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The Power Sums Involving Fibonacci Polynomials and Their Applications

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Received: 28 March 2019 ; Accepted: 25 April 2019; Published: 6 May 2019

Abstract: The Girard and Waring formula and mathematical induction are used to study a problem involving the sums of powers of Fibonacci polynomials in this paper, and we give it interesting divisible properties. As an application of our result, we also prove a generalized conclusion proposed by R. S. Melham.

Keywords: Fibonacci polynomials; Lucas polynomials; sums of powers; divisible properties; R. S. Melham's conjectures

MSC: 11B39

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1. Introduction

For any integer $n \geq 0$, the famous Fibonacci polynomials $\{F_n(x)\}$ and Lucas polynomials $\{L_n(x)\}$ are defined as $F_0(x) = 0$, $F_1(x) = 1$, $L_0(x) = 2$, $L_1(x) = x$ and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$ for all $n \geq 0$. Now, if we let $\alpha = \frac{x+\sqrt{x^2+4}}{2}$ and $\beta = \frac{x-\sqrt{x^2+4}}{2}$, then it is easy to prove that

$$F_n(x) = \frac{1}{\alpha - \beta}\left(\alpha^n - \beta^n\right) \text{ and } L_n(x) = \alpha^n + \beta^n \text{ for all } n \geq 0.$$

If $x = 1$, we have that $\{F_n(x)\}$ turns into Fibonacci sequences $\{F_n\}$, and $\{L_n(x)\}$ turns into Lucas sequences $\{L_n\}$. If $x = 2$, then $F_n(2) = P_n$, the nth Pell numbers, they are defined by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geq 0$. In fact, $\{F_n(x)\}$ is a second-order linear recursive polynomial, when x takes a different value x_0, then $F_n(x_0)$ can become a different sequence.

Since the Fibonacci numbers and Lucas numbers occupy significant positions in combinatorial mathematics and elementary number theory, they are thus studied by plenty of researchers, and have gained a large number of vital conclusions; some of them can be found in References [1–15]. For example, Yi Yuan and Zhang Wenpeng [1] studied the properties of the Fibonacci polynomials, and proved some interesting identities involving Fibonacci numbers and Lucas numbers. Ma Rong and Zhang Wenpeng [2] also studied the properties of the Chebyshev polynomials, and obtained some meaningful formulas about the Chebyshev polynomials and Fibonacci numbers. Kiyota Ozeki [3] got some identity involving sums of powers of Fibonacci numbers. That is, he proved that

$$\sum_{k=1}^{n} F_{2k}^{2m+1} = \frac{1}{5^m}\sum_{j=0}^{m}\frac{(-1)^j}{L_{2m+1-2j}}\binom{2m+1}{j}\left(F_{(2m+1-2j)(2n+1)} - F_{2m+1-2j}\right).$$

Helmut Prodinger [4] extended the result of Kiyota Ozeki [3].

In addition, regarding many orthogonal polynomials and famous sequences, Kim et al. have done a lot of important research work, obtaining a series of interesting identities. Interested readers can refer to References [16–22]; we will not list them one by one.

In this paper, our main purpose is to care about the divisibility properties of the Fibonacci polynomials. This idea originated from R. S. Melham. In fact, in [5], R. S. Melham proposed two interesting conjectures as follows:

Conjecture 1. *If $m \geq 1$ is a positive integer, then the summation*

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$$

can be written as $\left(F_{2n+1}-1\right)^2 P_{2m-1}\left(F_{2n+1}\right)$, where $P_{2m-1}(x)$ is an integer coefficients polynomial with degree $2m-1$.

Conjecture 2. *If $m \geq 0$ is an integer, then the summation*

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} L_{2k}^{2m+1}$$

can be written as $\left(L_{2n+1}-1\right) Q_{2m}\left(L_{2n+1}\right)$, where $Q_{2m}(x)$ is an integer coefficients polynomial with degree $2m$.

Wang Tingting and Zhang Wenpeng [6] solved Conjecture 2 completely. They also proved a weaker conclusion for Conjecture 1. That is,

$$L_1 L_3 L_5 \cdots L_{2m+1} \sum_{k=1}^{n} F_{2k}^{2m+1}$$

can be expressed as $\left(F_{2n+1}-1\right) P_{2m}\left(F_{2n+1}\right)$, where $P_{2m}(x)$ is a polynomial of degree $2m$ with integer coefficients.

Sun et al. [7] solved Conjecture 1 completely. In fact, Ozeki [3] and Prodinger [4] indicated that the odd power sum of the first several consecutive Fibonacci numbers of even order is equivalent to the polynomial estimated at a Fibonacci number of odd order. Sun et al. in [7] proved that this polynomial and its derivative both disappear at 1, and it can be an integer polynomial when a product of the first consecutive Lucas numbers of odd order multiplies it. This presents an affirmative answer to Conjecture 1 of Melham.

In this paper, we are going to use a new and different method to study this problem, and give a generalized conclusion. That is, we will use the Girard and Waring formula and mathematical induction to prove the conclusions in the following:

Theorem 1. *If n and h are positive integers, then we have the congruence*

$$L_1(x) L_3(x) \cdots L_{2n+1}(x) \sum_{m=1}^{h} F_{2m}^{2n+1}(x) \equiv 0 \bmod \left(F_{2h+1}(x)-1\right)^2 .$$

Taking $x = 1$ and $x = 2$ in Theorem 1, we can instantly infer the two corollaries:

Corollary 1. *Let F_n and L_n be Fibonacci numbers and Lucas numbers, respectively. Then, for any positive integers n and h, we have the congruence*

$$L_1 L_3 L_5 \cdots L_{2n+1} \sum_{m=1}^{h} F_{2m}^{2n+1} \equiv 0 \bmod \left(F_{2h+1}-1\right)^2 .$$

Corollary 2. *Let P_n be nth Pell numbers. Then, for any positive integers n and h, we have the congruence*

$$L_1(2)L_3(2)L_5(2)\cdots L_{2n+1}(2)\sum_{m=1}^{h} P_{2m}^{2n+1} \equiv 0 \bmod \left(P_{2h+1}-1\right)^2,$$

where $L_n(2) = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ is called nth Pell–Lucas numbers.

It is clear that our Corollary 1 gave a new proof for Conjecture 1.

2. Several Lemmas

In this part, we will give four simple lemmas, which are essential to prove our main results.

Lemma 1. *Let h be any positive integer; then, we have*

$$\left(x^2+4, F_{2h+1}(x)-1\right) = 1,$$

where x^2+4 and $F_{2h+1}(x)-1$ are said to be relatively prime.

Proof. From the definition of $F_n(x)$ and binomial theorem, we have

$$\begin{aligned} F_{2h+1}(x) &= \frac{1}{2^{2h+1}\sqrt{x^2+4}}\sum_{k=0}^{2h+1}\binom{2h+1}{k}x^k\left(x^2+4\right)^{\frac{2h+1-k}{2}} \\ &\quad -\frac{1}{2^{2h+1}\sqrt{x^2+4}}\sum_{k=0}^{2h+1}\binom{2h+1}{k}x^k(-1)^{2h+1-k}\left(x^2+4\right)^{\frac{2h+1-k}{2}} \\ &= \frac{1}{4^h}\sum_{k=0}^{h}\binom{2h+1}{2k}x^{2k}\left(x^2+4\right)^{h-k}. \end{aligned} \tag{1}$$

Thus, from Equation (1), we have the polynomial congruence

$$\begin{aligned} 4^h F_{2h+1}(x) = \sum_{k=0}^{h}\binom{2h+1}{2k}x^{2k}\left(x^2+4\right)^{h-k} \equiv (2h+1)x^{2h} \\ \equiv \quad (2h+1)\left(x^2+4-4\right)^h \equiv (2h+1)(-4)^h \bmod (x^2+4) \end{aligned}$$

or

$$F_{2h+1}(x)-1 \equiv (2h+1)(-1)^h - 1 \bmod (x^2+4). \tag{2}$$

Since x^2+4 is an irreducible polynomial of x, and $(2h+1)(-1)^h-1$ is not divisible by (x^2+4) for all integer $h \geq 1$, so, from (2), we can deduce that

$$\left(x^2+4, F_{2h+1}(x)-1\right) = 1.$$

Lemma 1 is proved. □

Lemma 2. *Let h and n be non-negative integers with $h \geq 1$; then, we have*

$$(x^2+4)F_{(2h+1)(2n+1)}(x) - L_{2n}(x) - L_{2n+2}(x) \equiv 0 \bmod \left(F_{2h+1}(x)-1\right).$$

Proof. We use mathematical induction to calculate the polynomial congruence for n. Noting $L_0(x) = 2$, $L_1(x) = x$, $L_2(x) = x^2 + 2$. Thus, if $n = 0$, then

$$\begin{aligned} & (x^2+4)F_{(2h+1)(2n+1)}(x) - L_{2n}(x) - L_{2n+2}(x) \\ = & (x^2+4)F_{2h+1}(x) - 2 - x^2 - 2 \\ = & (x^2+4)\left(F_{2h+1}(x) - 1\right) \equiv 0 \bmod \left(F_{2h+1}(x) - 1\right). \end{aligned}$$

If $n = 1$, then $L_2(x) + L_4(x) = x^2 + 2 + x^4 + 4x^2 + 2 = x^4 + 5x^2 + 4$. Note that the identity $F^3_{2h+1}(x) = \frac{1}{x^2+4}\left(F_{3(2h+1)}(x) + 3F_{2h+1}(x)\right)$, so we obtain the congruence

$$\begin{aligned} & (x^2+4)F_{(2h+1)(2n+1)}(x) - L_{2n}(x) - L_{2n+2}(x) \\ = & (x^2+4)F_{3(2h+1)}(x) - x^4 - 5x^2 - 4 \\ = & (x^2+4)\left[(x^2+4)F^3_{2h+1}(x) - 3F_{2h+1}(x)\right] - x^4 - 5x^2 - 4 \\ = & (x^2+4)^2\left[F^3_{2h+1}(x) - F_{2h+1}(x)\right] + (x^2+4)(x^2+1)F_{2h+1}(x) - x^4 - 5x^2 - 4 \\ \equiv & (x^2+4)^2\left(F^2_{2h+1}(x) + F_{2h+1}(x)\right)\left(F_{2h+1}(x) - 1\right) \equiv 0 \bmod \left(F_{2h+1}(x) - 1\right), \end{aligned}$$

which means that Lemma 2 is correct for $n = 0$ and 1.

Assume Lemma 2 is right for all integers $n = 0, 1, 2, \cdots, k$. Namely,

$$(x^2+4)F_{(2h+1)(2n+1)}(x) - L_{2n}(x) - L_{2n+2}(x) \equiv 0 \bmod \left(F_{2h+1}(x) - 1\right), \tag{3}$$

where $0 \le n \le k$.

Thus, $n = k + 1 \ge 2$, and we notice that

$$L_{2(2h+1)}(x)F_{(2h+1)(2k+1)}(x) = F_{(2h+1)(2k+3)}(x) + F_{(2h+1)(2k-1)}(x),$$

$$L_{2k+2}(x) + L_{2k+4}(x) = (x^2+2)L_{2k}(x) + (x^2+2)L_{2k+2}(x) - \left(L_{2k-2}(x) + L_{2k}(x)\right)$$

and

$$L_{2(2h+1)}(x) = (x^2+4)F^2_{2h+1}(x) - 2 \equiv x^2 + 2 \bmod \left(F_{2h+1}(x) - 1\right).$$

From inductive assumption (3), we have

$$\begin{aligned} & (x^2+4)F_{(2h+1)(2n+1)}(x) - L_{2n}(x) - L_{2n+2}(x) \\ = & (x^2+4)F_{(2h+1)(2k+3)}(x) - L_{2k+2}(x) - L_{2k+4}(x) \\ = & (x^2+4)L_{2(2h+1)}(x)F_{(2h+1)(2k+1)}(x) - (x^2+4)F_{(2h+1)(2k-1)} - L_{2k+2}(x) - L_{2k+4}(x) \\ \equiv & (x^2+4)(x^2+2)F_{(2h+1)(2k+1)}(x) - (x^2+2)L_{2k}(x) - (x^2+2)L_{2k+2}(x) \\ & -(x^2+4)F_{(2h+1)(2k-1)}(x) + L_{2k-2}(x) + L_{2k}(x) \\ \equiv & (x^2+2)\left[(x^2+4)F_{(2h+1)(2k+1)}(x) - L_{2k}(x) - L_{2k+2}(x)\right] \\ & -\left[(x^2+4))F_{(2h+1)(2k-1)}(x) - L_{2k-2}(x) - L_{2k}(x)\right] \\ \equiv & 0 \bmod \left(F_{2h+1}(x) - 1\right). \end{aligned}$$

Now, we have achieved the results of Lemma 2. □

Lemma 3. *Let h and n be non-negative integers with $h \geq 1$; then, we have the polynomial congruence*

$$\begin{aligned} & L_1(x)L_3(x)\cdots L_{2n+1}(x)\sum_{m=1}^{h}\left[F_{2m(2n+1)}(x)-(2n+1)F_{2m}(x)\right] \\ \equiv\ & 0 \bmod \left(F_{2h+1}(x)-1\right)^2. \end{aligned}$$

Proof. For positive integer n, first note that $\alpha\beta = -1$, $L_n(x) = \alpha^n + \beta^n$,

$$\begin{aligned} & \sum_{m=1}^{h} F_{2m(2n+1)}(x) = \frac{1}{\sqrt{x^2+4}}\sum_{m=1}^{h}\left[\alpha^{2m(2n+1)}-\beta^{2m(2n+1)}\right] \\ =\ & \frac{1}{\sqrt{x^2+4}}\left[\frac{\alpha^{2(2n+1)}\left(\alpha^{2h(2n+1)}-1\right)}{\alpha^{2(2n+1)}-1}-\frac{\beta^{2(2n+1)}\left(\beta^{2h(2n+1)}-1\right)}{\beta^{2(2n+1)}-1}\right] \\ =\ & \frac{1}{L_{2n+1}(x)}\left[F_{(2h+1)(2n+1)}(x)-F_{2n+1}(x)\right] \end{aligned} \tag{4}$$

and

$$\sum_{m=1}^{h} F_{2m}(x) = \frac{1}{\sqrt{x^2+4}}\sum_{m=1}^{h}\left[\alpha^{2m}-\beta^{2m}\right] = \frac{1}{L_1(x)}\left[F_{(2h+1)}(x)-1\right]. \tag{5}$$

Thus, from Labels (4) and (5), we know that, to prove Lemma 3, now we need to obtain the polynomial congruence

$$\begin{aligned} & L_1(x)\left(F_{(2h+1)(2n+1)}(x)-F_{2n+1}(x)\right)-(2n+1)L_{2n+1}(x)\left(F_{2h+1}(x)-1\right) \\ \equiv\ & 0 \bmod \left(F_{2h+1}(x)-1\right)^2. \end{aligned} \tag{6}$$

Now, we prove (6) by mathematical induction. If $n = 0$, then it is obvious that (6) is correct. If $n = 1$, we notice that $L_1(x) = x$, $F_{3(2h+1)}(x) = (x^2+4)F_{2h+1}^3(x) - 3F_{2h+1}(x)$ and $F_{2h+1}^3(x) \equiv \left(F_{2h+1}(x)-1+1\right)^3 \equiv 3F_{2h+1}(x) - 2 \bmod \left(F_{2h+1}(x)-1\right)^2$ we have

$$\begin{aligned} & L_1(x)F_{(2h+1)(2n+1)}(x) - L_1(x)F_{2n+1}(x) - (2n+1)L_{2n+1}(x)\left(F_{2h+1}(x)-1\right) \\ =\ & xF_{3(2h+1)}(x) - xF_3(x) - 3L_3(x)\left(F_{2h+1}(x)-1\right) \\ =\ & x(x^2+4)F_{2h+1}^3(x) - 3xF_{2h+1}(x) - x(x^2+1) - 3(x^3+3x)\left(F_{2h+1}(x)-1\right) \\ \equiv\ & (x^3+4x)\left(3F_{2h+1}(x)-2\right) - 3xF_{2h+1}(x) - (x^3+x) - 3(x^3+3x)\left(F_{2h+1}(x)-1\right) \\ \equiv\ & 3(x^3+3x)\left(F_{2h+1}(x)-1\right) - 3(x^3+3x)\left(F_{2h+1}(x)-1\right) \\ \equiv\ & 0 \bmod \left(F_{2h+1}(x)-1\right)^2. \end{aligned}$$

Thus, $n = 1$ is fit for (6). Assume that (6) is correct for all integers $n = 0, 1, 2, \cdots, k$. Namely,

$$\begin{aligned} & L_1(x)\left(F_{(2h+1)(2n+1)}(x)-F_{2n+1}(x)\right)-(2n+1)L_{2n+1}(x)\left(F_{2h+1}(x)-1\right) \\ \equiv\ & 0 \bmod \left(F_{2h+1}(x)-1\right)^2 \end{aligned} \tag{7}$$

for all $n = 0, 1, \cdots, k$.

Where $n = k+1 \geq 2$, we notice

$$L_{2(2h+1)}(x)F_{(2h+1)(2k+1)}(x) = F_{(2h+1)(2k+3)}(x) + F_{(2h+1)(2k-1)}(x)$$

and

$$\begin{aligned} & L_{2(2h+1)}(x) = (x^2+4)F_{2h+1}^2(x) - 2 = (x^2+4)\left(F_{2h+1}(x) - 1 + 1\right)^2 - 2 \\ = \ & (x^2+4)\left[(F_{2h+1}(x)-1)^2 + 2(F_{2h+1}(x)-1)\right] + x^2 + 2 \\ \equiv \ & 2(x^2+4)(F_{2h+1}(x)-1) + x^2 + 2 \bmod (F_{2h+1}(x)-1)^2 . \end{aligned}$$

From inductive assumption (7) and Lemma 2, we have

$$\begin{aligned} & xF_{(2h+1)(2n+1)}(x) - xF_{2n+1}(x) - (2n+1)L_{2n+1}(x)\left(F_{2h+1}(x)-1\right) \\ = \ & xF_{(2h+1)(2k+3)}(x) - xF_{2k+3}(x) - (2k+3)L_{2k+3}(x)\left(F_{2h+1}(x)-1\right) \\ = \ & xL_{2(2h+1)}(x)F_{(2h+1)(2k+1)}(x) - xF_{(2h+1)(2k-1)}(x) - xF_{2k+3}(x) \\ & -(2k+3)L_{2k+3}(x)\left(F_{2h+1}(x)-1\right) \\ \equiv \ & 2x(x^2+4)(F_{2h+1}(x)-1)F_{(2h+1)(2k+1)}(x) + x(x^2+2)F_{(2h+1)(2k+1)}(x) \\ & -xF_{(2h+1)(2k-1)}(x) - x(x^2+2)F_{2k+1}(x) + xF_{2k-1}(x) \\ & -(x^2+2)(2k+1)L_{2k+1}(x)\left(F_{2h+1}(x)-1\right) + (2k-1)L_{2k-1}(x)\left(F_{2h+1}(x)-1\right) \\ & -2x\left(L_{2k}(x) + L_{2k+2}(x)\right)\left(F_{2h+1}(x)-1\right) \\ \equiv \ & 2x(F_{2h+1}(x)-1)\left[(x^2+4)F_{(2h+1)(2k+1)}(x) - L_{2k}(x) - L_{2k+2}(x)\right] \\ & +(x^2+2)\left[xF_{(2h+1)(2k+1)}(x) - xF_{2k+1}(x) - (2k+1)L_{2k+1}(x)\left(F_{2h+1}(x)-1\right)\right] \\ & -\left[xF_{(2h+1)(2k-1)}(x) - xF_{2k-1}(x) - (2k-1)L_{2k-1}(x)\left(F_{2h+1}(x)-1\right)\right] \\ \equiv \ & 2x(F_{2h+1}(x)-1)\left[(x^2+4)F_{(2h+1)(2k+1)}(x) - L_{2k}(x) - L_{2k+2}(x)\right] \\ \equiv \ & 0 \bmod (F_{2h+1}(x)-1)^2 . \end{aligned}$$

Now, we attain Lemma 3 by mathematical induction. □

Lemma 4. *For all non-negative integers u and real numbers X, Y, we have the identity*

$$X^u + Y^u = \sum_{k=0}^{[\frac{u}{2}]} (-1)^k \frac{u}{u-k}\binom{u-k}{k}(X+Y)^{u-2k}(XY)^k ,$$

in which $[x]$ denotes the greatest integer $\leq x$.

Proof. This formula due to Waring [15]. It can also be found in Girard [14]. □

3. Proof of the Theorem

We will achieve the theorem by these lemmas. Taking $X = \alpha^{2m}$, $Y = -\beta^{2m}$ and $U = 2n+1$ in Lemma 4, we notice that $XY = -1$, from the expression of $F_n(x)$

$$\begin{aligned} F_{2m(2n+1)}(x) &= \sum_{k=0}^{n} (-1)^k \frac{2n+1}{2n+1-k}\binom{2n+1-k}{k}(x^2+4)^{n-k}F_{2m}^{2n+1-2k}(x)(-1)^k \\ &= \sum_{k=0}^{n} \frac{2n+1}{2n+1-k}\binom{2n+1-k}{k}(x^2+4)^{n-k}F_{2m}^{2n+1-2k}(x). \end{aligned} \tag{8}$$

For any integer $h \geq 1$, from (8), we get

$$\begin{aligned}
&\sum_{m=1}^{h} \left[F_{2m(2n+1)}(x) - (2n+1)F_{2m}(x) \right] \\
= &\sum_{k=0}^{n-1} \frac{2n+1}{2n+1-k} \binom{2n+1-k}{k} (x^2+4)^{n-k} \sum_{m=1}^{h} F_{2m}^{2n+1-2k}(x). \qquad (9)
\end{aligned}$$

If $n = 1$, then, from (9), we can get

$$L_1(x)L_3(x) \sum_{m=1}^{h} (F_{6m}(x) - 3F_{2m}(x)) = L_1(x)L_3(x)(x^2+4) \sum_{m=1}^{h} F_{2m}^3(x). \qquad (10)$$

From Lemma 1, we know that $(x^2+4, F_{2h+1}(x) - 1) = 1$, so, applying Lemma 3 and (10), we deduce that

$$L_1(x)L_3(x) \sum_{m=1}^{h} F_{2m}^3(x) \equiv 0 \bmod (F_{2h+1}(x) - 1)^2. \qquad (11)$$

This means that Theorem 1 is suitable for $n = 1$.

Assume that Theorem 1 is correct for all integers $n = 1, 2, \cdots, s$. Then,

$$L_1(x)L_3(x) \cdots L_{2n+1}(x) \sum_{m=1}^{h} F_{2m}^{2n+1}(x) \equiv 0 \bmod (F_{2h+1}(x) - 1)^2 \qquad (12)$$

for all integers $1 \leq n \leq s$.

When $n = s+1$, from (9), we obtain

$$\begin{aligned}
&\sum_{m=1}^{h} \left(F_{2m(2s+3)}(x) - (2s+3)F_{2m}(x) \right) \\
= &\sum_{k=0}^{s} \frac{2s+3}{2s+3-k} \binom{2s+3-k}{k} (x^2+4)^{s+1-k} \sum_{m=1}^{h} F_{2m}^{2s+3-2k}(x) \\
= &\sum_{k=1}^{s} \frac{2s+3}{2s+3-k} \binom{2s+3-k}{k} (x^2+4)^{s+1-k} \sum_{m=1}^{h} F_{2m}^{2s+3-2k}(x) \\
&+(x^2+4)^{s+1} \sum_{m=1}^{h} F_{2m}^{2s+3}(x). \qquad (13)
\end{aligned}$$

From Lemma 3, we have

$$\begin{aligned}
&L_1(x)L_3(x) \cdots L_{2s+3}(x) \sum_{m=1}^{h} \left[F_{2m(2s+3)}(x) - (2s+3)F_{2m}(x) \right] \\
\equiv \ &0 \bmod (F_{2h+1}(x) - 1)^2. \qquad (14)
\end{aligned}$$

Applying inductive hypothesis (12), we obtain

$$\begin{aligned}
&L_1(x)L_3(x) \cdots L_{2s+1}(x) \sum_{k=1}^{s} \frac{2s+3}{2s+3-k} \binom{2s+3-k}{k} \\
&\times (x^2+4)^{s+1-k} \sum_{m=1}^{h} F_{2m}^{2s+3-2k}(x) \equiv 0 \bmod (F_{2h+1}(x) - 1)^2. \qquad (15)
\end{aligned}$$

Combining (13), (14), (15) and Lemma 3, we have the conclusion

$$\begin{aligned} & L_1(x)L_3(x)\cdots L_{2s+3}(x)\cdot(x^2+4)^{s+1}\sum_{m=1}^{h} F_{2m}^{2s+3}(x) \\ \equiv\ & 0 \bmod \left(F_{2h+1}(x)-1\right)^2. \end{aligned} \tag{16}$$

Note that $\left(x^2+4, F_{2h+1}(x)-1\right)=1$, so (16) indicates the conclusion

$$L_1(x)L_3(x)\cdots L_{2s+3}(x)\cdot\sum_{m=1}^{h} F_{2m}^{2s+3}(x) \equiv 0 \bmod \left(F_{2h+1}(x)-1\right)^2.$$

Now, we apply mathematical induction to achieve Theorem 1.

Author Contributions: Conceptualization, L.C.; methodology, L.C and X.W.; validation, L.C. and X.W.; formal analysis, L.C.; investigation, X.W.; resources, L.C.; writing—original draft preparation, L.C.; writing—review and editing, X.W.; visualization,L.C.; supervision, L.C.; project administration, X.W.; all authors have read and approved the final manuscript.

Funding: This work is supported by the N. S. F. (11771351) and (11826205) of P. R. China.

Acknowledgments: The authors would like to thank the referees for their very helpful and detailed comments, which have significantly improved the presentation of this paper.

Conflicts of Interest: The authors state that there are no conflicts of interest concerning the publication of this paper.

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