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Asymptotic Properties of Solutions of Fourth-Order Delay Differential Equations

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Received: 28 March 2019; Accepted: 25 April 2019; Published: 3 May 2019

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Abstract: In the paper, we study the oscillation of fourth-order delay differential equations, the present authors used a Riccati transformation and the comparison technique for the fourth order delay differential equation, and that was compared with the oscillation of the certain second order differential equation. Our results extend and improve many well-known results for oscillation of solutions to a class of fourth-order delay differential equations. Some examples are also presented to test the strength and applicability of the results obtained.

Keywords: fourth-order; nonoscillatory solutions; oscillatory solutions; delay differential equations

1. Introduction

In this work, we consider a fourth-order delay differential equation

$$Lz + q\left(y\right) f\left(z\left(\sigma\left(y\right)\right)\right) = 0, \tag{1}$$

where

$$Lz := \left[m_3\left(y\right)\left(m_2\left(y\right)\left[m_1\left(y\right)z'\left(y\right)\right]'\right)'\right]'.$$

We assume $m_i,\ q,\ \sigma \in C\left(\left[y_0,\infty\right),\mathbb{R}\right),\ m_i\left(y\right) > 0,\ i = 1,\ 2,\ 3,\ \lim\limits_{y\to\infty}\dfrac{m_3\left(y\right)}{m_1\left(y\right)} > 0,\ q > 0,\ \sigma\left(y\right) \le y$ and $\lim\limits_{y\to\infty}\sigma\left(y\right) = \infty$, $f \in C\left(\mathbb{R},\mathbb{R}\right),\ f\left(u\right)/u \ge k > 0$ for $u \neq 0$.

By a solution of (1) we mean a function $z \in C\left(\left(\sigma\left(y_z\right),\infty\right)\right)$, which has the property $m_1\left(y\right)z'\left(y\right),\ m_2\left(y\right)\left[m_1\left(y\right)z'\left(y\right)\right]',\ m_3\left(y\right)\left(m_2\left(y\right)\left[m_1\left(y\right)z'\left(y\right)\right]'\right)' \in C^1[y_z,\infty)$, and satisfies (1) on $[y_z,\infty)$. We consider only those solutions z of (1) which satisfy $\sup\left\{\left|z\left(y\right)\right| : y \ge y_z\right\} > 0$, for all $y > y_z$. Such a solution is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory otherwise.

The study of differential equations with deviating argument was initiated in 1918, appearing in the first quarter of the twentieth century as an area of mathematics that has since received a lot of attention. It has been created in order to unify the study of differential and functional differential equations. Since then, there has been much research activity concerning the oscillation of solutions of various classes of differential and functional differential equations. Many authors have contributed on various aspects of this theory, see ([1–9]) .

The problem of the oscillation of higher and fourth order differential equations have been widely studied by many authors, who have provided many techniques used for obtaining oscillatory criteria for higher and fourth order differential equations. We refer the reader to the related books (see [4,10–13]) and to the papers (see [11,14–18]). Because of the above motivating factors for the study of fourth-order differential equations, as well as because of the theoretical interest in generalizing and extending some known results from those given for lower-order equations, the study of oscillation of such equations has received a considerable amount of attention. For a systematic summary of the most significant efforts made as regards this theory, the reader is referred to the monographs of [19–22].

Especially, second and fourth order delay differential equations are of great interest in biology in explaining self-balancing of the human body and in robotics in constructing biped robots.

One of the traditional tools in the study of oscillation of equations which are special cases of (1) has been based on a reduction of order and the comparison with oscillation of first-order delay differential equations. Another widely used technique, applicable also in the above-mentioned case, involves the Riccati type transformation which has been used to reduce Equation (1) to a first-order Riccati inequality see (see [2]) .

Moaaz et al. [11] improved and extended the Riccati transformation to obtain new oscillatory criteria for the fourth order delay differential equations

$$\left(\pi\left(y\right)\left(z'''\left(y\right)\right)^{\alpha}\right)'+\int_{a}^{m}q\left(y,\xi\right)f\left(z\left(\Phi\left(y,\xi\right)\right)\right)d\sigma\left(\xi\right)=0,\qquad y\geq y_{0}.$$

Elabbasy et al. [7] studied the equation

$$\left[m\left(y\right)\left(z^{(n-1)}\left(y\right)\right)^{\gamma}\right]'+\sum_{i=1}^{m}q_{i}\left(y\right)f\left(z\left(\sigma_{i}\left(y\right)\right)\right)=0,\qquad y\geq y_{0}.$$

Agarwal et al. [1] and the present authors in [18] used the comparison technique for the fourth order delay differential equation

$$\left[m\left(y\right)\left(z^{(n-1)}\left(y\right)\right)^{\gamma}\right]'+q\left(y\right)z^{\gamma}\left(\sigma\left(y\right)\right)=0,\qquad y\geq y_{0},$$

that was compared with the oscillation of certain first order differential equation and under the conditions

$$\int_{y_0}^{\infty}\frac{1}{m^{\frac{1}{\gamma}}\left(y\right)}dy=\infty,$$

and

$$\int_{y_0}^{\infty}\frac{1}{m^{\frac{1}{\gamma}}\left(y\right)}dy<\infty.$$

However, the authors of this paper used the comparison technique for the fourth order delay differential equation and that was compared with the oscillation of certain second order differential equation.

To the best of our knowledge, there is nothing known about the oscillation of (1) to be oscillatory under the

$$\int_{y_0}^{\infty}\frac{1}{m_i\left(y\right)}dy=\infty. \tag{2}$$

Our primary goal is to fill this gap by presenting simple criteria for the oscillation of all solutions of (1). So the main advantage of studying (1) essentially lies in the direct application of the well-known Kiguradze lemma [23] (Lemma 1), which allows one to classify the set of possible nonoscillatory solutions.

In what follows, all occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. As usual and without loss of generality, we can deal only with eventually positive solutions of (1).

2. Main Results

In this section, we state some oscillation criteria for (1). For convenience, we denote

$$\begin{aligned} \pi_i(y) &= \int_{y_1}^{y} \frac{1}{m_i(s)} ds, \quad i = 1, 2, 3, \quad I_2(y) = \int_{y_1}^{y} \frac{1}{m_1(s)} \pi_2(s)\, ds. \\ A_2(y) &= \int_{y_1}^{y} \frac{1}{m_2(s)} \pi_3(s)\, ds\,, \; A_3(y) = \int_{y_1}^{y} \frac{1}{m_1(s)} A_2(s)\, ds. \\ \bar{E}_0 z(y) &= z(y)\,, \; \bar{E}_i z(y) = m_i \left(\bar{E}_{i-1} z(y)\right)'\,, \; i = 1, 2, 3, \; \bar{E}_4 z(y) = \left(\bar{E}_3 z(y)\right)'. \end{aligned}$$

where y_1 is sufficiently large.

The main step to study Equation (1) is to determine the derivatives sign $\bar{E}_i z(y)$ according to Kiguradze's lemma [23]

$$\bar{E}_4 z(y) + q(y) f(z(\sigma(y))) = 0,$$

the set Φ of nonoscillatory solutions can be divided into two parts

$$\Phi = \Phi_1 \cup \Phi_3,$$

say positive solution $z(y)$ satisfies

$$z(y) \in \Phi_1 \Longleftrightarrow \bar{E}_1 z(y) > 0, \quad \bar{E}_2 z(y) < 0, \quad \bar{E}_3 z(y) > 0, \quad \bar{E}_4 z(y) < 0,$$

or

$$z(y) \in \Phi_3 \Longleftrightarrow \bar{E}_1 z(y) > 0, \quad \bar{E}_2 z(y) > 0, \quad \bar{E}_3 z(y) > 0, \quad \bar{E}_4 z(y) < 0.$$

Theorem 1. *Let (2) hold. Assume that $z(y)$ be a positive solution of Equation (1). If*

(i) $z(y) \in \Phi_1$, *then* $\dfrac{z(y)}{\pi_1(y)}$ *is decreasing.*

(ii) $z(y) \in \Phi_3$, *then* $\dfrac{z(y)}{A_3(y)}$ *is decreasing and* $\bar{E}_1 z(y) \geq A_2(y) \bar{E}_3 z(y)$.

Proof. Let $z(y)$ be a positive solution of (1) and $z(y) \in \Phi_1$. It follows from the monotonicity of $\bar{E}_1 z(y)$ that

$$\begin{aligned} z(y) &> z(y) - z(y_1) \\ &= \int_{y_1}^{y} \frac{1}{m_1(s)} \bar{E}_1 z(s)\, ds, \\ &\geq \bar{E}_1 z(y) \int_{y_1}^{y} \frac{1}{m_1(s)} ds, \\ &\geq \bar{E}_1 z(y) \pi_1(y) > m_1(y) z'(y) \pi_1(y). \end{aligned}$$

Therefore,

$$\left(\frac{z(y)}{\pi_1(y)} \right)' = \frac{z'(y) \pi_1(y) - z(y) \frac{1}{m_1(y)}}{(\pi_1(y))^2} < 0, \tag{3}$$

case (i) is proved. Now let $z(y) \in \Phi_3$. Since

$$\begin{aligned} \bar{E}_2 z(y) &= \bar{E}_2 z(y_1) + \int_{y_1}^{y} \frac{1}{m_3(s)} \bar{E}_3 z(s)\, ds \\ &> \bar{E}_3 z(y)\, \pi_3(y) \end{aligned}$$

then

$$\left(\frac{\bar{E}_2 z(y)}{\pi_3(y)} \right)' = \frac{\bar{E}_2' z(y)\, \pi_3(y) - \bar{E}_2 z(y) \frac{1}{m_3(y)}}{(\pi_3(y))^2} < 0. \tag{4}$$

Thus $\frac{\bar{E}_2 z(y)}{\pi_3(y)}$ is decreasing. Moreover,

$$\begin{aligned} \bar{E}_1 z(y) &= \bar{E}_1 z(y_1) + \int_{y_1}^{y} \frac{\pi_3(s)}{m_2(s)} \frac{\bar{E}_2 z(s)}{\pi_3(s)} ds, \\ &> \frac{\bar{E}_2 z(y)}{\pi_3(y)} A_2(y). \end{aligned}$$

we obtain $\bar{E}_1 z(y) \geq A_2(y) \bar{E}_3 z(y)$ and

$$\left(\frac{\bar{E}_1 z(y)}{A_2(y)} \right)' = \frac{\bar{E}_1' z(y)\, A_2(y) - \frac{1}{m_2(y)} \pi_3(y) \bar{E}_1 z(y)}{(A_2(y))^2} < 0. \tag{5}$$

Thus $\frac{\bar{E}_1 z(y)}{A_2(y)}$ is decreasing. On the other hand,

$$\begin{aligned} z(y) &= z(y_1) + \int_{y_1}^{y} \frac{A_2(s)}{m_1(s)} \frac{\bar{E}_1 z(s)}{A_2(s)} ds, \\ &> \frac{\bar{E}_1 z(y)}{A_2(y)} A_3(y), \end{aligned}$$

which implies

$$\left(\frac{z(y)}{A_3(y)} \right)' = \frac{z'(y)\, A_3(y) - \frac{1}{m_1(y)} A_2(y) z(y)}{(A_3(y))^2} < 0. \tag{6}$$

So that $\frac{z(y)}{A_3(y)}$ is decreasing. Theorem is proved. □

Let

$$\delta(y) = \frac{1}{m_1(y)} \left(\int_{y}^{\sigma^{-1}(y)} \frac{1}{m_2(s)} \int_{s}^{\sigma^{-1}(y)} \frac{1}{m_3(v)} dv ds \int_{\sigma^{-1}(y)}^{\infty} kq(s)\, ds \right).$$

Theorem 2. *Let (2) hold. Let $z(y)$ be a positive solution of Equation (1). If*

(i) $z(y) \in \Phi_1$, *then* $z'(y) \geq \delta(y) z(y)$.
(ii) $z(y) \in \Phi_3$, *then* $z'(y) \geq \frac{1}{m_1(y)\pi_1(y)} z(y)$.

Proof. Assume that $z(y)$ is a positive solution of (1) and $z(y) \in \Phi_1$. For any $u > y$, we have $\bar{E}_1 z(y)$ that

$$\begin{aligned} -\bar{E}_2 z(y) &= \bar{E}_2 z(u) - \bar{E}_2 z(y), \\ &= \int_{y}^{u} \frac{1}{m_3(s)} \bar{E}_3 z(s)\, ds, \\ &> \bar{E}_3 z(u) \int_{y}^{u} \frac{1}{m_3(s)} ds. \end{aligned} \tag{7}$$

Multiplying by $\frac{1}{m_2(s)}$ and then integrating from y to u, one gets

$$\begin{aligned} \bar{E}_1 z(y) &\geq \int_y^u \frac{\bar{E}_3 z(y)}{m_2(s)} \int_s^u \frac{1}{m_3(v)} dv ds, \\ &> \bar{E}_3 z(u) \int_y^u \frac{1}{m_2(s)} \int_s^u \frac{1}{m_3(v)} dv ds. \end{aligned} \tag{8}$$

An integration of (1) from u to ∞, yields

$$\begin{aligned} \bar{E}_3 z(u) &\geq \int_u^\infty kq(s) z(\sigma(s)) ds, \\ &\geq z(\sigma(s)) \int_u^\infty kq(s) ds. \end{aligned}$$

Combining (7) together with (8) and setting $u = \sigma^{-1}(y)$, we get

$$\begin{aligned} z'(y) &\geq \frac{1}{m_1(y)} \left(\int_y^{\sigma^{-1}(y)} \frac{1}{m_2(s)} \int_s^{\sigma^{-1}(y)} \frac{1}{m_3(v)} dv ds \int_{\sigma^{-1}(y)}^\infty kq(s) ds \right) z(y). \\ &\geq \delta(y) z(y). \end{aligned} \tag{9}$$

and case (i) is proved. Now let $z(y) \in \Phi_3$. Employing (H_2), the monotonicity of $\bar{E}_1 z(y)$ and the fact that $\bar{E}_1 z(y) \to \infty$ as $y \to \infty$, we get

$$\begin{aligned} z(y) &= z(y_1) + \int_{y_1}^y \frac{1}{m_1(s)} \bar{E}_1 z(s) ds, \\ &\leq z(y_1) + \bar{E}_1 z(y) \int_{y_1}^y \frac{1}{m_1(s)} ds, \\ &= z(y_1) - \bar{E}_1 z(y) \int_0^{y_1} \frac{1}{m_1(s)} ds + \bar{E}_1 z(y) \int_0^y \frac{1}{m_1(s)} ds, \\ &\leq \bar{E}_1 z(s) \int_0^y \frac{1}{m_1(s)} ds. \end{aligned} \tag{10}$$

The proof is complete now. $\square$

Now, we apply the results of the previous cases to obtain the oscillation conditions of Equation (1). We denote

$$\begin{aligned} \delta_1(y) &= q(y) \frac{\pi_1(\sigma(y))}{\pi_1(y)}, \\ \delta_2(y) &= kq(y) \frac{A_3(\sigma(y))}{A_3(y)}. \end{aligned}$$

Theorem 3. *Let (2) hold. Assume there exists a positive continuously differentiable functions ρ, $\vartheta \in C([y_0, \infty))$ such that*

$$\limsup_{y \to \infty} \int_{y_1}^\infty \left[\frac{\rho(v)}{m_2(v)} \int_v^\infty \frac{1}{m_3(u)} \int_u^\infty \delta_1(s) ds du - \frac{m_1(v)(\rho'(v))^2}{4\rho(v)} \right] dv = \infty, \tag{11}$$

and

$$\limsup_{y \to \infty} \int_{y_1}^\infty \left[\delta_2(v) \vartheta(s) - \frac{m_1(s)(\vartheta'(v))^2}{4\rho(v) A_2(s)} \right] ds = \infty. \tag{12}$$

Then every solution of Equation (1) is oscillatory.

Proof. Assume that (1) has a nonoscillatory solution $z(y)$. Without loss of generality, we can assume that $z(y)$ is a positive solution of (1). Then either $z(y) \in \Phi_1$ or $z(y) \in \Phi_3$. Now assume that $z(y) \in \Phi_1$. Theorem 1 implies that

$$z(\sigma(y)) \geq \frac{\pi_1(\sigma(y))}{\pi_1(y)} z(y)$$

On the other hand, it follows from Theorem 2 that

$$z'(y) \geq \delta(y) z(y).$$

Setting both estimates into (1), we get

$$\bar{E}_4 z(y) + \delta_1(y) \leq 0.$$

Integrating from y to ∞ one gets

$$\begin{aligned} -\bar{E}_3 z(y) &\geq \int_y^\infty \delta_1(s) z(s)\, ds, \\ &\geq z(y) \int_y^\infty \delta_1(s)\, ds. \end{aligned} \tag{13}$$

Integrating once more, we have

$$\bar{E}_2 z(y) + \left(\int_y^\infty \frac{1}{m_3(u)} \int_u^\infty \delta_1(s)\, ds du \right) z(y) \leq 0. \tag{14}$$

Define the function $\omega(y)$ by

$$\omega(y) := \rho(y) \frac{\bar{E}_1 z(y)}{z(y)}, \tag{15}$$

then $\omega(y) > 0$ and

$$\begin{aligned} \omega'(y) &= \rho'(y) \frac{\bar{E}_1 z(y)}{z(y)} + \rho(y) \frac{\bar{E}_2 z(y)}{m_2(y) z(y)} - \rho(y) \frac{\bar{E}_1 z(y) z'(y)}{z^2(y)} \\ &\leq -\frac{\rho(y)}{m_2(y) z(y)} \int_y^\infty \frac{1}{m_3(u)} \int_u^\infty \delta_1(s)\, ds du + \frac{\rho'(y)}{\rho(y)} \omega(y) - \frac{\omega^2(y)}{m_1(y) \rho(y)} \\ &\leq -\frac{\rho(y)}{m_2(y) z(y)} \int_y^\infty \frac{1}{m_3(u)} \int_u^\infty \delta_1(s)\, ds du + \frac{m_1(y) (\rho'(y))^2}{4\rho(y)}. \end{aligned} \tag{16}$$

Integration of the previous inequality yields

$$\int_{y_1}^y \left[\frac{\rho(v)}{m_1(v)} \int_v^\infty \frac{1}{m_3(u)} \int_u^\infty \delta_1(s)\, ds du - \frac{m_1(v) (\rho'(v))^2}{4\rho(v)} \right] dv \leq \omega(y_1),$$

this contradicts with (11) as $y \to \infty$. Now assume that $z(y) \in \Phi_3$. Theorems 1 and 2 guarantee that

$$z(\sigma(y)) \geq \frac{A_3(\sigma(y))}{A_3(y)} z(y), \quad z'(y) \geq \frac{1}{m_1(y) \pi_1(y)} z(y), \quad \bar{E}_1 z(y) \geq A_2(y) \bar{E}_3 z(y),$$

what in view of (1) provides

$$\bar{E}_4 z(y) + \delta_2(y) \leq 0.$$

Now define $\psi(y)$ by

$$\psi(y) := \vartheta(y) \frac{\bar{E}_3 z(y)}{z(y)}, \tag{17}$$

then $\psi(y) > 0$ and

$$\begin{aligned} \psi'(y) &= \vartheta'(y)\frac{\bar{E}_3 z(y)}{z(y)} + \vartheta(y)\frac{\bar{E}_4 z(y)}{z(y)} - \vartheta(y)\frac{\bar{E}_3 z(y) z'(y)}{z^2(y)} \\ &\le -\vartheta(y)\delta_2(y) + \frac{\vartheta'(y)}{\vartheta(y)}\psi(y) - \frac{A_2(y)\psi^2(y)}{m_1(y)\vartheta(y)} \\ &\le -\vartheta(y)\delta_2(y) + \frac{m_1(y)(\vartheta'(y))^2}{4\vartheta(y)A_2}. \end{aligned} \tag{18}$$

Integrating from $y_1$to y and letting $y \to \infty$, we get

$$\int_{y_1}^{\infty}\left[\delta_2(v)\vartheta(s) - \frac{m_1(s)(\vartheta'(v))^2}{4\rho(v)A_2(s)}\right]ds \le \psi(y_1),$$

which contradicts with (12) and the proof is complete. □

Corollary 1. *Let (2) hold and*

$$\limsup_{y\to\infty}\int_{y_1}^{\infty}\left[\frac{\pi_1(v)}{m_2(v)}\int_v^{\infty}\frac{1}{m_3(u)}\int_u^{\infty}\delta_1(s)\,dsdu - \frac{1}{4m_1(v)\pi_1(v)}\right]dv = \infty, \tag{19}$$

$$\limsup_{y\to\infty}\int_{y_1}^{\infty}\left[kq(s)A_3(\sigma(s)) - \frac{A_2(s)}{4m_1(s)A_3(s)}\right]ds = \infty. \tag{20}$$

Then every solution of Equation (1) is oscillatory.

Now, we use the comparison method to obtain other oscillation results. It is well known (see [10]) that the differential equation

$$[a(y)(z'(y))]' + q(y)z(\sigma(y)) = 0, \quad y \ge y_0, \tag{21}$$

where $a, q \in C[y_0,\infty)$, $a(y), q(y) > 0$, is nonoscillatory if and only if there exists a number $y \ge y_0$, and a function $v \in C^1[y,\infty)$, satisfying the inequality

$$v'(y) + \alpha a^{-1}(y)v^2(y) + q(y) \le 0, \quad \text{on } [y,\infty).$$

Lemma 1 (see [10])**.** *Let*

$$\int_{y_0}^{\infty}\frac{1}{a(s)}ds = \infty$$

holds, then the condition

$$\liminf_{y\to\infty}\left(\int_{y_0}^{\infty}\frac{1}{a(s)}ds\right)\int_y^{\infty}q(s)\,ds > \frac{1}{4}.$$

Theorem 4. *Let (2) hold. Assume that the equation*

$$[m_1(y)z'(y)]' + \left(\frac{1}{m_2(y)}\int_y^{\infty}\frac{1}{m_3(u)}\int_u^{\infty}\delta_1(s)\,dsdu\right)z(y) = 0, \tag{22}$$

and

$$\left(\frac{m_1(y)}{\pi_3(y)}z'(y)\right)' + \delta_2(y)z(y) = 0, \tag{23}$$

are oscillatory, then every solution of (1) is oscillatory.

Proof. Proceeding as in proof of the Theorem 3. We get (16). If we set $\rho(y) = 1$ in (16), then we obtain

$$\omega'(y) + \frac{1}{m_1(y)}\omega^2(y) - \frac{1}{m_2(y)}\int_y^\infty \frac{1}{m_3(u)}\int_u^\infty \delta_1(s)\,dsdu \le 0.$$

Thus, we can see that Equation (22) is nonoscillatory for every constant $\lambda_1 \in (0, 1)$, which is a contradiction. If we now set $\vartheta(y) = 1$ in (18), then we find

$$\psi'(y) + \frac{A_2(y)}{m_1(y)}\psi^2(y) + \delta_2(y) \le 0.$$

Hence, Equation (23) is nonoscillatory, which is a contradiction.
Theorem 4 is proved. □

In view of Lemma 1, oscillation criteria for (1) of Hille–Nehari-type are easily acquired. Please note that

Corollary 2. *Assume that*

$$\liminf_{y\to\infty} \pi_1(y)\int_y^\infty \frac{1}{m_2(v)}\int_v^\infty \frac{1}{m_3(u)}\int_u^\infty \delta_1(s)\,dsdu > \frac{1}{4},$$

$$\liminf_{y\to\infty}\left(\int_{y_0}^y \frac{A_2(s)}{m_1(s)}ds\right)\int_y^\infty \delta_2(s)\,ds > \frac{1}{4}.$$

Then every solution of (1) is oscillatory.

3. Example

In this section, we give the following example to illustrate our main results.

Example 1. *Let us consider the fourth-order differential equation of type*

$$\left(y^{1\backslash 2}\left(y^{1\backslash 2}z'(y)\right)''\right)' + \frac{1}{y^3}z(\beta y) = 0,\ y \ge 1, \tag{24}$$

where $0 < \beta < 1$ *is a constant. Let*

$$m_3(y) = y^{1\backslash 2},\ m_2(y) = y^{1\backslash 2},\ m_1(y) = 1 > 0,\ q(y) = \frac{1}{y^3},\ \sigma(y) = \beta y,$$

and

$$\pi_i(s) := \int_{y_\circ}^\infty \frac{1}{m_i(s)}ds = \infty.$$

If we now set $k = 1$, It is easy to see that all conditions of Corollary 1 are satisfied.

$$\begin{aligned} A_3(\sigma(s)) &= \int_{\sigma_1(s)}^{\sigma(s)} \frac{1}{m_1(\sigma(s))}A_2(\sigma(s))\,ds \\ &= \int_{\sigma_1(s)}^{\sigma(s)}\left(\int_{\sigma_1(s)}^{\sigma(s)} \frac{1}{m_2(\sigma(s))}\pi_3(\sigma(s))\,ds\right)ds \\ &= \int_{\sigma_1(s)}^{\sigma(s)}\left(\int_{\sigma_1(s)}^{\sigma(s)} \frac{1}{(\beta s)^{1/2}}\left(\int_{\sigma_1(s)}^\infty \frac{1}{(\beta s)^{1/2}}ds\right)ds\right)ds \end{aligned}$$

$$\begin{aligned} A_2(s) &= \int_{s_1}^{s} \frac{1}{m_2(s)} \pi_3(s)\, ds \\ &= \int_{s_1}^{s} \frac{1}{s^{1/2}} \left(\int_{s_1}^{\infty} \frac{1}{s^{1/2}} ds \right) ds \end{aligned}$$

now

$$\begin{aligned} &\limsup_{y\to\infty} \int_{y_1}^{\infty} \left[kq(s) A_3(\sigma(s)) - \frac{A_2(s)}{4m_1(s) A_3(s)} \right] ds \\ &\limsup_{y\to\infty} \int_{y_1}^{\infty} \left[\frac{1}{s^3} \int_{\sigma_1(s)}^{\sigma(s)} \left(\int_{\sigma_1(s)}^{\sigma(s)} \frac{1}{(\beta s)^{1/2}} \left(\int_{\sigma_1(s)}^{\infty} \frac{1}{(\beta s)^{1/2}} ds \right) ds \right) ds \right] ds \\ = \;& \infty \end{aligned}$$

and

$$\limsup_{y\to\infty} \int_{y_1}^{\infty} \left[\frac{\pi_1(\nu)}{m_2(\nu)} \int_{\nu}^{\infty} \frac{1}{m_3(u)} \int_{u}^{\infty} \delta_1(s)\, ds du - \frac{1}{4m_1(\nu)\pi_1(\nu)} \right] d\nu = \infty.$$

Hence, by Corollary 1, every solution of Equation (25) is oscillatory.

Example 2. *Consider a differential equation*

$$\left(y \left(y \left(yz'(y) \right)' \right)' \right)' + yz(y) = 0,\ y \geq 1, \tag{25}$$

We see

$$m_3(y) = m_2(y) = m_1(y) = y > 0,\ q(y) = y,\ \sigma(y) = y,$$

and

$$\pi_i(s) := \int_{y_\circ}^{\infty} \frac{1}{m_i(s)} ds = \infty.$$

If we now set $k = 1$, It is easy to see that all conditions of Corollary 1 are satisfied.

$$\limsup_{y\to\infty} \int_{y_1}^{\infty} \left[\frac{\pi_1(\nu)}{m_2(\nu)} \int_{\nu}^{\infty} \frac{1}{m_3(u)} \int_{u}^{\infty} \delta_1(s)\, ds du - \frac{1}{4m_1(\nu)\pi_1(\nu)} \right] d\nu = \infty,$$

$$\limsup_{y\to\infty} \int_{y_1}^{\infty} \left[kq(s) A_3(\sigma(s)) - \frac{A_2(s)}{4m_1(s) A_3(s)} \right] ds = \infty.$$

Hence, by Corollary 1, every solution of Equation (25) is oscillatory.

4. Conclusions

The results of this paper are presented in a form which is essentially new and of high degree of generality. To the best of our knowledge, there is nothing known about the oscillation of (1) under the assumption (2), our primary goal is to fill this gap by presenting simple criteria for the oscillation of all solutions of (1) by using the generalized Riccati transformations and comparison technique, so the main advantage of studying (1) essentially lies in the direct application of the well-known Kiguradze lemma [23] (Lemma 1). Further, we can consider the case of $\sigma(y) \geq y$ in the future work.

Author Contributions: The authors claim to have contributed equally and significantly in this paper. All authors read and approved the final manuscript.

Funding: The authors declare that they have not received funds from any institution.

Acknowledgments: The authors thank the referee for carefully reading the manuscript and suggesting very useful comments which improve the content of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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