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Generalized Mittag-Leffler Input Stability of the Fractional Differential Equations

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Abstract: The behavior of the analytical solutions of the fractional differential equation described by the fractional order derivative operators is the main subject in many stability problems. In this paper, we present a new stability notion of the fractional differential equations with exogenous input. Motivated by the success of the applications of the Mittag-Leffler functions in many areas of science and engineering, we present our work here. Applications of Mittag-Leffler functions in certain areas of physical and applied sciences are also very common. During the last two decades, this class of functions has come into prominence after about nine decades of its discovery by a Swedish Mathematician Mittag-Leffler, due to the vast potential of its applications in solving the problems of physical, biological, engineering, and earth sciences, to name just a few. Moreover, we propose the generalized Mittag-Leffler input stability conditions. The left generalized fractional differential equation has been used to help create this new notion. We investigate in depth here the Lyapunov characterizations of the generalized Mittag-Leffler input stability of the fractional differential equation with input.

Keywords: fractional differential equations with input; Mittag-Leffler stability; left generalized fractional derivative; ρ -Laplace transforms

1. Introduction

The behavior of the analytical solutions of the fractional differential equation described by the fractional order derivative operators is the main subject in stability problems [1]. There exist many stability notions introduced in fractional calculus. Some examples are asymptotic stability, global asymptotic uniform stability, synchronization problems, stabilization problems, Mittag-Leffler stability and fractional input stability. In this paper, we extend the Mittag-Leffler input stability in the context of the fractional differential equations described by the left generalized fractional derivative. We note here that the left generalized fractional derivative is the generalization of the Liouville-Caputo fractional derivative and the Riemann-Liouville fractional derivative [2]. There exist many works related to stability problems. In [3], Souahi et al. propose some new Lyapunov characterizations of fractional differential equations described by the conformable fractional derivative. In [4], Sene proposes a new stability notion and introduce the Lyapunov characterization of the conditional asymptotic stability. In [5,6], Sene proposes some applications of the fractional input stability to the electrical circuits described the Liouville-Caputo fractional derivative and the Riemann-Liouville fractional derivative. In [7], Li et al. introduce the Mittag-Leffler stability of the fractional differential equations described by the Liouville-Caputo fractional derivative [8]. In [9], Song et al. analyze the stability of the fractional differential equations with time variable impulses. In [10], Tuan et al. propose a novel

methodology for studying the stability of the fractional differential equations using the Lyapunov direct method. In [11], Makhlof studies the stability with respect to part of the variables of nonlinear Caputo fractional differential equations. In [12], Alidousti et al. propose a new stability analysis of the fractional differential equation described by the Liouville–Caputo fractional derivative. Many other works related to the stability analysis exist in literature, we direct our readers to the References section for more related literature.

The generalized Mittag-Leffler input stability is a new stability notion. This new stability notion studies the behavior of the analytical solution of the fractional differential equations with exogenous input described by the left generalized fractional derivative [13]. We know from previous work in stability problems, it is not trivial to get analytical solutions. The issue is to propose a method to analyze the stability of the fractional differential equations with exogenous input. Classically, the most popular method is the Lyapunov direct method as given in [14–18]. We propose the Lyapunov characterization of the generalized Mittag-Leffler input stability here in this work. As we will be able to show, the generalized Mittag-Leffler input stability generates three properties:

- the converging-input converging-state
- the bounded-input bounded-state
- the uniform global asymptotic stability of the trivial solution of the unforced fractional differential equation (fractional differential equation without exogenous input).

We note here that the fractional differential equation with exogenous input is said to be generalized Mittag-Leffler input stable when the Euclidean norm of its solution is bounded, by a generalized Mittag-Leffler function, plus a quantity which is proportional to the exogenous input bounded when the input is bounded and converging when the input converges in time. The fractional input stability and its consequences are a good compromise in stability problems of the fractional differential equations described by the fractional order derivative operators.

We organize the rest of the paper as follows. In Section 2, we recall the definition of the fractional derivative operators with or without singular kernels. In Section 3, we propose our motivations for studying the generalized Mittag-Leffler input stability. In Section 4, we give the Lyapunov characterizations for the generalized Mittag-Leffler input stability of the fractional differential equations with exogenous inputs. In Section 5, we provide numerical examples for illustrating the main results of this paper. Finally, we finish with some concluding remarks in Section 6.

2. Background on Fractional Derivatives

Let us first recall the fractional derivative operators and the comparison functions [19]. We will use them throughout this paper. There exist many fractional derivative operators in fractional calculus. There exist two types of fractional derivative operators. The first is fractional derivatives with singular kernels and the second is fractional derivatives without singular kernels. With regards to fractional derivatives with singular kernels, we cite the Riemann–Liouville fractional derivative [2], the Liouville–Caputo fractional derivative [2], the Hilfer fractional derivative [20], the Hadamard fractional derivative [2], and Erdélyi–Kober fractional derivative [21]. We note here that all previous fractional derivatives are associated to their fractional integrals [2,20]. As fractional derivatives without singular kernels we cite the Atangana–Baleanu–Liouville–Caputo derivative [22], the Caputo–Fabrizio fractional derivative [23], and the Prabhakar fractional derivative [24]. We note here that all previous fractional derivatives are associated to their fractional integrals [21–24]. Recently, the generalization of the Riemann–Liouville and the Liouville–Caputo fractional derivative were introduced in the literature by Udit [25]. Namely, the generalized fractional derivative and the Liouville–Caputo generalized fractional derivative. Let us now observe the comparison functions used in this paper.

Definition 1. The class \mathcal{PD} function denotes the set of all continuous functions $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\alpha(0) = 0$, and $\alpha(s) > 0$ for all $s > 0$. A class \mathcal{K} function is an increasing \mathcal{PD} function. The class \mathcal{K}_{∞} represents the set of all unbounded \mathcal{K} functions [17].

Definition 2. A continuous function $\beta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class \mathcal{L} if β is non-increasing and tends to zero as its arguments tend to infinity [17].

Definition 3. Let the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Liouville-Caputo derivative of the function f of order α is expressed in the form

$$D_{\alpha}^c f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^{\alpha}} ds, \quad (1)$$

for all $t > 0$, where the order $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function [2,26–29].

Definition 4. Let the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Riemann-Liouville derivative of the function f of order α is expressed in the form

$$D_{\alpha}^{RL} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^{\alpha}} ds, \quad (2)$$

for all $t > 0$, where the order $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function [2,26–30].

Definition 5. Let the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Liouville-Caputo generalized derivative of the function f of order α is expressed in the form

$$\left(D_c^{\alpha, \rho} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{-\alpha} f'(s) \frac{ds}{s^{1-\rho}}, \quad (3)$$

for all $t > 0$, where the order $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function [2,26,28,29,31].

Definition 6. Let the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the left generalized derivative of the function f of order α is expressed in the form

$$\left(D^{\alpha, \rho} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_0^t \left(\frac{t^{\rho} - s^{\rho}}{\rho}\right)^{-\alpha} f(s) \frac{ds}{s^{1-\rho}}, \quad (4)$$

for all $t > 0$, where the order $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function [2,26,28,29,31].

Definition 7. Let us take the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Caputo-Fabrizio fractional derivative of the function f of order α is expressed in the form

$$D_{\alpha}^{CF} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(s) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds, \quad (5)$$

for all $t > 0$, where the order $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function [22].

Definition 8. Let the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Caputo-Fabrizio fractional derivative of the function f of order α is expressed in the form

$$D_{\alpha}^{ABC} f(t) = \frac{AB(\alpha)}{1-\alpha} \int_0^t f'(s) E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds, \quad (6)$$

for all $t > 0$, where the order $\alpha \in (0, 1)$ and $\Gamma(\cdot)$ is the gamma function [22,30].

Definition 9. Let us consider the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Erdélyi-Kober fractional integral of the function f of order $\alpha > 0$, $\eta > 0$ and $\gamma \in \mathbb{R}$ is expressed in the form

$$I_{\eta}^{\gamma, \alpha} f(t) = \frac{t^{-\eta(\gamma+\alpha)}}{\Gamma(\alpha)} \int_0^t \tau^{\eta\gamma} (t^\eta - \tau^\eta)^{\alpha-1} f(\tau) d(\tau^\eta), \quad (7)$$

for all $t > 0$, and $\Gamma(\cdot)$ is the gamma function [21].

Definition 10. Let us consider the function $f : [0, +\infty[\rightarrow \mathbb{R}$, the Erdélyi-Kober fractional derivative of the function f of order $\alpha > 0$, $\eta > 0$ and $\gamma \in \mathbb{R}$ is expressed in the form

$$D_{\eta}^{\gamma, \alpha} f(t) = \prod_{j=1}^n \left(\gamma + j + \frac{1}{\eta} \frac{d}{dt} \right) \left(I_{\eta}^{\gamma+\alpha, n-\mu} f(t) \right), \quad (8)$$

for all $t > 0$, and where $n - 1 < \alpha \leq n$ [21].

Some special cases can be recovered with the above definitions. In Definition 8, when $\rho = 1$, we recover the Liouville-Caputo fractional derivative. In Definition 9, when $\rho = 1$, we recover the Riemann-Liouville fractional derivative. In Definition 10, when $\gamma = -\alpha$ and $\eta = 1$, we obtain the relation existing between Erdélyi-Kober fractional derivative and Riemann-Liouville fractional derivative expressed in the form $D_1^{-\alpha, \alpha} f(t) = t^{\alpha} D^{\alpha, 1} f(t)$.

The Laplace transform will be used for solving a class of the fractional differential equations. The ρ -Laplace transform was recently introduced by Fahd et al. in order to solve differential equations in the frame of conformable derivatives to extend the possibility of working in a large class of functions [2]. We encourage readers to refer to [2] for more detailed information about ρ -Laplace transforms and their applications.

The ρ -Laplace transform of the generalized fractional derivative in the Liouville-Caputo sense is expressed in the following form

$$\mathcal{L}_{\rho} \left\{ \left(D_c^{\alpha, \rho} f \right) (t) \right\} = s^{\alpha} \mathcal{L}_{\rho} \{ f(t) \} - s^{\alpha-1} f(0), \quad (9)$$

The ρ -Laplace transform of the function f is given in the form

$$\mathcal{L}_{\rho} \{ f(t) \} (s) = \int_0^{\infty} e^{-s \frac{t^{\rho}}{\rho}} f(t) \frac{dt}{t^{1-\rho}}. \quad (10)$$

Definition 11. The Mittag-Leffler function with two parameters is defined as the following series

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (11)$$

where $\alpha > 0$, $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$. The classical exponential function is obtained with $\alpha = \beta = 1$. Here we see that when α and β are strictly positive, the series is convergent [14].

3. New Stability Notion of the Fractional Differential Equations

In this section, we introduce a new stability notion for the fractional differential equation with exogenous input described by the left generalized fractional derivative. Historically, the fractional input stability and the Mittag-Leffler input stability of the fractional differential equation represented by the Liouville-Caputo fractional derivative were stated in previous works [5,18]. Moreover, the idea of a discrete version of fractional derivatives is studied in the seminal work [32]. The Lyapunov characterizations of these new stability notions have been provided in [15,18]. In this section, we extend the Mittag-Leffler input stability involving the left generalized fractional derivative. We provide

some modifications in the structure of the definitions, however the idea is not modified. The new stability notion addressed in this paper is called the generalized Mittag-Leffler input stability. In the literature there exist many stability notions related to the fractional differential equations without exogenous inputs such as the asymptotic stability [7,14], the practical stability [12,33], the Mittag-Leffler stability [7] and many others notions. Let us consider the fractional differential equations with exogenous inputs. In fractional calculus, we have not seen a lot of work related to the stability of the fractional differential equations with inputs. The stabilization problems [3] of the fractional differential equations with exogenous inputs is one of the most popular notion existing in the known literature. The challenge consists of finding possible values of the input under which the trivial solution of the obtained fractional differential equation is asymptotically stable. In this paper, we adopt a new method. Let us consider the fractional differential equation with exogenous input described by the left generalized fractional derivative

$$D^{\alpha,\rho}x = Ax + Bu, \tag{12}$$

where $x \in \mathbb{R}^n$ is a state variable, the matrix $A \in \mathbb{R}^{n \times n}$ satisfies the property $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$, the matrix $B \in \mathbb{R}^{n \times n}$ and $u \in \mathbb{R}^n$ represents the exogenous input. The initial boundary condition is defined by $(I^{1-\alpha,\rho}x)(0) = x_0$. Firstly, we give the analytical solution of the fractional differential equation with exogenous input described by the left generalized fractional derivative defined by Equation (12). Applying the ρ -Laplace transform to both sides of Equation (12), we obtain

$$\begin{aligned} \mathcal{L}_\rho(D^{\alpha,\rho}x(t)) - (I^{1-\alpha,\rho}x)(0) &= A\mathcal{L}_\rho(x(t)) + \mathcal{L}_\rho(Bu) \\ s^\alpha \bar{x}(s) - x_0 &= A\bar{x}(s) + B\bar{u}(s) \\ \bar{x}(s) - x_0(s^\alpha I_n - A)^{-1} &= (s^\alpha I_n - A)^{-1} B\bar{u}(s), \end{aligned} \tag{13}$$

where \bar{x} denotes the Laplace transform of the function x and \bar{u} denotes the Laplace transform of the function u . Applying the inverse of the ρ -Laplace transform to both sides of Equation (13), we obtain

$$\begin{aligned} x(t) &= x_0 \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \\ &+ \int_{t_0}^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) Bu(s) \frac{ds}{s^{1-\rho}}. \end{aligned} \tag{14}$$

Applying the Euclidean norm to both sides of Equation (14), we obtain the following relationship

$$\begin{aligned} \|x(t)\| &\leq \|x_0\| \left\| \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right\| \\ &+ \|B\| \|u\| \int_{t_0}^t \left\| \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \frac{ds}{s^{1-\rho}} \right\|. \end{aligned} \tag{15}$$

From assumption $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$, there exist a positive number $M > 0$ [4,18,34] such that, we have

$$\int_{t_0}^t \left\| \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \frac{ds}{s^{1-\rho}} \right\| \leq M. \tag{16}$$

This inequality is a classic condition in stability analysis of fractional derivatives shown in [34]. Finally, the solution of the fractional differential Equation (12) described by the left generalized fractional derivative with exogenous input satisfies the following relationship

$$\|x(t)\| \leq \|x_0\| \left\| \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right\| + \|B\| \|u\| M. \quad (17)$$

We first notice, when the exogenous input of the fractional differential Equation (12) described by the left generalized fractional derivative is null $\|u\| = 0$. The solution obtained in Equation (17) becomes

$$\|x(t)\| \leq \|x_0\| \left\| \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right\|. \quad (18)$$

It corresponds to the classical Mittag-Leffler stability of the trivial solution of the fractional differential equation without input $D^{\alpha,\rho}x = Ax$ described by the left generalized fractional derivative.

Secondly, let us consider the exogenous input converging to zero when t tends to infinity. We know when the identity $|\arg(\lambda(A))| > \frac{\alpha\pi}{2}$ is held, we have

$$E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \rightarrow 0. \quad (19)$$

From which we obtain $\|x(t)\| \rightarrow 0$. Summarizing, we have the following

$$\|u\| \rightarrow 0 \implies \|x(t)\| \rightarrow 0. \quad (20)$$

In other words, a converging input generates a converging state. This property is called the CICS property, derived in [15,18].

Finally, let us consider the exogenous input bounded ($\|u\| \leq \eta$). The solution of the fractional differential Equation (12) described by the left generalized fractional derivative satisfies the following relationship

$$\|x(t)\| \leq \|x_0\| \left\| \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right\| + \|B\| \eta M. \quad (21)$$

Furthermore, we consider the function $\left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \in \mathcal{L}$, thus there exist $\sigma > 0$ such that we have the following relationship

$$\left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(A \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \leq \sigma. \quad (22)$$

Thus Equation (21) can be expressed in the following form

$$\|x(t)\| \leq \|x_0\| \sigma + \|B\| \eta M. \quad (23)$$

Thus, the solution of the fractional differential given in Equation (12) described by the left generalized fractional derivative is bounded as well. A bounded input for Equation (12), we obtain a bounded state for Equation (12). This property is called the BIBS property, created in [15,18]. The objective in this paper is to introduce a new stability notion taking into account a few things; namely the converging input, the converging state, the bounded input bounded state and the generalized Mittag-Leffler stability of the trivial solution of the unforced fractional differential equation. This stability notion we refer to as the generalized Mittag-Leffler input stability. In other words, the fractional differential

equation described by the Left generalized fractional derivative is said generalized Mittag-Leffler stable, when its solution is bounded by a class \mathcal{KL} function (contain a Mittag-Leffler function) plus a class \mathcal{K}_∞ function proportional to the input of the given fractional differential equation. A similar derivation leading to Equation (23) has also recently been applied to the study of fixed-time stability in [35].

In this section, we introduce new stability notion in the context of the fractional differential equations described by the left generalized fractional derivative operator. The fractional differential equation under consideration is expressed in the following form

$$D^{\alpha,\rho}x = f(t, x, u) \quad (24)$$

where the function $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuous locally Lipschitz function, the function $x \in \mathbb{R}^n$ is a state variable, and furthermore the condition $f(t, 0, 0) = 0$ is held. Given an initial condition $x_0 \in \mathbb{R}^n$, the solution of the fractional differential Equation (24) starting at x_0 at time t_0 is represented by $x(\cdot) = x(\cdot, x_0, u)$.

Definition 12. The solution $x = 0$ of the fractional differential equation described by the left generalized fractional derivative defined by Equation (24) is said to be generalized Mittag-Leffler stable if, for any initial condition $\|x_0\|$ and initial time t_0 , its solution satisfies the following condition

$$\|x(t, \|x_0\|)\| \leq \left[m(\|x_0\|) \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(\eta \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right]^a, \quad (25)$$

where $a > 0$, $\eta < 0$ and the function m is locally Lipschitz on a domain contained in \mathbb{R}^n and satisfies $m(0) = 0$ [7,14].

In the following definition, we introduce the definition of the generalized Mittag-Leffler input stability in the context of the fractional differential equation described by the left generalized fractional derivative operator.

Definition 13. The fractional differential equation described by the left generalized fractional derivative defined by Equation (24) is said to be generalized Mittag-Leffler input stable if, there exist a class $\gamma \in \mathcal{K}_\infty$ function such that for any initial condition $\|x_0\|$, its solution satisfies the following condition

$$\|x(t, \|x_0\|)\| \leq \left[m(\|x_0\|) \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha,\alpha} \left(\eta \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^\alpha \right) \right]^a + \gamma(\|u\|_\infty), \quad (26)$$

where $a > 0$ and $\eta < 0$.

From the condition $\gamma \in \mathcal{K}_\infty$, we get $\gamma(0) = 0$. We recover Definition 13. That is, the generalized Mittag-Leffler input stability of the fractional differential given in Equation (24) implies the generalized Mittag-Leffler stability of the trivial solution of the fractional differential equation with no input defined by $D^{\alpha,\rho}x = f(t, x, 0)$. From the fact $\gamma \in \mathcal{K}_\infty$, when the input is bounded implies the function $\gamma(\|u\|_\infty)$ is bounded as well. Thus the state of the fractional differential Equation (24) is bounded too. We thus recover BIBS. From the fact $\gamma \in \mathcal{K}_\infty$, a converging input causes $\gamma(\|u\|_\infty)$ to converge. Thus the state of the fractional differential Equation (24) is converging as well. We thus recover CICS. In conclusion we can say that Definition 12 is well posed.

4. Lyapunov Characterizations of the Generalized Mittag-Leffler input Stability

In this section, we give the Lyapunov characterization of the generalized Mittag-Leffler input stability of the fractional differential equation. We know, it is not always trivial to get the analytical solution of the fractional differential equation with exogenous inputs. An alternative is to propose a method of analyzing the Mittag-Leffler input stability. The method consist of calculating the fractional energy of the fractional differential equation along the trajectories. In other words, we use the Lyapunov direct method.

Theorem 1. *Let us consider that there exists a positive function $V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ continuous and differentiable, and a class \mathcal{K}_∞ function χ_1 and class \mathcal{K} functions χ_2, χ_3 satisfying the following assumptions:*

1. $\|x\|^a \leq V(t, x) \leq \chi_1(\|x\|)$.
2. *If for any $\|x\| \geq \chi_2(\|u\|) \implies D_c^{\alpha, \rho} V(t, x) \leq -\chi_3(\|x\|)$.*

Then the fractional differential Equation (24) described by the left generalized fractional derivative is generalized Mittag-Leffler input stable.

Proof. Summarizing [18], combining Assumption (1) and Assumption (2), we have

$$\begin{aligned} \|x\|^a &\leq V(x, t) \leq \alpha_1 \circ \alpha_2(\|u\|) \\ \|x\| &\leq (\alpha_1 \circ \alpha_2(\|u\|))^{1/a} \\ \|x\| &\leq \gamma(\|u\|), \end{aligned} \quad (27)$$

where the function $\gamma(\|u\|) = (\alpha_1 \circ \alpha_2(\|u\|))^{1/a} \in \mathcal{K}_\infty$.

From Assumption (2), using an exponential form of the Lyapunov function in, there exist positive constant such that, we have

$$\begin{aligned} \|x\| \geq \chi_2(\|u\|) &\implies D_c^{\alpha, \rho} V(t, x) \leq -\chi_3(\|x\|) \\ &\implies D_c^{\alpha, \rho} V(t, x) \leq -\chi_3(\|x\|) \leq -kV(x, t). \end{aligned} \quad (28)$$

It follows from Equation (28), the following inequality

$$\begin{aligned} \|x\|^a &\leq V(t, x) \leq V(\|x_0\|) \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha, \alpha} \left(-k \left(\frac{t^\rho - t_0^\rho}{\rho} \right) \right) \\ \|x\| &\leq \left\{ V(\|x_0\|) \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha, \alpha} \left(-k \left(\frac{t^\rho - t_0^\rho}{\rho} \right) \right) \right\}^{1/a}. \end{aligned} \quad (29)$$

Finally, combining Equations (27) and (29), we obtain

$$\|x\| \leq \max \left\{ \left\{ V(\|x_0\|) \left(\frac{t^\rho - t_0^\rho}{\rho} \right)^{\alpha-1} E_{\alpha, \alpha} \left(-k \left(\frac{t^\rho - t_0^\rho}{\rho} \right) \right) \right\}^{1/a}; \gamma(\|u\|) \right\}. \quad (30)$$

Thus the fractional differential equation defined by Equation (24) is generalized Mittag-Leffler input stable. \square

The second characterization is a consequence of the first theorem. It is more simple to be applied in many cases. We have the following characterization.

Theorem 2. *Let there exist a positive function $V : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ continuous and differentiable, and a class \mathcal{K}_∞ of functions χ_1 and class \mathcal{K}_∞ function γ satisfying the following assumption:*

1. $\|x\|^a \leq V(t, x) \leq \chi_1 (\|x\|)$.
2. $D_c^{\alpha, \rho} V(t, x) \leq -kV(x, t) + \gamma (\|u\|)$.

Then fractional differential Equation (24) described by the left generalized fractional derivative is generalized Mittag-Leffler input stable stable.

Proof. From Assumption (2), we have the following relationships

$$\begin{aligned} D_c^{\alpha, \rho} V(t, x) &\leq -kV(x, t) + \gamma (\|u\|) \\ D_c^{\alpha, \rho} V(t, x) &\leq -(1 - \theta)kV(x, t) - \theta kV(x, t) + \gamma (\|u\|), \end{aligned} \quad (31)$$

where $\theta \in (0, 1)$. We have

$$\begin{aligned} -\theta kV(x, t) + \gamma (\|u\|) \leq 0 &\implies D_c^{\alpha, \rho} V(t, x) \leq -(1 - \theta)kV(x, t) \\ V(x, t) \geq \frac{\gamma (\|u\|)}{\theta k} &\implies D_c^{\alpha, \rho} V(t, x) \leq -(1 - \theta)kV(x, t). \end{aligned} \quad (32)$$

From first assumption, it yields that

$$\theta k \chi_1 (\|x\|) \geq \gamma (\|u\|) \implies D_c^{\alpha, \rho} V(t, x) \leq -(1 - \theta)kV(x, t).$$

Thus the fractional differential equation described by the left generalized fractional derivative is Mittag-Leffler input stable. \square

5. Practical Applications

In this section, we give many practical applications of the Mittag-Leffler input stability of the fractional differential equation described by the generalized fractional derivative using the Lyapunov characterizations.

Let us consider the fractional differential equation described by the left generalized fractional differential equation defined by

$$\begin{cases} D_c^{\alpha, \rho} x_1 = -x_1 + \frac{1}{2}x_2 + \frac{1}{2}u_1 \\ D_c^{\alpha, \rho} x_2 = -x_2 + \frac{1}{2}u_2 \end{cases} \quad (33)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $u = (u_1, u_2) \in \mathbb{R}^2$ represents the exogenous input. Let us take the Lyapunov function defined by $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha, \rho} V(t, x) &= -x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_1u_1 - x_2^2 + \frac{1}{2}x_2u_2 \\ &\leq -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 + \frac{1}{4}\|u\|^2 \\ &\leq -V(x) + \frac{1}{4}\|u\|^2 \end{aligned} \quad (34)$$

Consider $\gamma(\|u\|) = \frac{1}{4}\|u\|^2 \in \mathcal{K}_\infty$. It follows from Theorem 2, the fractional differential equation described by the left generalized fractional derivative given in Equation (33) is Mittag-Leffler input stable. Thus, the origin of the unforced fractional differential equation obtained with $u = (u_1, u_2) = (0, 0)$

$$\begin{cases} D_c^{\alpha, \rho} x_1 = -x_1 + \frac{1}{2}x_2 \\ D_c^{\alpha, \rho} x_2 = -x_2 \end{cases} \quad (35)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, is Mittag-Leffler stable.

Let us consider the fractional differential equation described by the left generalized fractional differential equation defined by

$$\begin{cases} D_c^{\alpha,\rho} x_1 = -x_1 + x_2 + u_1 \\ D_c^{\alpha,\rho} x_2 = -x_2 + u_2 \end{cases}, \quad (36)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $u = (u_1, u_2) \in \mathbb{R}^2$ represents the exogenous input. Let the Lyapunov function defined by $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha,\rho} V(t, x) &= -x_1^2 + x_1 x_2 + x_1 u_1 - x_2^2 + x_2 u_2 \\ &\leq -x_1^2 + \frac{1}{2} x_1^2 + \frac{1}{2} x_1^2 - x_2^2 + \frac{1}{2} x_2^2 + \frac{1}{2} x_2^2 + \|u\|^2 \\ &\leq \|u\|^2. \end{aligned} \quad (37)$$

Let $\gamma(\|u\|) = \frac{1}{4} \|u\|^2 \in \mathcal{K}_\infty$. It follows from Theorem 2, the fractional differential equation described by the left generalized fractional derivative in Equation (35) is bounded as well [36].

Let us consider the electrical RL circuit described by the left generalized fractional differential equation defined by

$$D_c^{\alpha,\rho} x = -\frac{\sigma^{1-\alpha} R}{L} x + u \quad (38)$$

with the initial boundary condition defined by $x(0) = x_0$. The parameter σ is associated with the temporal components in the differential equation. u represents the exogenous input. Let us take the Lyapunov function defined by $V(x) = \frac{1}{2} \|x\|^2$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha,\rho} V(t, x) &= \frac{\sigma^{1-\alpha} R}{L} x^2 + xu \\ &\leq -\frac{\sigma^{1-\alpha} R}{L} \|x\|^2 + \frac{1}{2} \|x\|^2 + \frac{1}{2} \|u\| \\ &\leq -\left(\frac{\sigma^{1-\alpha} R}{L} - \frac{1}{2}\right) \|x\|^2 + \frac{1}{2} \|u\|. \end{aligned} \quad (39)$$

Let us consider $k = \frac{\sigma^{1-\alpha} R}{L} - \frac{1}{2}$ and $\theta \in (0, 1)$. We have the following relationship

$$D_c^{\alpha,\rho} V(t, x) \leq -(1-\theta)k \|x\|^2 + k\theta \|x\|^2 + \frac{1}{2} \|u\| \quad (40)$$

From Theorem 1, if $\|x\| \geq \frac{\|u\|}{2k\theta}$, we have $D_c^{\alpha,\rho} V(t, x) \leq -(1-\theta)k \|x\|^2$. Thus, the electrical RL circuit (36) is Mittag-Leffler input stable form.

Let us consider the fractional differential equation described in [4] by the left generalized fractional differential equation defined by

$$D_c^{\alpha,\rho} x = -x + xu, \quad (41)$$

where $x \in \mathbb{R}^n$ is a state variable. u represents the exogenous input. Let's the Lyapunov function defined by $V(x) = \frac{1}{2} \|x\|^2$. The left generalized fractional derivative of the Lyapunov function along the trajectories is given by

$$\begin{aligned} D_c^{\alpha,\rho} V(t, x) &= -x^2 + x^2 u \\ &\leq -\|x\|^2 + \|x\|^2 \|u\| \\ &\leq -(1 - \|u\|) \|x\|^2. \end{aligned} \quad (42)$$

We can observe, when we pick $\|u\| > 1$, using α -integration, the state x of Equation (42) diverge as t tends to infinity. Then the fractional differential equation is not BIBS. Thus, the fractional differential Equation (41) is not, in general, Mittag-Leffler input stable.

6. Conclusions

In this paper, the Mittag-Leffler input stability has been thoroughly investigated. We have tried to motivate this study with its connection to many real world applications known to use Mittag-Leffler functions. We also address the Lyapunov characterization of the fractional differential equations. In doing so, we have created a further Lyapunov characterization which is more useful. Finally, we give some numerical examples to help illustrate the work that was accomplished in this paper. Analyzing the generalized Mittag-Leffler input stability of the fractional differential equations without decomposing it can be non trivial. The possible issue is to decompose it as a cascade of triangular equations and to find a method to analyze the generalized Mittag-Leffler input stability of the obtained fractional differential equation. In other words, finding the conditions for the generalized Mittag-Leffler input stability of the fractional differential cascade equations will be subject of future works.

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