Results

First of all, we collect some notions and notations to state the main theorems.

Definition 1. *Given a metric space (M, d) and $\alpha : M \times M \to [0, \infty)$, let $\mathcal{T}$ be a self-mapping on M. Denote the set $\mathcal{A} \subseteq M \times M$ by:*

$$\mathcal{A}(\mathcal{T}, \alpha) = \{(u, v) : d(\mathcal{T}u, \mathcal{T}v) > 0 \text{ and } \alpha(u, v) \geq 1\}.$$

Then, $\mathcal{T}$ is called an (α-σ-ψ)-contraction, if there exist $k \in (0, 1)$, $\psi \in \Psi$ and $\sigma \in \Xi$ in order that:

$$\sigma(d(\mathcal{T}u, \mathcal{T}v)) \leq [\sigma(\psi(d(u, v)))]^k, \quad \text{for all } (u, v) \in \mathcal{A}(\mathcal{T}, \alpha). \tag{2}$$

Remark 1. *Let (M, d) be a metric space. If $\mathcal{T} : M \to M$ is an (α-σ-ψ)-contraction, then by (2), we deduce:*

$$ln[\sigma(d(\mathcal{T}u, \mathcal{T}v))] \leq k \ln[\sigma(\psi(d(u, v)))] < \ln[\sigma(\psi(d(u, v)))].$$

Using $(\sigma 1)$, we have that:

$$d(\mathcal{T}u, \mathcal{T}v) < \psi(d(u, v)), \quad \text{for all } (u, v) \in \mathcal{A}(\mathcal{T}, \alpha).$$

The last inequality gives us that:

$$u, v \in M, \quad \alpha(u, v) \geq 1 \Longrightarrow d(\mathcal{T}u, \mathcal{T}v) \leq \psi(d(u, v)).$$

Now, we can have the main theorem of this study.

Theorem 3. *Let (M, d) be a complete metric space and $\mathcal{T} : M \to M$ be an (α-σ-ψ)-contraction. Assume that the following conditions are satisfied:*

- (i) *$\mathcal{T}$ is α-admissible;*
- (ii) *there exists $\xi_0 \in M$ in order that $\alpha(\xi_0, \mathcal{T}\xi_0) \geq 1$;*
- (iii) *$\mathcal{T}$ is continuous or;*
- (iv) *for every $\{\xi_n\}_{n \in \mathbb{N}} \subset M$ in order that $\xi_n \to u \in M$ and $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha(\xi_n, u) \geq 1$ for all $n \in \mathbb{N}$.*

Then, $\mathcal{T}$ possesses a fixed point. Moreover, if $\alpha(\xi, u) \geq 1$ for all $\xi, u \in Fix(\mathcal{T})$, then we have a unique fixed point.

Proof. By virtue of the assertion (ii), then there exists $\xi_0 \in M$ in order that $\alpha(\xi_0, \mathcal{T}\xi_0) \geq 1$. Define a sequence $\{\xi_n\}$ in M by $\xi_n = \mathcal{T}\xi_{n-1} = \mathcal{T}^n\xi_0$ for each $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ in order that $\xi_{n_0} = \xi_{n_0+1}$, then $\xi_{n_0} = \mathcal{T}\xi_{n_0}$. This finishes the proof. Due to this reason, we suppose that $\xi_n \neq \xi_{n+1}$, for all n, that is,

$$d(\mathcal{T}\xi_{n-1}, \mathcal{T}\xi_n) > 0, \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

Since $\alpha(\xi_0, \xi_1) = \alpha(\xi_0, \mathcal{T}\xi_0) \geq 1$ and $\mathcal{T}$ is α-admissible, we obtain:

$$\alpha(\xi_n, \xi_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4}$$

Combining (3) and (4), we deduce that:

$$(\xi_n, \xi_{n+1}) \in \mathcal{A}(\mathcal{T}, \alpha), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{5}$$

Taking (2) and (5) into consideration, we obtain:

$$\sigma(d(\xi_n, \xi_{n+1})) = \sigma(d(\mathcal{T}\xi_{n-1}, \mathcal{T}\xi_n)) \leq [\sigma(\psi(d(\xi_{n-1}, \xi_n)))]^k, \quad \text{for all } n \in \mathbb{N}.$$

Since σ is non-decreasing, we have:

$$\sigma(d(\xi_n, \xi_{n+1})) < [\sigma(d(\xi_{n-1}, \xi_n))]^k, \quad \text{for all } n \in \mathbb{N}.$$

Letting $u_n := d(\xi_n, \xi_{n+1})$ for all $n \in \mathbb{N}$, from the above inequality, we infer:

$$\sigma(u_n) < [\sigma(u_{n-1})]^k < [\sigma(u_{n-2})]^{k^2} < \cdots < [\sigma(u_0)]^{k^n}.$$

Thus, for all $n \in \mathbb{N}$, we have:

$$1 < \sigma(u_n) < [\sigma(u_0)]^{k^n}. \tag{6}$$

Taking the limit of (6) as $n \to \infty$, we obtain:

$$\lim_{n\to\infty} \sigma(u_n) = 1,$$

which implies by Lemma 1 that:

$$\lim_{n\to\infty} u_n = 0. \tag{7}$$

To prove that $\{\xi_n\}$ is a Cauchy sequence, let us consider condition $(\sigma 3)$. Then, there exist $r \in (0,1)$ and $\lambda \in (0, \infty]$ in order that:

$$\lim_{n\to\infty} \frac{\sigma(u_n) - 1}{(u_n)^r} = \lambda. \tag{8}$$

Take $\delta \in (0, \lambda)$. By the definition of the limit, there exists $n_1 \in \mathbb{N}$ in order that:

$$(u_n)^r \leq \delta^{-1}(\sigma(u_n) - 1), \quad \text{for all } n > n_1.$$

Using (6) and the above inequality, we deduce:

$$n(u_n)^r \leq \delta^{-1} n([\sigma(u_0)]^{k^n} - 1), \quad \text{for all } n > n_1.$$

This implies that:

$$\lim_{n\to\infty} n(u_n)^r = \lim_{n\to\infty} n(d(\xi_n, \xi_{n+1}))^r = 0.$$

Thence, there exists $n_2 \in \mathbb{N}$ in order that:

$$d(\xi_n, \xi_{n+1}) \leq \frac{1}{n^{1/r}}, \quad \text{for all } n > n_2. \tag{9}$$

Let $m > n > n_2$. Then, using the triangular inequality and (9), we have:

$$d(\xi_n, \xi_m) \leq \sum_{k=n}^{m-1} d(\xi_k, \xi_{k+1}) \leq \sum_{k=n}^{m-1} \frac{1}{k^{1/r}} \leq \sum_{k=n}^{\infty} \frac{1}{k^{1/r}}$$

and hence, $\{\xi_n\}$ is a Cauchy sequence in M. From the completeness of (M, d), then there exists $u \in M$ in order that $\xi_n \to u$ as $n \to \infty$. If $\mathcal{T}$ is continuous, then $\xi_{n+1} = \mathcal{T}\xi_n \to \mathcal{T}u$. The uniqueness of the limit yields that $u = \mathcal{T}u$.

Now, assume that the assumption (iv) holds. Then, $\alpha(\xi_n, u) \geq 1$ for all $n \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ in order that $d(\xi_{k+1}, \mathcal{T}u) = 0$, then from the uniqueness of the limit, $u = \mathcal{T}u$. Therefore, the proof is completed. Hence, there exists $n_3 \in \mathbb{N}$ in order that $d(\mathcal{T}\xi_n, \mathcal{T}u) > 0$ for all $n > n_3$. Thus, $(\xi_n, u) \in \mathcal{A}(\mathcal{T}, \alpha)$ for all $n > n_3$. By considering Remark 1 (i), we have:

$$d(\xi_{n+1}, \mathcal{T}u) = d(\mathcal{T}\xi_n, \mathcal{T}u) \leq \psi(d(\xi_n, u)),$$

and so:

$$0 < d(\xi_{n+1}, \mathcal{T}u) < d(\xi_n, u), \quad \text{for all } n > n_3.$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain $d(u, \mathcal{T}u) = 0$, and so, $u = \mathcal{T}u$. Now, we prove that the fixed point of $\mathcal{T}$ is unique. Suppose that $\mathfrak{p}, \mathfrak{q} \in Fix(\mathcal{T})$ in order that $d(\mathfrak{p}, \mathfrak{q}) > 0$. Then, $d(\mathcal{T}\mathfrak{p}, \mathcal{T}\mathfrak{q}) > 0$, and by the hypothesis, $\alpha(\mathfrak{p}, \mathfrak{q}) \geq 1$. Hence, we deduce that $(\mathfrak{p}, \mathfrak{q}) \in \mathcal{A}(\mathcal{T}, \alpha)$. Regarding Remark 1 (i), we obtain:

$$d(\mathfrak{p}, \mathfrak{q}) = d(\mathcal{T}\mathfrak{p}, \mathcal{T}\mathfrak{q}) \leq \psi(d(\mathfrak{p}, \mathfrak{q})) < d(\mathfrak{p}, \mathfrak{q}),$$

which implies that $\mathfrak{p} = \mathfrak{q}$. $\square$

The following example shows that Theorem 3 is a proper generalization of Theorems 1 and 2.

Example 3. *Let $M = [0, \infty)$ with the usual metric $d(u, v) = |u - v|$ for all $u, v \in M$. Consider:*

$$\mathcal{T}u = \begin{cases} \frac{1}{2}e^{-2}u, & \text{if } u \in [0,4], \\ 2u, & \text{if } u > 4, \end{cases} \quad \text{and} \quad \alpha(u, v) = \begin{cases} 8, & \text{if } u, v \in [0,4], \\ 0, & \text{otherwise.} \end{cases}$$

Here, we infer that:

$$\begin{aligned} \mathcal{A}(\mathcal{T}, \alpha) &= \{(u, v) \in M \times M : d(\mathcal{T}u, \mathcal{T}v) > 0 \text{ and } \alpha(u, v) \geq 1\} \\ &= \{(u, v) \in M \times M : u \neq v \text{ and } u, v \in [0,4]\}. \end{aligned}$$

Firstly, we claim that $\mathcal{T}$ is an (α-σ-ψ)-contraction with $k = e^{-1}$, $\psi(t) = \frac{t}{2}$ and $\sigma(t) = e^{\sqrt{te^t}}$. For all $u, v \in \mathcal{A}(\mathcal{T}, \alpha)$, that is, for all $u, v \in [0,4]$ with $u \neq v$,

$$\begin{aligned}\sigma\left(d\left(\mathcal{T}u,\mathcal{T}v\right)\right) &= \sigma\left(e^{-2}\frac{|u-v|}{2}\right) \\ &= e^{\sqrt{e^{-2}\frac{|u-v|}{2}e^{e^{-2}\frac{|u-v|}{2}}}} \\ &\leq e^{e^{-1}\sqrt{\frac{|u-v|}{2}e^{\frac{|u-v|}{2}}}} \\ &= e^{e^{-1}\sqrt{\psi(d(u,v))e^{\psi(d(u,v))}}} \\ &= [\sigma(\psi(d(u,v)))]^{k}.\end{aligned}$$

This means that $\mathcal{T}$ is an (α-σ-ψ)-contraction.

Now, let $u,v \in M$ in order that $\alpha\left(u,v\right) \geq 1$. Then, $u,v \in [0,4]$ implies that $\mathcal{T}u,\mathcal{T}v \in [0,4]$, and so, $\alpha\left(\mathcal{T}u,\mathcal{T}v\right) \geq 1$. Hence, the contraction $\mathcal{T}$ is α-admissible. Moreover, there exists $\xi_0 = 4$ in order that $\alpha\left(\xi_0,\mathcal{T}\xi_0\right) \geq 1$.

Let $\{\xi_n\}$ be a sequence in order that $\xi_n \to u$ and $\alpha\left(\xi_n,\xi_{n+1}\right) \geq 1$ for all n. Then, $\xi_n \in [0,4]$ for all n, and so, $u \in [0,4]$ as $\xi_n \to u$. Thus, $\alpha\left(\xi_n,u\right) \geq 1$ for all n.

Consequently, all hypotheses of Theorem 3 are fulfilled. Here, $u = 0$ is the unique fixed point.

Furthermore, for $u = 0$ and $v = 5$, we have:

$$\sigma\left(d\left(\mathcal{T}u,\mathcal{T}v\right)\right) = \sigma\left(d\left(\mathcal{T}0,\mathcal{T}5\right)\right) = \sigma\left(10\right) > \left[\sigma\left(5\right)\right]^{k} = \left[\sigma\left(d(u,v)\right)\right]^{k},$$

for all $\sigma \in \Xi$ and $k \in (0,1)$. Therefore, $\mathcal{T}$ does not verify the axioms of σ-contractions, i.e., Theorem 1 cannot be utilized in this example.

Furthermore, for $u = 0$ and $v = 4$, we obtain:

$$\alpha(u,v)d\left(\mathcal{T}u,\mathcal{T}v\right) = \alpha(0,4)d\left(\mathcal{T}0,\mathcal{T}4\right) = 8\cdot\frac{1}{2}e^{-2}4 > \frac{4}{2} = \psi(d(u,v)).$$

Thus, $\mathcal{T}$ is not an α-ψ-contraction, and hence, Theorem 2 cannot be applied in this example either.

Corollary 1. *Let $\mathcal{T} : M \to M$ be an α-admissible self-mapping on a complete metric space (M,d). Suppose that:*

- (i) *there exists $\xi_0 \in M$ in order that $\alpha\left(\xi_0,\mathcal{T}\xi_0\right) \geq 1$;*
- (ii) *$\mathcal{T}$ is continuous or;*
- (iii) *for every $\{\xi_n\}_{n\in\mathbb{N}} \subset M$ in order that $\xi_n \to x \in M$ and $\alpha\left(\xi_n,\xi_{n+1}\right) \geq 1$ for all $n \in \mathbb{N}$, then $\alpha\left(\xi_n,x\right) \geq 1$ for all $n \in \mathbb{N}$;*
- (iv) *there exist $k \in (0,1)$, $\psi \in \Psi$ and $\sigma \in \Xi$ in order that:*

$$u,v \in M, \quad \mathcal{T}u \neq \mathcal{T}v \Longrightarrow \sigma(\alpha(u,v)d(\mathcal{T}u,\mathcal{T}v)) \leq [\sigma(\psi(d(u,v)))]^{k}. \tag{10}$$

Then, there exists a fixed point of $\mathcal{T}$. Moreover, if $\alpha(u,v) \geq 1$ for all $u,v \in Fix(\mathcal{T})$, then such a fixed point is unique.

Proof. Let $u,v \in M$ in order that $\alpha\left(u,v\right) \geq 1$ and $d(\mathcal{T}u,\mathcal{T}v) > 0$. Then, $(u,v) \in \mathcal{A}(\mathcal{T},\alpha)$. Using $(\sigma 1)$ and (10), we have:

$$\sigma(d(\mathcal{T}u,\mathcal{T}v)) \leq \sigma(\alpha(u,v)d(\mathcal{T}u,\mathcal{T}v)) \leq [\sigma(\psi(d(u,v)))]^{k},$$

and hence:

$$\sigma(d(\mathcal{T}u,\mathcal{T}v)) \leq [\sigma(\psi(d(u,v)))]^k, \quad \text{for all } (u,v) \in \mathcal{A}(\mathcal{T},\alpha).$$

This yields that (2) is satisfied. Thus, the rest of the proof follows from Theorem 3. □

Remark 2. *Let $\mathcal{T}$ be a self-mapping on a metric space (M,d) fulfilling the inequality (10). Then:*

$$\alpha(u,v)d(\mathcal{T}u,\mathcal{T}v) < \psi(d(u,v)),$$

for all $u,v \in M$ with $d(\mathcal{T}u,\mathcal{T}v) > 0$. Hence, we infer that:

$$\alpha(u,v)d(\mathcal{T}u,\mathcal{T}v) \leq \psi(d(u,v)), \quad \textit{for all } u,v \in M.$$

Corollary 2. *Let $\mathcal{T} : M \to M$ be a self-mapping on a complete metric space (M,d). If there exist $k \in (0,1)$, $\psi \in \Psi$ and $\sigma \in \Xi$ in order that:*

$$u,v \in M, \quad d(\mathcal{T}u,\mathcal{T}v) > 0 \Longrightarrow \sigma(d(\mathcal{T}u,\mathcal{T}v)) \leq [\sigma(\psi(d(u,v)))]^k.$$

Then, there exists a unique fixed point of $\mathcal{T}$.

Proof. It is enough to take $\alpha(u,v) = 1$ in Corollary 1. □

Corollary 3. *Let (M,d) be a complete metric space and $\mathcal{T} : M \to M$ be a specified mapping. If there exist $k,c \in (0,1)$, and $\sigma \in \Xi$ in order that:*

$$u,v \in M, \quad d(\mathcal{T}u,\mathcal{T}v) > 0 \Longrightarrow \sigma(d(\mathcal{T}u,\mathcal{T}v)) \leq [\sigma(cd(u,v))]^k,$$

then the mapping $\mathcal{T}$ has a unique fixed point.

Proof. It follows from Corollary 2 with $\psi(t) = ct$. □

3. Applications

Applying our obtained results, we will:

- present some results for graphic contractions;
- ensure the existence a solution for a functional equation originating in dynamic programming.

3.1. Some Results for Graphic Contractions

First, Jachymski [16] provided fixed point results when considering graphic contractions. For other details, see [12,17–23].

We start with the following.

Definition 2 ([16])**.** *The self-mapping $\mathcal{T}$ on M is called a Banach G-contraction or just a G-contraction, if:*

$$\forall u,v \in M, \quad (u,v) \in E(G) \Rightarrow (\mathcal{T}u,\mathcal{T}v) \in E(G) \tag{11}$$

and $\mathcal{T}$ decreases the weights of edges of G as follows:

$$\exists k \in (0,1),\ \forall u,v \in M, \quad (u,v) \in E(G) \Rightarrow d(\mathcal{T}u,\mathcal{T}v) \leq kd(u,v). \tag{12}$$

Definition 3 ([16]). *One says that $\mathcal{T} : M \to M$ is G-continuous, if for u, $\{\xi_n\}$ in M such that $\xi_n \to u$ when n tends to infinity and $(\xi_n, \xi_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ implies $\mathcal{T}\xi_n \to \mathcal{T}u$ as $n \to \infty$.*

Note that if $\mathcal{T}$ is G-continuous, then $\mathcal{T}$ is continuous. However, the converse of the statement is not true in general.

Definition 4. *We endow a metric space (M, d) with a graph G. Given $\mathcal{T} : M \to M$. Denote by $\mathcal{G} \subseteq M \times M$ the set:*

$$\mathcal{G}(\mathcal{T}, G) = \{(u, v) : d(\mathcal{T}u, \mathcal{T}v) > 0 \text{ and } (u, v) \in E(G)\}.$$

Such $\mathcal{T}$ is stated to be an (α-σ-ψ)-G-contraction, if there exist $k \in (0, 1)$, $\psi \in \Psi$ and $\sigma \in \Xi$ in order that:

$$\sigma(d(\mathcal{T}u, \mathcal{T}v)) \leq [\sigma(\psi(d(u, v)))]^k, \quad \text{for all } (u, v) \in \mathcal{G}(\mathcal{T}, G). \tag{13}$$

Theorem 4. *Let (M, d) be a complete metric space endowed with a graph G and $\mathcal{T} : M \to M$ be an (α-σ-ψ)-G-contraction. Assume that the following conditions are satisfied:*

- (i) *$\mathcal{T}$ preserves edges of G;*
- (ii) *there exists $\xi_0 \in M$ in order that $(\xi_0, \mathcal{T}\xi_0) \in E(G)$;*
- (iii) *$\mathcal{T}$ is G-continuous or;*
- (iv) *G satisfies the property (C), that is, for every $\{\xi_n\}_{n\in\mathbb{N}} \subset V(G)$ with $\xi_n \to x$ as $n \to \infty$ and $(\xi_n, \xi_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ implies that $(\xi_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then, there exists a fixed point of $\mathcal{T}$. Moreover, if $(u, v) \in E(G)$ for all $u, v \in Fix(\mathcal{T})$, then such a fixed point is unique.

Proof. Define the function $\alpha : M \times M \to [0, \infty)$ by:

$$\alpha(u, v) = \begin{cases} 1, & \text{if } (u, v) \in E(G), \\ 0, & \text{otherwise} \end{cases}$$

for all $u, v \in M$. Let $(u, v) \in \mathcal{A}(\mathcal{T}, \alpha)$. Then, $d(\mathcal{T}u, \mathcal{T}v) > 0$ and $\alpha(u, v) \geq 1$. By the definition of α, we have $d(\mathcal{T}u, \mathcal{T}v) > 0$ and $(u, v) \in E(G)$, that is, $(u, v) \in \mathcal{G}(\mathcal{T}, G)$. Since $\mathcal{T}$ is an (α-σ-ψ)-G-contraction, we obtain:

$$\sigma(d(\mathcal{T}u, \mathcal{T}v)) \leq [\sigma(\psi(d(u, v)))]^k,$$

that is,

$$\sigma(d(\mathcal{T}u, \mathcal{T}v)) \leq [\sigma(\psi(d(u, v)))]^k, \quad \text{for all } (u, v) \in \mathcal{A}(\mathcal{T}, \alpha).$$

This means that $\mathcal{T}$ satisfies the inequality (2). To prove that $\mathcal{T}$ is α-admissible, let $\alpha(u, v) \geq 1$ for all $u, v \in M$. Then, $(u, v) \in E(G)$. By the virtue of (i), we get $(\mathcal{T}u, \mathcal{T}v) \in E(G)$, and hence, $\alpha(\mathcal{T}u, \mathcal{T}v) \geq 1$. This proves that $\mathcal{T}$ is α-admissible. Furthermore, clearly, (iii) together with (iv) yield (iii) and (iv) of Theorem 3. Thus, all hypotheses of Theorem 3 hold, so $\mathcal{T}$ has a fixed point. We claim that such a fixed point is unique. On the contrary, assume that $u, v \in Fix(\mathcal{T})$. Then, by the hypothesis, $(u, v) \in E(G)$, and so, $\alpha(u, v) \geq 1$. Therefore, from Theorem 3, $\mathcal{T}$ has a unique fixed point. □

Example 4. *Following Example 2.8 in [21], consider* $M = [0,1]$ *is endowed with the usual metric. Let G be a graph with* $V(G) = M$ *and* $E(G) = \Delta \cup \left\{ \left(\frac{1}{n}, \frac{1}{n+1} \right) : n \in \mathbb{N} \right\} \cup \left\{ \left(\frac{1}{8}, \frac{1}{4} \right) \right\} \cup \left\{ \left(\frac{1}{n}, 0 \right) : n \in \mathbb{N} \right\}$. *Consider:*

$$\mathcal{T}u = \begin{cases} \dfrac{1}{4}, & \text{if } 0 \leq u < 1, \\ \dfrac{1}{8}, & \text{if } u = 1. \end{cases}$$

Now, we prove that $\mathcal{T}$ *is an* $(\alpha$-σ-$\psi)$*-G-contraction with* $k = \frac{1}{2}$, $\psi(t) = \frac{3t}{4}$ *and* $\sigma(t) = 1 + t$. *Note that* $(u, v) \in \mathcal{G}(\mathcal{T}, G)$ *if and only if* $u = 1$ *and* $v \in \{0, \frac{1}{2}\}$. *Then, we need to check the following cases:*

Case 1. *If* $u = 1$ *and* $v = 0$, *we have:*

$$\left.\begin{array}{l} \sigma\left(d\left(\mathcal{T}1, \mathcal{T}0\right)\right) = \sigma\left(\left|\dfrac{1}{8} - \dfrac{1}{4}\right|\right) = \sigma\left(\dfrac{1}{8}\right) = \dfrac{9}{8} = 1.125 \\ \left[\sigma\left(\psi\left(d\left(1, 0\right)\right)\right)\right]^k = \left[\sigma\left(\dfrac{3}{4}\right)\right]^{\frac{1}{2}} = \left(\dfrac{7}{4}\right)^{\frac{1}{2}} = 1.3228 \end{array}\right\}$$
$$\Longrightarrow \sigma\left(d\left(\mathcal{T}1, \mathcal{T}0\right)\right) \leq \left[\sigma\left(\psi\left(d\left(1, 0\right)\right)\right)\right]^k.$$

Case 2. *If* $u = 1$ *and* $v = \frac{1}{2}$, *we get:*

$$\left.\begin{array}{l} \sigma\left(d\left(\mathcal{T}1, \mathcal{T}\frac{1}{2}\right)\right) = \sigma\left(\left|\dfrac{1}{8} - \dfrac{1}{4}\right|\right) = \sigma\left(\dfrac{1}{8}\right) = \dfrac{9}{8} = 1.125 \\ \left[\sigma\left(\psi\left(d\left(1, \frac{1}{2}\right)\right)\right)\right]^k = \left[\sigma\left(\dfrac{3}{8}\right)\right]^{\frac{1}{2}} = \left(\dfrac{11}{8}\right)^{\frac{1}{2}} = 1.1726 \end{array}\right\}$$
$$\Longrightarrow \sigma\left(d\left(\mathcal{T}1, \mathcal{T}\frac{1}{2}\right)\right) \leq \left[\sigma\left(\psi\left(d\left(1, \frac{1}{2}\right)\right)\right)\right]^k.$$

Thus, $\mathcal{T}$ *is an* $(\alpha$-σ-$\psi)$*-G-contraction in all possible cases. Furthermore, it is easy to see that:*

(i) $\mathcal{T}$ *preserves edges of G;*
(ii) $(\xi_0, \mathcal{T}\xi_0) \in E(G)$ *for* $\xi_0 \in \left\{ \dfrac{1}{4}, \dfrac{1}{3} \right\}$;
(iii) *G satisfies the property* (C).

All hypotheses of Theorem 4 are verified. Here, $Fix(\mathcal{T}) = \left\{ \frac{1}{4} \right\}$.

3.2. Existence Theorem for a Solution of a Functional Equation

It is known that dynamic programming provides useful tools for people working in the fields of optimization and computer programming. In particular, consider the following functional equation:

$$p(u) = \sup_{v \in D} \left\{ f\left(u, v\right) + K\left(u, v, p\left(\vartheta\left(u, v\right)\right)\right) \right\}, \quad u \in S, \tag{14}$$

where $f : S \times D \to \mathbb{R}$ and $K : S \times D \times \mathbb{R} \to \mathbb{R}$ are bounded; $\vartheta : S \times D \to S$, S and D are Banach spaces; S is a state space; and D is a decision space. We refer the reader to [17,24–28] for more details.

Here, we discuss the existence of a bounded solution of the functional Equation (14) by using the obtained results in the previous section.

Denote by $B(S)$ the set of all real bounded functions defined on S. For $h \in B(S)$, define $\|h\| = \sup_{u \in S} |h(u)|$. Given the Banach space $(B(S), \|\cdot\|)$ where:

$$d(h,k) = \sup_{u \in S} |h(u) - k(u)|,$$

for all $h, k \in B(S)$, represents a metric on $B(S)$. We also define the self-operator $\mathcal{T}$ on $B(S)$ as:

$$\mathcal{T}h(u) = \sup_{v \in D} \{f(u,v) + K(u,v,h(\vartheta(u,v)))\}, \quad u \in S,\ h \in B(S).$$

Consider the following assumptions:

$(A1)$ there exists a function $\eta : B(S) \times B(S) \to \mathbb{R}$ in order that if $\eta(h, h_1) \geq 0$ for all $h, h_1 \in B(S)$ with $h \neq h_1$, we have:

$$|K(u,v,h(u)) - K(u,v,h_1(u))| \leq \left[\left[1 + \sqrt{\psi(|h(u) - h_1(u)|)}\right]^k - 1\right]^2,$$

where $(u,v) \in S \times D$, $k \in (0,1)$ and $\psi \in \Psi$;

$(A2)$ for all $h, h_1 \in B(S)$, $\eta(h, h_1) \geq 0$ implies that $\eta(\mathcal{T}h, \mathcal{T}h_1) \geq 0$;

$(A3)$ there exists $h_0 \in B(S)$ in order that $\eta(h_0, \mathcal{T}h_0) \geq 0$;

$(A4)$ if $\{h_n\}$ is a sequence in $B(S)$ in order that $h_n \to h \in B(S)$ and $\eta(h_n, h_{n+1}) \geq 0$, $n \in \mathbb{N}$, then $\eta(h_n, h) \geq 0$, $n \in \mathbb{N}$.

Theorem 5. *Suppose that the assumptions $(A1)$–$(A4)$ are satisfied. Then, the functional Equation (14) has at least one bounded solution.*

Proof. Let $u \in S$ and $h_1, h_2 \in B(S)$ with $\eta(h_1, h_2) \geq 0$ and $\mathcal{T}h_1 \neq \mathcal{T}h_2$. Then, from $(A1)$, there exist $v \in D$ in order that:

$$\begin{aligned}
d(\mathcal{T}h_1, \mathcal{T}h_2) &= \sup_{u \in S} |\mathcal{T}h_1(u) - \mathcal{T}h_2(u)| \\
&= \sup_{u \in S} \left| \sup_{v \in D} \{f(u,v) + K(u,v,h_1(\vartheta(u,v)))\} \right. \\
&\qquad \left. - \sup_{v \in D} \{f(u,v) + K(u,v,h_2(\vartheta(u,v)))\} \right| \\
&\leq \sup_{u \in S} \left\{ \sup_{v \in D} |K(u,v,h_1(\vartheta(u,v))) - K(u,v,h_2(\vartheta(u,v)))| \right\} \\
&\leq \sup_{x \in S} \left\{ \sup_{v \in D} \left\{ \left[\left[1 + \sqrt{\psi(|h_1(\vartheta(u,v)) - h_2(\vartheta(u,v))|)}\right]^k - 1\right]^2 \right\} \right\} \\
&\leq \sup_{u \in S} \left\{ \left[\left[1 + \sqrt{\psi(\|h_1 - h_2\|)}\right]^k - 1\right]^2 \right\} \\
&\leq \left[\left[1 + \sqrt{\psi(d(h_1, h_2))}\right]^k - 1\right]^2,
\end{aligned}$$

and so:

$$d(\mathcal{T}h_1, \mathcal{T}h_2) \leq \left[\left[1 + \sqrt{\psi(d(h_1, h_2))}\right]^k - 1\right]^2. \tag{15}$$

From the above inequality, we obtain:

$$1 + \sqrt{d(\mathcal{T}h_1, \mathcal{T}h_2)} \leq \left[1 + \sqrt{\psi(d(h_1, h_2))}\right]^k. \tag{16}$$

By setting $\sigma \in \Xi$ by $\sigma(t) = 1 + \sqrt{t}$ for all $t > 0$ and using (16), we infer:

$$\sigma(d(\mathcal{T}h_1, \mathcal{T}h_2)) \leq [\sigma(\psi(d(h_1, h_2)))]^k, \tag{17}$$

for all $h_1, h_2 \in B(S)$ with $\eta\,(h_1, h_2) \geq 0$ and $\mathcal{T}h_1 \neq \mathcal{T}h_2$.

Now, define $\alpha : B(S) \times B(S) \to [0, \infty)$ by:

$$\alpha\,(h_1, h_2) = \begin{cases} 1, & \text{if } \eta\,(h_1, h_2) \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, it follows from (17) that:

$$\sigma(d(\mathcal{T}h_1, \mathcal{T}h_2)) \leq [\sigma(\psi(d(h_1, h_2)))]^k,$$

for all $h_1, h_2 \in B(S)$ with $\alpha\,(h_1, h_2) \geq 1$ and $d(\mathcal{T}h_1, \mathcal{T}h_2) > 0$. This means that $\mathcal{T}$ is an (α-σ-ψ)-contraction. Furthermore, the assertions $(A2), (A3), (A4)$ imply the conditions $(i), (ii)$ and (iv) of Theorem 3, respectively. Consequently, there exists a fixed point of $\mathcal{T}$. Hence, there exists a solution in $B(S)$ for the functional Equation (14). □

By using the same method in the proof of Theorems 5 and 3 together with the function $\sigma \in \Xi$ defined by $\sigma(t) = e^{\sqrt{t}}$, we get the following result.

Theorem 6. *In Theorem 5, replace the assumption $(A1)$ by the following, besides retaining the rest:*

$(A1^*)$ *there exists a function $\eta : B(S) \times B(S) \to \mathbb{R}$ in order that if $\eta(h, h_1) \geq 0$ for all $h, h_1 \in B(S)$ with $h \neq h_1$, we have:*

$$|K\,(u, v, h\,(u)) - K\,(u, v, h_1\,(u))| \leq e^{-\tau}(\psi(|h - h_1|)),$$

where $(u, v) \in S \times D$, $\tau \in (0, \infty)$ and $\psi \in \Psi$.

Then, Equation (14) has at least one bounded solution.

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