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Identities of Symmetry for Type 2 Bernoulli and Euler Polynomials

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Abstract: The main purpose of this paper is to give several identities of symmetry for type 2 Bernoulli and Euler polynomials by considering certain quotients of bosonic p-adic and fermionic p-adic integrals on $\mathbb{Z}_p$, where p is an odd prime number. Indeed, they are symmetric identities involving type 2 Bernoulli polynomials and power sums of consecutive odd positive integers, and the ones involving type 2 Euler polynomials and alternating power sums of odd positive integers. Furthermore, we consider two random variables created from random variables having Laplace distributions and show their moments are given in terms of the type 2 Bernoulli and Euler numbers.

Keywords: type 2 Bernoulli polynomials; type 2 Euler polynomials; identities of symmetry; Laplace distribution

1. Introduction

In this section, we are going to review some known results. We first recall the definitions of Bernoulli and Euler polynomials together with their type 2 polynomials. Then, we introduce the bosonic p-adic integrals and the fermionic p-adic integrals on $\mathbb{Z}_p$ that we need for the derivation of an identity of symmetry. As is well known, the Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1}$$

(see [1,2]).

In particular, the Bernoulli numbers are the constant terms $B_n=B_n(0)$ of the Bernoulli polynomials. By making use of (1), we can deduce that

$$\sum_{l=0}^{n-1}l^k=\frac{1}{k+1}\left(B_{k+1}(n)-B_{k+1}\right),\ \text{for } k=0,1,2,\cdots. \tag{2}$$

The type 2 Bernoulli polynomials are defined by generating function

$$\frac{t}{e^t-e^{-t}}e^{xt}=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!}, \tag{3}$$

(see [3,4]).

In particular, $b_n = b_n(0)$ are called type 2 Bernoulli numbers. From (3), it can be seen that

$$b_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^{n-k}, \tag{4}$$

(see [3,4]).

Analogously to (2), we observe that

$$\begin{aligned} \sum_{l=0}^{n-1} e^{(2l+1)t} &= \frac{1}{e^t - e^{-t}}(e^{2nt} - 1) \\ &= \sum_{k=0}^{\infty} \left(\frac{b_{k+1}(2n) - b_{k+1}}{k+1} \right) \frac{t^k}{k!}. \end{aligned} \tag{5}$$

Thus, by (5), we get

$$\sum_{l=0}^{n-1} (2l+1)^k = \frac{1}{k+1}(b_{k+1}(2n) - b_{k+1}), \quad k = 0, 1, 2, \cdots. \tag{6}$$

Let p be a fixed odd prime number. Throughout this paper, we will use the notations $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$, and $\mathbb{C}$ to denote the ring of p-adic rational integers, the field of p-adic rational numbers, the completion of an algebraic closure of $\mathbb{Q}_p$, and the field of complex numbers, respectively. The normalized valuation in $\mathbb{C}_p$ is denoted by $|\cdot|_p$, with $|p|_p = \frac{1}{p}$. For a uniformly differentiable function f on $\mathbb{Z}_p$, the bosonic p-adic integral on $\mathbb{Z}_p$ (or p-adic invariant integral on $\mathbb{Z}_p$) is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_0(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_0(x + p^N\mathbb{Z}_p) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x). \tag{7}$$

Then, by (7), we easily get

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_0(x) - \int_{\mathbb{Z}_p} f(x) d\mu_0(x) = f'(0), \tag{8}$$

(see [5,6]).

The fermionic integral on $\mathbb{Z}_p$ is defined by Kim [6] as

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x + p^N\mathbb{Z}_p) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{9}$$

From (9), we can show that

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0), \tag{10}$$

(see [4,7–10]).

It is well known that the Euler polynomials are defined by

$$\frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n^*(x) \frac{t^n}{n!}. \tag{11}$$

We denote the Euler numbers by $E_n^* = E_n^*(0), \ (n \geq 0)$. Clearly, we have

$$2\sum_{l=0}^{n-1}(-1)^l e^{lt} = \frac{2}{e^t+1}(e^{nt}+1), \quad \text{where } n \equiv 1 \ (\text{mod } 2). \tag{12}$$

From (11) and (12), we obtain that

$$2\sum_{l=0}^{n-1}(-1)^l l^k = E_k^*(n) + E_k^*, \tag{13}$$

where n is a positive odd integer.

Now, we consider the type 2 Euler polynomials which are given by

$$\frac{2}{e^t+e^{-t}}e^{xt} = \operatorname{sech}(t)e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{14}$$

In particular, when $x = 0$, $E_n = E_n(0)$ are called the type 2 Euler numbers.

In this paper, we obtain some identities of symmetry involving the type 2 Bernoulli polynomials, the type 2 Euler polynomials, power sums of odd positive integers and alternating power sums of odd positive integers which are derived from certain quotients of bosonic p-adic and fermionic p-adic integrals on $\mathbb{Z}_p$. In the following section, we will construct two random variables from random variables having Laplace distributions whose moments are closely related to the type 2 Bernoulli and Euler numbers. All the results in Sections 2 and 3 are newly developed. Finally, we note that the results here have applications in such diverse areas as combinatorics, probability, algebra and analysis (see [11–13]).

2. Some Identities of Symmetry for Type 2 Bernoulli and Euler Polynomials

In virtue of (8), we readily see that

$$\frac{1}{2}\int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_0(x) = \frac{t}{e^t - e^{-t}}. \tag{15}$$

Hence, by (15), we get

$$\frac{1}{2}\int_{\mathbb{Z}_p} (2x+1)^n d\mu_0(x) = b_n, \quad (n \geq 0). \tag{16}$$

In addition, it follows from (15) that

$$\frac{1}{2}\int_{\mathbb{Z}_p} e^{(2y+x+1)t}d\mu_0(y) = \frac{t}{e^t - e^{-t}}e^{xt} = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}. \tag{17}$$

Hence, by (17), we get

$$\frac{1}{2}\int_{\mathbb{Z}_p} (2y+x+1)^n d\mu_0(y) = b_n(x), \quad (n \geq 0). \tag{18}$$

Using (15) and (17), one can easily check that

$$\frac{1}{2}\left(\int_{\mathbb{Z}_p} e^{(2x+2n+1)t}d\mu_0(x) - \int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_0(x)\right) = t\sum_{l=0}^{n-1} e^{(2l+1)t}. \tag{19}$$

Next, we let $T_k(n) = \sum_{l=0}^{n}(2l+1)^k,\ (k \in \mathbb{N} \cup \{0\})$. Note that $T_k(n)$ represents the kth power sums of consecutive positive odd integers. By (19), we easily get

$$\int_{\mathbb{Z}_p} e^{(2x+1+2n)t} d\mu_0(x) - \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_0(x) = \frac{2nt \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_0(x)}{\int_{\mathbb{Z}_p} e^{2nxt} d\mu_0(x)}. \tag{20}$$

Let $w_1,\ w_2$ be positive integers. Then, we observe that

$$\begin{aligned} \frac{w_1 \int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_0(x)}{\int_{\mathbb{Z}_p} e^{2w_1xt} d\mu_0(x)} &= \sum_{l=0}^{w_1-1} e^{(2l+1)t} \\ &= \sum_{k-0}^{\infty} \sum_{l=0}^{w_1-1} (2l+1)^k \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} T_k(w_1-1)\frac{t^k}{k!}. \end{aligned} \tag{21}$$

Now, we consider the next quotient of bosonic p-adic integrals on $\mathbb{Z}_p$ from which the identities of symmetry for the type 2 Bernoulli polynomials follow:

$$I(w_1, w_2) = \frac{w_1w_2}{2} \frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} e^{(2w_1x_1+w_1+2w_2x_2+w_2+w_1w_2x)t} d\mu_0(x_1) d\mu_0(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1w_2xt} d\mu_0(x)}. \tag{22}$$

From (22), we have

$$\begin{aligned} I(w_1, w_2) &= \frac{w_2}{2} \int_{\mathbb{Z}_p} e^{(2x_1+w_2x+1)w_1t} d\mu_0(x) \ \frac{w_1 \int_{\mathbb{Z}_p} e^{(2w_2x_2+w_2)t} d\mu_0(x_2)}{\int_{\mathbb{Z}_p} e^{2w_1w_2xt} d\mu_0(x)} \\ &= w_2 \sum_{k=0}^{\infty} b_k(w_2x)\frac{w_1^k}{k!} t^k \ \sum_{l=0}^{\infty} T_l(w_1-1)\frac{w_2^l}{l!} t^l \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} b_k(w_2x) T_{n-k}(w_1-1) w_1^k w_2^{n-k+1} \frac{t^n}{n!}. \end{aligned} \tag{23}$$

We note from (22) that $I(w_1, w_2) = I(w_2, w_1)$. Interchanging w_1 and w_2, we get

$$I(w_2, w_1) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} b_k(w_1x) T_{n-k}(w_2-1) w_2^k w_1^{n-k+1} \frac{t^n}{n!}. \tag{24}$$

Therefore, by (23) and (24), we obtain the following theorem.

Theorem 1. *For $w_1, w_2 \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} b_k(w_2x) T_{n-k}(w_1-1) w_1^k w_2^{n-k+1} = \sum_{k=0}^{n} \binom{n}{k} b_k(w_1x) T_{n-k}(w_2-1) w_2^k w_1^{n-k+1}.$$

Setting $x = 0$ in Theorem 1, we obtain the following corollary.

Corollary 1. *For $w_1, w_2 \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$\sum_{k=0}^{n} \binom{n}{k} b_k T_{n-k}(w_1 - 1) w_1^k w_2^{n-k+1} = \sum_{k=0}^{n} \binom{n}{k} b_k T_{n-k}(w_2 - 1) w_2^k w_1^{n-k+1}.$$

Furthermore, let us take $w_2 = 1$ in Corollary 1. Then, we have

$$\sum_{k=0}^{n} \binom{n}{k} b_k w_1^{n-k+1} = \sum_{k=0}^{n} \binom{n}{k} b_k T_{n-k}(w_1 - 1) w_1^k. \tag{25}$$

Therefore, by (4) and (25), we obtain the following corollary.

Corollary 2. *For $w_1 \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$b_n(w_1) = \sum_{k=0}^{n} \binom{n}{k} b_k T_{n-k}(w_1 - 1) w_1^{k-1} = \sum_{k=0}^{n} \binom{n}{k} b_k w_1^{k-1} \sum_{l=0}^{w_1-1} (2l+1)^{n-k}.$$

From (22), we observe that

$$\begin{aligned} I(w_1, w_2) &= \frac{w_2}{2} e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} e^{2w_1 x_1 t + w_1 t} d\mu_0(x_1) \, \frac{w_1 \int_{\mathbb{Z}_p} e^{(2w_2 x_2 + w_2)t} d\mu_0(x_1)}{\int_{\mathbb{Z}_p} e^{2w_1 w_2 x t} d\mu_0(x)} \\ &= \frac{w_2}{2} e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} e^{(2w_1 x_1 + w_1)t} d\mu_0(x_1) \sum_{l=0}^{w_1-1} e^{(2l+1) w_2 t} \\ &= \frac{w_2}{2} \sum_{l=0}^{w_1-1} \int_{\mathbb{Z}_p} e^{\left(2x_1 + 1 + w_2 x + (2l+1)\frac{w_2}{w_1}\right) w_1 t} d\mu_0(x_1) \\ &= \sum_{n=0}^{\infty} w_2 \sum_{l=0}^{w_1-1} b_n \left(w_2 x + (2l+1) \frac{w_2}{w_1} \right) \frac{w_1^n t^n}{n!}. \end{aligned} \tag{26}$$

By interchanging w_1 and w_2, we obtain the following equation:

$$I(w_2, w_1) = \sum_{n=0}^{\infty} w_1 \sum_{l=0}^{w_2-1} b_n \left(w_1 x + (2l+1) \frac{w_1}{w_2} \right) \frac{w_2^n t^n}{n!}. \tag{27}$$

As $I(w_1, w_2) = I(w_2, w_1)$, the following theorem is immediate from (26) and (27).

Theorem 2. *For $w_1, w_2 \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$w_1^n w_2 \sum_{l=0}^{w_1-1} b_n \left(w_2 x + (2l+1) \frac{w_2}{w_1} \right) = w_2^n w_1 \sum_{l=0}^{w_2-1} b_n \left(w_1 x + (2l+1) \frac{w_1}{w_2} \right).$$

Example 1. *We check the result in Theorem 2 in the case of $n = 2, w_1 = 3$, and $w_2 = 7$. We first note that $b_2(x) = \frac{1}{2}(x^2 - \frac{1}{3})$. This can be obtained from $B_2(x) = x^2 - x + \frac{1}{6}$ and the relation $b_n(x) = 2^{n-1} B_n(\frac{x+1}{2})$ which follows from (1) and (3). Thus, we have to see that*

$$\sum_{l=0}^{2} \left\{ \left(7x + \frac{7}{3}(2l+1)\right)^2 - \frac{1}{3} \right\} = \frac{7}{3} \sum_{l=0}^{6} \left\{ \left(3x + \frac{3}{7}(2l+1)\right)^2 - \frac{1}{3} \right\}. \tag{28}$$

Now, we can easily show that both the left and the right side of (28) *are equal to* $147x^2 + 294x + \frac{1706}{9}$.

Let us take $w_1 = 1$. Then, by Theorem 2, we get

$$w_2 b_n(w_2 x + w_2) = w_2^n \sum_{l=0}^{w_2-1} b_n\left(x + (2l+1)\frac{1}{w_2}\right). \tag{29}$$

Equivalently, by (29), we have

$$b_n(w_2 x + w_2) = w_2^{n-1} \sum_{l=0}^{w_2-1} b_n\left(x + (2l+1)\frac{1}{w_2}\right). \tag{30}$$

Similarly to (13), we observe that

$$2\sum_{l=0}^{n-1}(-1)^l e^{(2l+1)t} = \sum_{k=0}^{\infty}\left(E_k + E_k(2n)\right)\frac{t^k}{k!}, \tag{31}$$

where $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$. Thus, by (31), we get

$$2\sum_{l=0}^{n-1}(-1)^l(2l+1)^k = E_k + E_K(2n), \tag{32}$$

where $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

From (14), we easily note that

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}E_k x^{n-k}, \quad (n \geq 0). \tag{33}$$

By (10), we get

$$\int_{\mathbb{Z}_p} e^{(2y+x+1)t}d\mu_{-1}(y) = \frac{2}{e^t + e^{-t}}e^{xt} = \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{34}$$

Thus, we have

$$\int_{\mathbb{Z}_p}(2y + x + 1)^n d\mu_{-1}(y) = E_n(x), \quad (n \geq 0).$$

The next equation follows immediately from (10):

$$\int_{\mathbb{Z}_p} e^{(2y+2n+1)t}d\mu_{-1}(y) + \int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_{-1}(x) = 2\sum_{l=0}^{n-1}e^{(2l+1)t}(-1)^l, \tag{35}$$

where $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

Now, we let $A_k(n) = \sum_{l=0}^{n}(-1)^l(2l+1)^k$, $(k \in \mathbb{N} \cup \{0\})$. Here we note that $A_k(n)$ is the alternating kth power sums of consecutive odd positive integers. From (35), we have

$$\int_{\mathbb{Z}_p} e^{(2x+2n+1)t}d\mu_{-1}(x) + \int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_{-1}(x) = \frac{2\int_{\mathbb{Z}_p} e^{(2x+1)t}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{2nxt}d\mu_{-1}(x)}. \tag{36}$$

Let a, b be positive integers with $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 2$. Then, by using the fermionic p-adic integral on $\mathbb{Z}_p$, we get

$$\begin{aligned}\frac{2\int_{\mathbb{Z}_p} e^{(2x+1)t} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p} e^{2axt} d\mu_{-1}(x)} =& 2\sum_{l=0}^{a-1} e^{(2l+1)t}(-1)^l \\ =& \sum_{k=0}^{\infty} 2\sum_{l=0}^{a-1}(2l+1)^k(-1)^l \frac{t^k}{k!} \\ =& \sum_{k=0}^{\infty} 2A_k(a-1)\frac{t^k}{k!}.\end{aligned} \tag{37}$$

We now consider the next quotient of the fermionic p-adic integrals on $\mathbb{Z}_p$ from which the identities of symmetry for the type 2 Euler polynomials follow:

$$J(a,b) = \frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} e^{(2ax_1+a+2bx_2+b+abx)t} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2abxt} d\mu_{-1}(x)}. \tag{38}$$

From (38), we can derive the following equation given by

$$\begin{aligned}J(a,b) =& \frac{1}{2}\int_{\mathbb{Z}_p} e^{a(2x_1+1+bx)t} d\mu_{-1}(x_1)\ \frac{2\int_{\mathbb{Z}_p} e^{(2bx_2+b)t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2abxt} d\mu_{-1}(x)} \\ =& \frac{1}{2}\sum_{k=0}^{\infty} E_k(bx)\frac{a^k t^k}{k!}\ 2\sum_{l=0}^{\infty} A_l(a-1)\frac{b^l t^l}{l!} \\ =& \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} E_k(bx) A_{n-k}(a-1) a^k b^{n-k}\frac{t^n}{n!}.\end{aligned} \tag{39}$$

We note from (38) that $J(a,b) = J(b,a)$. Interchanging a and b, we get

$$J(b,a) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} E_k(ax) A_{n-k}(b-1) b^k a^{n-k}\frac{t^n}{n!}. \tag{40}$$

The following theorem is an immediate consequence of (39) and (40).

Theorem 3. *For $n \geq 0$, $a, b \in \mathbb{N}$ with $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 2$, we have*

$$\sum_{k=0}^{n}\binom{n}{k} E_k(bx) A_{n-k}(a-1) a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k} E_k(ax) A_{n-k}(b-1) b^k a^{n-k}.$$

The next corollary is now obtained by setting $x = 0$ in Theorem 3.

Corollary 3. *For $n \geq 0$, $a, b \in \mathbb{N}$, with $a \equiv 1 \pmod 2$ and $b \equiv 1 \pmod 2$, we have*

$$\sum_{k=0}^{n}\binom{n}{k} E_k A_{n-k}(a-1) a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k} E_k A_{n-k}(b-1) b^k a^{n-k}.$$

Taking $b = 1$ in Corollary 3 gives us the following identities.

Corollary 4. *For $n \geq 0$, $a \in \mathbb{N}$ with $a \equiv 1$ (mod 2), we have*

$$\begin{aligned} E_n(a) &= \sum_{k=0}^{n} \binom{n}{k} E_k A_{n-k}(a-1) a^k \\ &= \sum_{k=0}^{n} \binom{n}{k} E_k a^k \sum_{l=0}^{a-1} (-1)^l (2l+1)^{n-k}. \end{aligned}$$

From (38), we have

$$\begin{aligned} J(a,b) =& \frac{e^{abxt}}{2} \int_{\mathbb{Z}_p} e^{(2ax_1+a)t} d\mu_{-1}(x_1) \ \frac{2\int_{\mathbb{Z}_p} e^{(2bx_2+b)t} d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p} e^{2abxt} d\mu_{-1}(x)} \\ =& \frac{e^{abxt}}{2} \int_{\mathbb{Z}_p} e^{(2ax_1+a)t} d\mu_{-1}(x_1) \ 2\sum_{l=0}^{a-1} (-1)^l e^{(2l+1)bt} \\ =& \sum_{l=0}^{a-1} (-1)^l \int_{\mathbb{Z}_p} e^{(2x_1+1+bx+(2l+1)\frac{b}{a})at} d\mu_{-1}(x_1) \\ =& \sum_{n=0}^{\infty} a^n \sum_{l=0}^{a-1} (-1)^l E_n(bx + (2l+1)\frac{b}{a}) \frac{t^n}{n!}, \end{aligned} \tag{41}$$

where $a, b \in \mathbb{N}$ with $a \equiv 1$ (mod 2) and $b \equiv 1$ (mod 2). Interchanging a and b, we get

$$J(b,a) = \sum_{n=0}^{\infty} b^n \sum_{l=0}^{b-1} (-1)^l E_n(ax + (2l+1)\frac{a}{b}) \frac{t^n}{n!}. \tag{42}$$

As $J(a,b) = J(b,a)$, by (41) and (42), we obtain the following theorem.

Theorem 4. *For $n \geq 0$, $a, b \in \mathbb{N}$ with $a \equiv 1$ (mod 2) and $b \equiv 1$ (mod 2), we have*

$$a^n \sum_{l=0}^{a-1} (-1)^l E_n(bx + (2l+1)\frac{b}{a}) = b^n \sum_{l=0}^{b-1} (-1)^l E_n(ax + (2l+1)\frac{a}{b}).$$

Let us take $a = 1$ in Theorem 4. Then, we have

$$E_n(bx + b) = b^n \sum_{l=0}^{b-1} (-1)^l E_n(x + (2l+1)\frac{1}{b}).$$

Example 2. *Here, we illustrate Theorem 2 in the case of $n = 2, a = 7$, and $b = 3$. First, we note that $E_2(x) = x^2 - 1$. This follows from $E_2^*(x) = x^2 - x$ and the relation $E_n(x) = 2^n E_n^*(\frac{x+1}{2})$ that can be deduced from (11) and (14). Here, we need to show that*

$$\sum_{l=0}^{6} (-1)^l \left\{ \left(3x + \frac{3}{7}(2l+1)\right)^2 - 1 \right\} = (\frac{3}{7})^2 \sum_{l=0}^{2} (-1)^l \left\{ \left(7x + \frac{7}{3}(2l+1)\right)^2 - 1 \right\}. \tag{43}$$

Indeed, we can easily check that both the left- and right-hand side of (43) are equal to $9x^2 + 18x + \frac{824}{49}$.

3. Further Remarks

For $s \in \mathbb{C}$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (Re(s) > 1),$$

(see [14–16]).

It is well known that

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} \pi^{2n} B_{2n}, \quad (n \geq 0), \tag{44}$$

(see [14,16]).

By (44), we get

$$\begin{aligned}
z\cot(z) &= z\frac{\cos(z)}{\sin(z)} \\
&= z\frac{\frac{e^{iz}+e^{-iz}}{2}}{\frac{e^{iz}-e^{-iz}}{2i}}, \quad (i=\sqrt{-1}) \\
&= iz\left(1+\frac{2}{e^{2iz}-1}\right) \\
&= iz + \sum_{k=0}^{\infty} B_k \frac{(2iz)^k}{k!} \\
&= 1 + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} 2^{2k} i^{2k} z^{2k} \\
&= 1 - 2\sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k} \\
&= 1 - 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \frac{z^{2k}}{(n\pi)^{2k}}\right) \\
&= 1 - 2\sum_{n=1}^{\infty}\left(\frac{z}{n\pi}\right)^2\left(1-\left(\frac{z}{n\pi}\right)^2\right)^{-1}.
\end{aligned} \tag{45}$$

Thus, by (45), we get

$$\cot(z) - \frac{1}{z} = -\sum_{n=1}^{\infty} \frac{2z}{(n\pi)^2}\left(1-\left(\frac{z}{n\pi}\right)^2\right)^{-1}. \tag{46}$$

From (39), we easily note that

$$\frac{d}{dz}\left(\log(\sin(z)) - \log(z)\right) = \sum_{n=1}^{\infty} \frac{d}{dz}\left(\log\left(1-\left(\frac{z}{n\pi}\right)^2\right)\right). \tag{47}$$

By (47), we easily get

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty}\left(1-\left(\frac{z}{n\pi}\right)^2\right). \tag{48}$$

It is not difficult to show that

$$z\cot(z) - 2z\cot(2z) = z\tan(z). \tag{49}$$

From (45) and (49), we have

$$\begin{aligned} z\tan(z) &= z\cot(z) - 2z\cot(2z) \\ &= 2\sum_{n=1}^{\infty}\left(\frac{2z}{n\pi}\right)^2\left(1-\left(\frac{2z}{n\pi}\right)^2\right)^{-1} - 2\sum_{n=1}^{\infty}\left(\frac{z}{n\pi}\right)^2\left(1-\left(\frac{z}{n\pi}\right)^2\right)^{-1}. \end{aligned} \tag{50}$$

By (50), we get

$$\frac{d}{dz}\left(-\log(\cos(z))\right) = -\sum_{n=1}^{\infty}\frac{d}{dz}\left(\log\left(1-\frac{4z^2}{(n\pi)^2}\right)\right) + \sum_{n=1}^{\infty}\frac{d}{dz}\left(\log\left(1-\left(\frac{z}{n\pi}\right)^2\right)\right). \tag{51}$$

Thus, from (51), we have

$$\sec(z) = \prod_{n=1}^{\infty}\left(\frac{1-(\frac{z}{n\pi})^2}{1-(\frac{2z}{n\pi})^2}\right) = \prod_{n=1}^{\infty}\left(1-\left(\frac{2z}{(2n-1)\pi}\right)^2\right)^{-1}, \tag{52}$$

which is equivalent to

$$\cos(z) = \prod_{n=1}^{\infty}\left(1-\left(\frac{2z}{(2n-1)\pi}\right)^2\right). \tag{53}$$

A random variable has the Laplace distribution with positive parameter μ and b if its probability density function is

$$f(x|\mu,b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right), \tag{54}$$

(see [17]).

The shorthand notation $X \sim \text{Laplace}(\mu, b)$ is used to indicate that the random variable X has the Laplace distribution with positive parameters μ and b. If $\mu = 0$ and $b = 1$, the positive half-time is exactly an exponential scaled by $\frac{1}{2}$.

We assume that the independent random variables $X_1, X_2, X_3, \cdots$ have the Laplace distribution with parameters 0 and 1, (i.e., $X_k \sim \text{Laplace}(0,1)$, $k \in \mathbb{N}$). Let us put

$$Y = \sum_{k=1}^{\infty}\frac{X_k}{(2k-1)\pi}. \tag{55}$$

Then, the characteristic function of Y is given by

$$\begin{aligned} \sum_{n=0}^{\infty}E[Y^n]\frac{(2it)^n}{n!} &= E\Big[\sum_{n=0}^{\infty}Y^n\frac{(2it)^n}{n!}\Big] \\ &= E[e^{2iYt}] \\ &= E\Big[e^{(\sum_{k=1}^{\infty}\frac{X_k}{(2k-1)\pi})2it}\Big] \\ &= \prod_{k=1}^{\infty}E\Big[e^{\frac{X_k}{(2k-1)\pi}2it}\Big]. \end{aligned} \tag{56}$$

Now, we observe that

$$\begin{aligned}E\Big[e^{\frac{X_k}{(2k-1)\pi}2it}\Big] &= \int_{-\infty}^{\infty}\frac{1}{2}e^{\left(\frac{2it}{(2k-1)\pi}\right)x}e^{-|x|}dx \\ &= \frac{1}{2}\int_{-\infty}^{0}e^{\left(\frac{2it}{(2k-1)\pi}\right)x}e^{x}dx+\frac{1}{2}\int_{0}^{\infty}e^{\left(\frac{2it}{(2k-1)\pi}\right)x}e^{-x}dx \\ &= \frac{1}{2}\frac{1}{1+\frac{2it}{(2k-1)\pi}}+\frac{1}{2}\frac{1}{1-\frac{2it}{(2k-1)\pi}} \\ &= \left(1+\left(\frac{2t}{(2k-1)\pi}\right)^2\right)^{-1}.\end{aligned} \tag{57}$$

By (53), (56) and (57), we get

$$\begin{aligned}\sum_{n=0}^{\infty}E[Y^n]\frac{(2it)^n}{n!} &= \prod_{k=1}^{\infty}E\Big[e^{\left(\frac{X_k}{(2k-1)\pi}\right)2it}\Big] \\ &= \prod_{k=1}^{\infty}\left(1+\left(\frac{2t}{(2k-1)\pi}\right)^2\right)^{-1} \\ &= \frac{2}{e^t+e^{-t}} \\ &= \sum_{n=0}^{\infty}E_n\frac{t^n}{n!}.\end{aligned} \tag{58}$$

Therefore, by comparing the coefficients on both sides of (58), we get

$$2^n i^n E[Y^n] = E_n, \quad (n \geq 0). \tag{59}$$

Now, we assume that

$$Z = \sum_{k=1}^{\infty}\frac{X_k}{2k\pi}. \tag{60}$$

Then, the characteristic function of Z is given by

$$\begin{aligned}\sum_{n=0}^{\infty}E[Z^n]\frac{(it)^n}{n!} &= E\Big[\sum_{n=0}^{\infty}Z^n\frac{(it)^n}{n!}\Big] \\ &= E[e^{Zit}] \\ &= E\Big[e^{\sum_{k=1}^{\infty}\left(\frac{X_k}{2k\pi}\right)it}\Big] \\ &= \prod_{k=1}^{\infty}E\Big[e^{\left(\frac{X_k}{2k\pi}\right)it}\Big].\end{aligned} \tag{61}$$

Now, we note that

$$
\begin{aligned}
E\left[e^{\left(\frac{X_k}{2k\pi}\right)it}\right] &= \frac{1}{2}\int_{-\infty}^{\infty} e^{\left(\frac{it}{2k\pi}\right)x} e^{-|x|}dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{\left(\frac{it}{2k\pi}\right)x} e^{x}dx + \frac{1}{2}\int_{0}^{\infty} e^{\left(\frac{it}{2k\pi}\right)x} e^{-x}dx \\
&= \frac{1}{2}\left(\frac{1}{1+\frac{it}{2k\pi}}\right) + \frac{1}{2}\left(\frac{1}{1-\frac{it}{2k\pi}}\right) \\
&= \frac{1}{1+\left(\frac{t}{k\pi}\right)^2}.
\end{aligned}
\tag{62}
$$

From (61) and (62), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} E[Z^n]\frac{(it)^n}{n!} &= \prod_{k=1}^{\infty} E\left[e^{\left(\frac{X_k}{2k\pi}\right)it}\right] \\
&= \prod_{k=1}^{\infty}\left(1+\left(\frac{t}{2k\pi}\right)^2\right)^{-1}.
\end{aligned}
\tag{63}
$$

On the other hand, by (48), we get

$$
\prod_{n=1}^{\infty}\left(1+\left(\frac{t}{n\pi}\right)^2\right)^{-1} = \frac{it}{\sin(it)} = \frac{2t}{e^t - e^{-t}}. \tag{64}
$$

By replacing t by $\frac{t}{2}$, we have

$$
\begin{aligned}
\prod_{n=1}^{\infty}\left(1+\left(\frac{t}{2n\pi}\right)^2\right)^{-1} &= \frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n-1} b_n \frac{t^n}{n!}.
\end{aligned}
\tag{65}
$$

Therefore, by (63) and (65), we obtain the following equation

$$
i^n E[Z^n] = \left(\frac{1}{2}\right)^{n-1} b_n, \quad (n \geq 0). \tag{66}
$$

4. Conclusions

In this paper, we obtained several identities of symmetry for the type 2 Bernoulli and Euler polynomials (see Theorems 1–4). Indeed, they are symmetric identities involving type 2 Bernoulli polynomials and power sums of consecutive odd positive integers, and the ones involving type 2 Euler polynomials and alternating power sums of odd positive integers. For the derivation of those identities, we introduced certain quotients of bosonic p-adic and fermionic p-adic integrals on $\mathbb{Z}_p$, which have built-in symmetries. We note that this idea of using certain quotients of p-adic integrals has produced abundant symmetric identities (see [5,7,8,18–21] and references therein).

We emphasize here that, even though there have been many results on symmetric identities relating to some special numbers and polynomials, this paper is the first one that deals with symmetric identities involving type 2 Bernoulli polynomials, type 2 Euler polynomials, power sums of odd positive integers and alternating power sums of odd positive integers.

In [22,23], we derived some identities involving special numbers and moments of random variables by using the generating functions of the moments of certain random variables. The related special numbers are Stirling numbers of the first and second kinds, degenerate Stirling numbers of the first and second kinds, derangement numbers, higher-order Bernoulli numbers and Bernoulli numbers of the second kind.

In this paper, we considered two random variables created from random variables having Laplace distributions and showed that their moments are closely connected with the type 2 Bernoulli and Euler numbers. Again, this is the first paper that interprets the type 2 Bernoulli and Euler numbers as the moments of certain random variables.

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References

1. Kim, Y.-H.; Hwang, K.-W. Symmetry of power sum and twisted Bernoulli polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2009**, *18*, 127–133.
2. Simsek, Y. Identities on the Changhee numbers and Apostol-type Daehee polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2017**, *27*, 199–212.
3. Jang, G.-W.; Kim, T. A note on type 2 degenerate Euler and Bernoulli polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2019**, *29*, 147–159.
4. Kim, T.; Kim, D.S. A note on type 2 Changhee and Daehee polynomials. *RACSAM* **2019**. [CrossRef]
5. Kim, D.S.; Lee, N.; Na, J.; Park, K.H. Abundant symmetry for higher-order Bernoulli polynomials (I). *Adv. Stud. Contemp. Math. (Kyungshang)* **2013**, *23*, 461–482.
6. Kim, T. q-Volkenborn integration. *Russ. J. Math. Phys.* **2002**, *9*, 288–299.
7. Kim, D.S. Symmetry identities for generalized twisted Euler polynomials twisted by unramified roots of unity. *Proc. Jangjeon Math. Soc.* **2012**, *15*, 303–316.
8. Kim, D.S.; Lee, N.; Na, J.; Park, K.H. Identities of symmetry for higher-order Euler polynomials in three variables (I). *Adv. Stud. Contemp. Math. (Kyungshang)* **2012**, *22*, 51–74. [CrossRef]
9. Kim, T. A Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$. *Russ. J. Math. Phys.* **2009**, *16*, 93–96. [CrossRef]
10. Kim, T. Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$. *Russ. J. Math. Phys.* **2009**, *16*, 484–491. [CrossRef]
11. Bagarello, F.; Trapani, C.; Triolo, S. Representable states on quasilocal quasi *-algebras. *J. Math. Phys.* **2011**, *52*, 013510. [CrossRef]
12. Bongiorno, B.; Trapani, C.; Triolo, S. Extensions of positive linear functionals on a topological *-algebra. *Rocky Mt. J. Math.* **2010**, *40*, 1745–1777. [CrossRef]
13. Trapani, C.; Triolo, S. Representations of modules over a *-algebra and related seminorms. *Stud. Math.* **2008**, *184*, 133–148. [CrossRef]
14. Kim, T. Euler numbers and polynomials associated with zeta functions. *Abstr. Appl. Anal.* **2008**, *52*, 581582. [CrossRef]
15. Andrews, G.E.; Askey, R.; Roy, R. Special functions. In *Encyclopedia of Mathematics and its Applications*; Cambridge University Press: Cambridge, UK, 1999.

16. Whittaker, E.T.; Watson, G.N. *A Course of Modern Analysis. an Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions*; Reprint of the 4th (1927) edition; Cambridge Mathematical Library: Cambridge, UK, 2017.
17. Kotz, S.; Kozubowski, T.J.; Podgórski, K. The Laplace distribution and generalizations. In *A Revisit with Applications to Communications, Economics, Engineering, and Finance*; Birkhäuser Boston, Inc.: Boston, MA, USA, 2001.
18. Kwon, J.; Sohn, G.; Park, J.-W. Symmetric identities for (h, q)-extensions of the generalized higher order modified q-Euler polynomials. *J. Comput. Anal. Appl.* **2018**, *24*, 1431–1438.
19. Kwon, J.; Park, J.-W. A note on symmetric identities for twisted Daehee polynomials. *J. Comput. Anal. Appl.* **2018**, *24*, 636–643.
20. Jeong, J.; Kang, D.-J.; Rim, S.-H. Symmetry identities of Changhee polynomials of type two. *Symmetry* **2018**, *10*, 740. [CrossRef]
21. Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. Symmetric Identities for Fubini Polynomials. *Symmetry* **2018**, *10*, 219. [CrossRef]
22. Kim, T.; Yao, Y.; Kim, D.S.; Kwon, H.-I. Some identities involving special numbers and moments of random variables. *To appear Rocky Mountain J. Math.* Available online: https://projecteuclid.org/euclid.rmjm/1538208042 (accessed on 30 April 2019).
23. Kim, T.; Kim, D.S. Extended Stirling numbers of the first kind associated with Daehee numbers and polynomials. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 1159–1171. [CrossRef]