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Particle Fluctuations in Mesoscopic Bose Systems

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Abstract: Particle fluctuations in mesoscopic Bose systems of arbitrary spatial dimensionality are considered. Both ideal Bose gases and interacting Bose systems are studied in the regions above the Bose–Einstein condensation temperature T_c , as well as below this temperature. The strength of particle fluctuations defines whether the system is stable or not. Stability conditions depend on the spatial dimensionality d and on the confining dimension D of the system. The consideration shows that mesoscopic systems, experiencing Bose–Einstein condensation, are stable when: (i) ideal Bose gas is confined in a rectangular box of spatial dimension $d > 2$ above T_c and in a box of $d > 4$ below T_c ; (ii) ideal Bose gas is confined in a power-law trap of a confining dimension $D > 2$ above T_c and of a confining dimension $D > 4$ below T_c ; (iii) the interacting Bose system is confined in a rectangular box of dimension $d > 2$ above T_c , while below T_c , particle interactions stabilize the Bose-condensed system, making it stable for $d = 3$; (iv) nonlocal interactions diminish the condensation temperature, as compared with the fluctuations in a system with contact interactions.

Keywords: Bose systems; asymptotic symmetry breaking; Bose–Einstein condensation; particle fluctuations; stability of Bose systems

1. Introduction

The theory of Bose systems has recently attracted high attention triggered by experimental studies of cold trapped atoms (see, e.g., the books and review articles [1–19]). Special attention has been paid to the study of particle fluctuations, mainly considering three-dimensional macroscopic Bose systems or harmonically-trapped atoms. The importance of this problem has been emphasized after the appearance of a number of papers claiming the occurrence of thermodynamically-anomalous particle fluctuations in the whole region below the condensation temperature T_c even for equilibrium three-dimensional interacting systems (a list of the papers containing such claims has been summarized in [20]). The origin of the arising fictitious anomalies and the ways of avoiding them have been discussed in detail in reviews [16–18].

It would not be strange if anomalously strong fluctuations would be found at the point of a second-order phase transition. This would be natural, since at the point of a phase transition, the system is unstable and fluctuations in a system can drastically increase. It is exactly the system instability that drives the phase transition and forces the system to transfer to another state. However, as soon as the transition to the other state has happened, the real system becomes stable and has to exhibit thermodynamically normal fluctuations. It is therefore more than strange how thermodynamically-anomalous fluctuations could arise in realistic three-dimensional interacting systems.

Moreover, Bose–Einstein condensation is necessarily accompanied by the spontaneous breaking of global gauge symmetry. From the mathematical point of view, the similar breaking of continuous symmetry occurs under magnetic phase transitions [21]; hence, anomalous fluctuations of the order

parameter should appear in magnets below T_c . However, thermodynamically-anomalous fluctuations imply the system instability [22]. Therefore, if such fluctuations would really arise in the whole range below T_c , then neither superfluids nor magnets would exist. Fortunately, it has been shown [23,24] that thermodynamically-anomalous fluctuations in interacting three-dimensional equilibrium systems, discussed in theoretical papers, are just calculational artifacts caused, briefly speaking, by the use of a second-order approximation for calculating fourth-order terms.

The aim of the present paper is to extend the investigation of particle fluctuations in Bose systems in several aspects: First, we consider mesoscopic systems that are finite, although containing many particles $N \gg 1$. Taking into account a finite number of particles requires modifying the definition of the Bose function by introducing a finite cutoff responsible for the existence of a minimal wave vector prescribed by the system geometry. Second, we analyze particle fluctuations above, as well as below T_c for the Bose systems of arbitrary dimensionality, which allows us to find the critical spatial dimension above which the system is stable. Third, we consider two types of Bose systems, confined either in a rectangular box or in a power-law trap. Fourth, the influence of nonlocal interactions on particle fluctuations is analyzed, as compared to that of local interactions.

Throughout the paper, the system of units is employed where the Planck and Boltzmann constants are set to one, $\hbar = 1$ and $k_B = 1$.

2. Particle Fluctuations and Stability

Here and in what follows, we consider mesoscopic systems that are finite, containing a finite number of particles N , although with this number is rather large, $N \gg 1$.

Observable quantities are given by statistical averages $\langle \hat{A} \rangle$ of Hermitian operators \hat{A} . Fluctuations of the observable quantities are characterized by the variance:

$$\text{var}(\hat{A}) \equiv \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2.$$

The observable is called extensive when:

$$\langle \hat{A} \rangle \propto N \quad (N \gg 1), \quad (1)$$

which is equivalent to the condition:

$$\frac{\langle \hat{A} \rangle}{N} \simeq \text{const} \quad (N \gg 1). \quad (2)$$

Fluctuations are termed thermodynamically normal if the inequalities:

$$0 \leq \frac{\text{var}(\hat{A})}{|\langle \hat{A} \rangle|} < \infty \quad (3)$$

are valid for any N , which can also be represented as the condition:

$$\frac{\text{var}(\hat{A})}{|\langle \hat{A} \rangle|} \simeq \text{const} \quad (N \gg 1). \quad (4)$$

When these conditions do not hold, the fluctuations are called thermodynamically anomalous. Sometimes, instead of the terms thermodynamically normal or thermodynamically anomalous, one says, for short, that fluctuations are just normal or anomalous.

Particle fluctuations, describing the fluctuations of the number of particles, characterized by the number-of-particles operator \hat{N} , are quantified by the relative variance:

$$\frac{\text{var}(\hat{N})}{N} = \frac{1}{N} \left(\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2 \right), \quad (5)$$

where $N = \langle \hat{N} \rangle$. The fluctuations are normal when:

$$0 \leq \frac{\text{var}(\hat{N})}{N} < \infty \quad (6)$$

for any N , or in other words, when:

$$\frac{\text{var}(\hat{N})}{N} \simeq \text{const} \quad (N \gg 1). \quad (7)$$

The strength of particle fluctuations characterizes the system stability, since these fluctuations are directly connected to the isothermal compressibility:

$$\kappa_T \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_{TN} = \frac{1}{\rho N} \left(\frac{\partial N}{\partial \mu} \right)_{TV} \quad (8)$$

by the equality:

$$\kappa_T = \frac{\text{var}(\hat{N})}{\rho T N} \quad \left(\rho \equiv \frac{N}{V} \right), \quad (9)$$

with ρ being the average particle density. The system stability requires that:

$$0 \leq \kappa_T < \infty \quad (10)$$

for any N , which yields Conditions (6) and (7). The above relations give us one of the ways for calculating the relative variance:

$$\frac{\text{var}(\hat{N})}{N} = \rho T \kappa_T = \frac{T}{N} \left(\frac{\partial N}{\partial \mu} \right)_{TV}. \quad (11)$$

3. Ideal Gas in a Rectangular Box

Bose systems in a rectangular box are not merely an interesting object allowing for detailed calculations, but it can also be realized experimentally inside box-shaped traps [25–27].

3.1. Modified Bose Function

The grand Hamiltonian of a gas in a rectangular box of volume V reads as:

$$H = \hat{H} - \mu \hat{N} = \int \psi^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} - \mu \right) \psi(\mathbf{r}) d\mathbf{r}, \quad (12)$$

where the integration is over the given volume V . Assuming periodic continuation of the box, the field operators can be expanded in plane waves,

$$\psi(\mathbf{r}) = \sum_k a_k \varphi_k(\mathbf{r}), \quad \varphi_k(\mathbf{r}) = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{r}}, \quad (13)$$

which gives:

$$H = \sum_k \omega_k a_k^\dagger a_k \quad \left(\omega_k = \frac{k^2}{2m} - \mu \right). \quad (14)$$

The total number of particles is the sum:

$$N = N_0 + N_1, \quad N_1 = \sum_{k \neq k_0} n_k, \quad (15)$$

where N_0 is the number of condensed particles, while N_1 is the number of uncondensed particles, with the momentum distribution:

$$n_k \equiv \langle a_k^\dagger a_k \rangle = \left(e^{\beta \omega_k} - 1 \right)^{-1}. \quad (16)$$

Here, $\beta = 1/T$ is the inverse temperature. For a large number of particles, the sums over momenta can be represented as the integrals,

$$\sum_k n_k \rightarrow V \int n_k \frac{d\mathbf{k}}{(2\pi)^d},$$

where d is spatial dimensionality. In the case of isotropic functions under the integrals, it is possible to pass to spherical coordinates. However, it is necessary to take into account that for a finite system, the values of the wave vectors start not from zero, but from a finite minimal momentum k_0 that can be estimated as:

$$k_0 = \frac{2\pi}{L} = \frac{2\pi}{aN^{1/d}}, \quad (17)$$

with the box volume:

$$V = L^d, \quad L = aN^{1/d},$$

where a is the mean interparticle distance. Thus, the integration over the momenta takes the form:

$$\int \frac{d\mathbf{k}}{(2\pi)^d} \rightarrow \frac{2}{(4\pi)^{d/2} \Gamma(d/2)} \int_{k_0}^{\infty} k^{d-1} dk, \quad (18)$$

where the lower limit is given by the cutoff prescribed by the minimal quantity k_0 . Then, the number of uncondensed particles becomes proportional to the modified Bose function:

$$g_n(z) \equiv \frac{1}{\Gamma(n)} \int_{u_0}^{\infty} \frac{zu^{n-1}}{e^u - z} du, \quad (19)$$

with $z \equiv \exp(\beta\mu)$ being the fugacity and where the lower limit is given by the cutoff:

$$u_0 = \frac{k_0^2}{2mT} = \frac{\varepsilon_0}{T} \quad (20)$$

defined by the minimal energy:

$$\varepsilon_0 = \frac{2\pi^2}{ma^2} N^{-2/d}. \quad (21)$$

Since the minimal energy (21) tends to zero for large N , it is admissible to keep in mind that:

$$u_0 \ll 1 \quad (N \gg 1). \quad (22)$$

In this way, the relative variance (11) can be expressed through the derivative of the modified Bose function (19). The latter differs from the standard Bose function by the existence of a nonzero lower integration limit defined by the minimal wave vector.

3.2. Fluctuations above the Condensation Temperature

At temperatures above the condensation point, there are no condensed particles, so that the total number of particles reads as:

$$N = \frac{V}{\lambda_T^d} g_{d/2}(z) \quad (T \geq T_c), \quad (23)$$

where:

$$\lambda_T \equiv \sqrt{\frac{2\pi}{mT}}$$

is the thermal wavelength. Hence, the relative variance (11) is:

$$\frac{\text{var}(\hat{N})}{N} = \frac{z}{\rho \lambda_T^d} \frac{\partial g_{d/2}(z)}{\partial z} \quad (T > T_c), \quad (24)$$

where $\rho \equiv N/V$ is the particle density.

Estimating the Bose function above T_c , where $z < 1$, we find:

$$g_n(z) = - \frac{z}{(1-z)\Gamma(1+n)} \left[u_0^n - \frac{nu_0^{1+n}}{(1+n)(1-z)} \right] \quad (n < 0, z < 1). \quad (25)$$

In particular,

$$g_{-1/2}(z) = - \frac{z}{\sqrt{\pi}(1-z)} \left(u_0^{-1/2} + \frac{u_0^{1/2}}{1-z} \right) \quad (z < 1) \quad (26)$$

and:

$$g_0(z) = - \frac{z}{1-z} \quad (z < 1). \quad (27)$$

Calculating the derivatives of the modified Bose functions requires being attentive, since some of the derivatives are different from those for the standard Bose functions. Generally, we have:

$$\frac{\partial g_n(z)}{\partial z} = \frac{1}{z} g_{n-1}(z) + \frac{u_0^{n-1}}{\Gamma(n)(1-z+u_0)}. \quad (28)$$

Using the smallness of u_0 , we can write:

$$\frac{u_0^{n-1}}{1-z+u_0} \simeq \frac{1}{1-z} \left(u_0^{n-1} - \frac{u_0^n}{1-z} \right) \quad (z < 1).$$

Therefore, for $n < 1$, we find:

$$\frac{\partial g_n(z)}{\partial z} = - \frac{u_0^n}{(1-z)^2 \Gamma(1+n)} \quad (n < 1, z < 1); \quad (29)$$

while for $n > 1$, keeping the main terms, we get:

$$\frac{\partial g_n(z)}{\partial z} = \frac{1}{z} g_{n-1}(z) \quad (n > 1, z < 1). \quad (30)$$

We shall also need the derivatives:

$$\frac{\partial g_{1/2}(z)}{\partial z} = - \frac{2u_0^{1/2}}{\sqrt{\pi}(1-z)^2} \quad (d = 1, z < 1),$$

$$\frac{\partial g_1(z)}{\partial z} = - \frac{u_0}{(1-z)^2} \quad (d = 2, z < 1),$$

$$\frac{\partial g_{3/2}(z)}{\partial z} = \frac{1}{z} g_{1/2}(z) \quad (d = 3, z < 1).$$

For the relative variance, depending on the space dimensionality, we obtain:

$$\frac{\text{var}(\hat{N})}{N} = - \frac{2z}{\sqrt{\pi}(1-z)^2 \rho \lambda_T^d} \left(\frac{\varepsilon_0}{T} \right)^{1/2} \quad (d = 1, T > T_c),$$

$$\frac{\text{var}(\hat{N})}{N} = - \frac{z}{(1-z)^2 \rho \lambda_T^d} \left(\frac{\varepsilon_0}{T} \right) \quad (d = 2, T > T_c),$$

$$\frac{\text{var}(\hat{N})}{N} = \frac{1}{\rho \lambda_T^3} g_{1/2}(z) \quad (d = 3, T > T_c). \quad (31)$$

The negative values of the variance for $d = 1$ and $d = 2$ show that these low-dimensional systems are unstable. However, the gas is stable in three dimensions.

As follows from the derivative:

$$\frac{\partial g_{d/2}(z)}{\partial z} = \frac{1}{z} g_{(d-2)/2}(z) \quad (d > 2, z < 1), \quad (32)$$

the system is stable for $d > 2$. That is, the critical spatial dimension, above which the uncondensed gas in a rectangular box is stable, is $d_c = 2$, so that the stability condition is:

$$d > d_c = 2 \quad (T > T_c). \quad (33)$$

3.3. Condensation Temperature of a Gas in a Rectangular Box

At the temperature of Bose condensation, the chemical potential becomes zero, $\mu = 0$, because of which $z = 1$. The total number of particles:

$$N = \frac{V}{\lambda_T^d} g_{d/2}(1) \quad (T = T_c) \quad (34)$$

defines the critical temperature:

$$T_c = \frac{2\pi}{m} \left[\frac{\rho}{g_{d/2}(1)} \right]^{2/d}. \quad (35)$$

For different dimensionalities, we have:

$$g_{1/2}(1) = \frac{2}{\sqrt{\pi}} u_0^{-1/2} \quad (d = 1),$$

$$g_1(1) = -\ln u_0 \quad (d = 2),$$

$$g_{3/2}(1) = \zeta(3/2) \quad (d = 3).$$

This gives the critical temperatures for a one-dimensional system:

$$T_c = \frac{\pi\rho}{\sqrt{2m}} \varepsilon_0^{1/2} \quad (d = 1), \quad (36)$$

and for a two-dimensional system:

$$T_c = \frac{2\pi\rho}{m \ln(T_c/\varepsilon_0)} \quad (d = 2). \quad (37)$$

Iterating the latter equation and taking into account that:

$$\frac{T_c}{\varepsilon_0} \ll \exp\left(\frac{2\pi\rho}{m\varepsilon_0}\right), \quad \varepsilon_0 \propto N^{-2/d} \quad (N \gg 1),$$

we obtain:

$$T_c = \frac{2\pi\rho}{m \ln(2\pi\rho/m\varepsilon_0)} \quad (d = 2). \quad (38)$$

For a three-dimensional box, we have the known result:

$$T_c = \frac{2\pi}{m} \left[\frac{\rho}{\zeta(3/2)} \right]^{2/3} \quad (d = 3). \quad (39)$$

The critical temperatures at a large $N \gg 1$ scale as:

$$\begin{aligned} T_c &\propto \frac{1}{N} & (d = 1), \\ T_c &\propto \frac{1}{\ln N} & (d = 2), \\ T_c &\propto \text{const} & (d = 3). \end{aligned} \quad (40)$$

For $d \leq 2$, the critical temperature diminishes to zero as N increases. It remains finite for $d > 2$. Recall that, as is found above, the system is unstable for $d \leq 2$. Thus, the Bose gas in a box is stable in the case where the critical temperature remains finite for large N .

3.4. Fluctuations below Critical Temperature

Below the critical temperature, there appears the Bose–Einstein condensate, so that the number-of-particle operator becomes the sum of the number of condensed particles N_0 and the number-of-particle operator \hat{N}_1 of uncondensed particles,

$$\hat{N} = N_0 + \hat{N}_1 \quad (T < T_c). \quad (41)$$

When the condensate function η is introduced by means of the Bogolubov shift [28–30] of the field operator:

$$\psi(\mathbf{r}) \rightarrow \eta(\mathbf{r}) + \psi_1(\mathbf{r}),$$

then particle fluctuations are defined by the fluctuations of uncondensed particles (see the detailed explanations in [3,9,16–18]),

$$\text{var}(\hat{N}) = \text{var}(\hat{N}_1).$$

The average number of uncondensed particles is:

$$N_1 = \frac{V}{\lambda_T^d} g_{d/2}(1) \quad (T < T_c). \quad (42)$$

Therefore, the relative particle variance reads as:

$$\frac{\text{var}(\hat{N})}{N} = \frac{1}{\rho \lambda_T^d} \lim_{z \rightarrow 1} \frac{\partial g_{d/2}(z)}{\partial z} \quad (T < T_c). \quad (43)$$

For a mesoscopic system, we have:

$$\lim_{z \rightarrow 1} \frac{\partial g_n(z)}{\partial z} = g_{n-1}(1) + \frac{u_0^{n-2}}{\Gamma(n)}. \quad (44)$$

Notice that the last term here would be absent for a macroscopic system. In particular,

$$\lim_{z \rightarrow 1} \frac{\partial g_{1/2}(z)}{\partial z} = g_{-1/2}(1) + \frac{u_0^{-3/2}}{\sqrt{\pi}}.$$

Since:

$$g_{-1/2}(1) = -\frac{1}{3\sqrt{\pi}} u_0^{-3/2},$$

we find:

$$\lim_{z \rightarrow 1} \frac{\partial g_{1/2}(z)}{\partial z} = \frac{2}{3\sqrt{\pi}} u_0^{-3/2}.$$

We also need the limits:

$$\lim_{z \rightarrow 1} \frac{\partial g_1(z)}{\partial z} = \frac{1}{u_0},$$

and:

$$\lim_{z \rightarrow 1} \frac{\partial g_{3/2}(z)}{\partial z} = g_{1/2}(1) + \frac{2}{\sqrt{\pi}} u_0^{-1/2}.$$

Using $g_{1/2}(1)$ from the previous subsection, we get:

$$\lim_{z \rightarrow 1} \frac{\partial g_{3/2}(z)}{\partial z} = \frac{4}{\sqrt{\pi}} u_0^{-1/2}.$$

Generally, from the expression:

$$\lim_{z \rightarrow 1} \frac{\partial g_{d/2}(z)}{\partial z} = g_{(d-2)/2}(1) + \frac{u_0^{(d-4)/2}}{\Gamma(d/2)} \tag{45}$$

we see that the last term here increases with N by a power-law, when $d < 4$, while it increases logarithmically for $d = 4$,

$$\lim_{z \rightarrow 1} \frac{\partial g_2(z)}{\partial z} = 1 - \ln u_0 \quad (d = 4).$$

For the relative variance (43), we obtain:

$$\begin{aligned} \frac{\text{var}(\hat{N})}{N} &= \frac{2}{3\sqrt{\pi}\rho\lambda_T} \left(\frac{T}{\epsilon_0}\right)^{3/2} \quad (d = 1), \\ \frac{\text{var}(\hat{N})}{N} &= \frac{1}{\rho\lambda_T^2} \left(\frac{T}{\epsilon_0}\right) \quad (d = 2), \\ \frac{\text{var}(\hat{N})}{N} &= \frac{4}{\sqrt{\pi}\rho\lambda_T^3} \left(\frac{T}{\epsilon_0}\right)^{1/2} \quad (d = 3), \\ \frac{\text{var}(\hat{N})}{N} &= \frac{1}{\rho\lambda_T^4} \left(\frac{T}{\epsilon_0}\right) \quad (d = 4). \end{aligned} \tag{46}$$

Keeping in mind that $\epsilon_0 \propto N^{-2/d}$, the scaling of these expressions with respect to N is as follows:

$$\begin{aligned} \frac{\text{var}(\hat{N})}{N} &\propto N^3 \quad (d = 1), \\ \frac{\text{var}(\hat{N})}{N} &\propto N \quad (d = 2), \\ \frac{\text{var}(\hat{N})}{N} &\propto N^{1/3} \quad (d = 3), \\ \frac{\text{var}(\hat{N})}{N} &\propto \ln N \quad (d = 4). \end{aligned} \tag{47}$$

This shows that for all dimensions below and including four, particle fluctuations are anomalous, corresponding to an unstable systems. In that sense, the dimension four is critical, implying that the stability condition for a condensed gas in a box is:

$$d > d_c = 4 \quad (T < T_c). \tag{48}$$

4. Ideal Gas in a Power-Law Trap

Power-law traps are the most often used devices for trapping particles. Here, we study particle fluctuations and the related stability of mesoscopic clouds in such traps.

4.1. Modified Semiclassical Approximation

The general form of confining potentials, employed in power-law traps, can be represented as:

$$U(\mathbf{r}) = \sum_{\alpha=1}^d \frac{\omega_{\alpha}}{2} \left| \frac{r_{\alpha}}{l_{\alpha}} \right|^{n_{\alpha}}, \quad (49)$$

where:

$$l_{\alpha} \equiv \frac{1}{\sqrt{m\omega_{\alpha}}}$$

is the effective trap radius in the α direction. As a whole, a trap can be characterized by the effective trap frequency ω_0 and effective length l_0 connected by the relations:

$$\omega_0 \equiv \left(\prod_{\alpha=1}^d \omega_{\alpha} \right)^{1/d} = \frac{1}{ml_0^2}, \quad l_0 \equiv \left(\prod_{\alpha=1}^d l_{\alpha} \right)^{1/d} = \frac{1}{\sqrt{m\omega_0}}. \quad (50)$$

In the limit $n_{\alpha} \rightarrow \infty$, we return to a rectangular box.

When the effective trap frequency is much lower than temperature,

$$\frac{\omega_0}{T} \ll 1, \quad (51)$$

it is possible to resort to the semiclassical approximation that, however, needs to be modified for considering mesoscopic systems [18,31].

In the semiclassical approximation, one defines the density of states:

$$\rho(\varepsilon) = \frac{(2m)^{d/2}}{(4\pi)^{d/2}\Gamma(d/2)} \int_{\mathbb{V}_{\varepsilon}} [\varepsilon - U(\mathbf{r})]^{d/2-1} d\mathbf{r},$$

in which:

$$\mathbb{V}_{\varepsilon} \equiv \{\mathbf{r} : U(\mathbf{r}) \leq \varepsilon\}$$

is the volume available for particle motion.

For trapped particles, an important notion is the confining dimension [18,31]:

$$D \equiv d + \sum_{\alpha=1}^d \frac{2}{n_{\alpha}}. \quad (52)$$

The density of states for the power-law potential (49) reduces to:

$$\rho(\varepsilon) = \frac{\varepsilon^{D/2-1}}{\gamma_D \Gamma(D/2)}, \quad (53)$$

where we use the notation:

$$\gamma_D \equiv \frac{\pi^{d/2}}{2^{D/2}} \prod_{\alpha=1}^d \frac{\omega_{\alpha}^{1/2+1/n_{\alpha}}}{\Gamma(1+1/n_{\alpha})}.$$

In the normal state above T_c , the number of particles is given by the formula:

$$N = \frac{T^{D/2}}{\gamma_D} g_{D/2}(z) \quad (T \geq T_c). \quad (54)$$

We again meet the Bose function that has to be modified according to the definition (19) by using the integral cutoff:

$$u_0 = \frac{\varepsilon_0}{T} \quad (\varepsilon_0 \sim \omega_0), \quad (55)$$

with ε_0 being the lowest energy level in the trap, which is of the order of ω_0 .

4.2. Condensation Temperature of a Gas in a Power-Law Trap

At the critical temperature T_c , we have $\mu = 0$ and $z = 1$. Then, Equation (54) yields:

$$T_c = \left[\frac{\gamma_D N}{g_{D/2}(1)} \right]^{2/D}. \quad (56)$$

The modified Bose function, depending on the confining dimension, takes the forms:

$$g_{D/2}(1) = \frac{2}{(2-D)\Gamma(D/2)} \left(\frac{T}{\varepsilon_0} \right)^{1-D/2} \quad (D < 2),$$

$$g_1(1) = \ln \frac{T}{\varepsilon_0} \quad (D = 2),$$

$$g_{D/2}(1) = \zeta \left(\frac{D}{2} \right) \quad (D > 2).$$

If $D < 2$, the spatial dimension can only be $d = 1$, when:

$$\gamma_D = \frac{\sqrt{\pi}}{\Gamma(D)} \left(\frac{\omega_0}{2} \right)^{D/2} \quad (d = 1).$$

Then, the critical temperature is:

$$T_c = \frac{\sqrt{\pi}}{\Gamma(D)} \left(1 - \frac{D}{2} \right) \Gamma \left(\frac{D}{2} \right) \left(\frac{\omega_0}{2\varepsilon_0} \right)^{D/2} N\varepsilon_0 \quad (D < 2, d = 1). \quad (57)$$

The confining dimension equals two, $D = 2$, when $d = 1$ and $n = 2$, so that:

$$\gamma_2 = \omega_0 \quad (D = 2, d = 1, n = 2).$$

This yields the critical temperature:

$$T_c = \frac{N\omega_0}{\ln(T_c/\varepsilon_0)} \quad (D = 2, d = 1). \quad (58)$$

For large N , one has:

$$\frac{T_c}{\varepsilon_0} \ll \exp \left(\frac{\omega_0}{\varepsilon_0} N \right),$$

since for a one-dimensional harmonic oscillator, $\varepsilon_0 = \omega_0/2$. Because of this, the critical temperature (58) can be simplified to:

$$T_c = \frac{N\omega_0}{\ln(2N)}. \quad (59)$$

For the confining dimension larger than two, the critical temperature is:

$$T_c = \left[\frac{\gamma_D N}{\zeta(D/2)} \right]^{2/D} \quad (D > 2). \quad (60)$$

In the case of harmonic traps, when $n_\alpha = 2$, hence $D = 2d$ and $\gamma_D = \omega_0^d$, the critical temperature becomes:

$$T_c = \left[\frac{N}{\zeta(d)} \right]^{1/d} \omega_0 \quad (n_\alpha = 2, d > 1).$$

4.3. Scaling with Respect to the Particle Number

As is explained in Section 2, extensive observables are proportional to the number of particles N , when this number is large. This definition prescribes the scaling of the system characteristics with respect to N . As a representative of an observable quantity, we may take, e.g., internal energy:

$$\langle \hat{H} \rangle = \langle H \rangle + \mu N. \quad (61)$$

This is an extensive quantity satisfying the condition:

$$\frac{\langle \hat{H} \rangle}{N} \simeq \text{const} \quad (N \gg 1). \quad (62)$$

For the considered case of a gas in a power-law trap, we have:

$$\frac{\langle \hat{H} \rangle}{N} = \frac{D g_{1+D/2}(z)}{2N\gamma_D} T^{1+D/2}. \quad (63)$$

The function $g_{1+D/2}(z)$ is finite for all $D > 0$ and all z . Hence, the condition (62) implies:

$$N\gamma_D \simeq \text{const} \quad (N \gg 1); \quad (64)$$

To make the consideration slightly less cumbersome, let us set the powers $n_\alpha = n$ for the trapping potential. Then, the confining dimension is:

$$D = \left(1 + \frac{2}{n} \right) d. \quad (65)$$

γ_D becomes:

$$\gamma_D = \frac{\pi^{d/2}}{\Gamma^d(1 + 1/n)} \left(\frac{\omega_0}{2} \right)^{D/2},$$

which tells us that:

$$\gamma_D \propto \omega_0^{D/2} \quad (N \gg 1).$$

Therefore, ω_0 scales as:

$$\omega_0 \propto \frac{1}{N^{2/D}} \quad (N \gg 1). \quad (66)$$

Using this scaling and the fact that $\omega_0 \sim \varepsilon_0$, we see that the critical temperatures from the previous subsection behave as:

$$\begin{aligned} T_c &\propto \frac{1}{N^{2/D-1}} \quad (D < 2), \\ T_c &\propto \frac{1}{\ln N} \quad (D = 2), \\ T_c &\propto \text{const} \quad (D > 2). \end{aligned} \quad (67)$$

4.4. Fluctuations above the Condensation Temperature

Particle fluctuations above the condensation temperature are described by the formula:

$$\frac{\text{var}(\hat{N})}{N} = \frac{T^{D/2}}{N\gamma_D} z \frac{\partial g_{D/2}(z)}{\partial z} \quad (T > T_c), \quad (68)$$

where $z < 1$. For the modified Bose function, we have:

$$\frac{\partial g_m(z)}{\partial z} = \frac{1}{z} g_{m-1}(z) + \frac{1}{(1-z)\Gamma(m)} \left(u_0^{m-1} - \frac{u_0^m}{1-z} \right) \quad (z < 1), \quad (69)$$

with the value:

$$g_{m-1}(z) = - \frac{z}{(1-z)\Gamma(m)} \left[u_0^{m-1} - \frac{m-1}{m(1-z)} u_0^m \right] \quad (m < 1, z < 1) \quad (70)$$

for $m < 1$. Summarizing, we have the derivatives:

$$\frac{\partial g_m(z)}{\partial z} = - \frac{u_0^m}{(1-z)^2 \Gamma(1+m)} \quad (m < 1, z < 1),$$

$$\frac{\partial g_1(z)}{\partial z} = - \frac{u_0}{(1-z)^2} \quad (m = 1, z < 1),$$

$$\frac{\partial g_m(z)}{\partial z} = \frac{1}{z} g_{m-1}(z) \quad (m > 1, z < 1).$$

From here, we find the relative variance:

$$\frac{\text{var}(\hat{N})}{N} = - \frac{zT^{D/2}}{(1-z)^2 N\gamma_D \Gamma(1+D/2)} \left(\frac{\varepsilon_0}{T} \right)^{D/2} \quad (D < 2, T > T_c),$$

$$\frac{\text{var}(\hat{N})}{N} = - \frac{zT}{(1-z)^2 N\gamma_2} \left(\frac{\varepsilon_0}{T} \right) \quad (D = 2, T > T_c),$$

$$\frac{\text{var}(\hat{N})}{N} = \frac{T^{D/2}}{N\gamma_D} g_{D/2-1}(z) \quad (D > 2, T > T_c), \quad (71)$$

characterizing particle fluctuations above the critical temperature. For $D \leq 2$, the variance is negative, which means instability. The system is stable only for $D > 2$, giving the stability condition:

$$d + \sum_{\alpha=1}^d \frac{2}{n_\alpha} > 2 \quad (T > T_c). \quad (72)$$

4.5. Fluctuations below the Condensation Temperature

Below the condensation temperature, where $\mu = 0$ and $z = 1$, the number of uncondensed particles reads as:

$$N_1 = \frac{T^{D/2}}{\gamma_D} g_{D/2}(1) \quad (T \leq T_c). \quad (73)$$

The variance of the total number of particles coincides with that of the uncondensed particles, which leads to:

$$\frac{\text{var}(\hat{N})}{N} = \frac{T^{D/2}}{N\gamma_D} \lim_{z \rightarrow 1} \frac{\partial g_{D/2}(z)}{\partial z} \quad (T < T_c). \quad (74)$$

For the derivative in the right-hand side of the above formula, we have:

$$\lim_{z \rightarrow 1} \frac{\partial g_{D/2}(z)}{\partial z} = g_{D/2-1}(1) + \frac{u_0^{D/2-2}}{\Gamma(D/2)}. \tag{75}$$

Employing the values:

$$g_{D/2-1}(1) = \frac{2}{(4-D)\Gamma(D/2-1)} u_0^{D/2-2} \quad (D < 4),$$

$$g_1(1) = -\ln u_0 \quad (D = 4),$$

we get the derivatives:

$$\lim_{z \rightarrow 1} \frac{\partial g_{D/2}(z)}{\partial z} = \left[\frac{2}{(4-D)\Gamma(D/2-1)} + \frac{1}{\Gamma(D/2)} \right] u_0^{D/2-2} \quad (D < 4),$$

$$\lim_{z \rightarrow 1} \frac{\partial g_2(z)}{\partial z} = -\ln u_0 \quad (D = 4),$$

$$\lim_{z \rightarrow 1} \frac{\partial g_{D/2}(z)}{\partial z} = g_{D/2-1}(1) = \zeta\left(\frac{D}{2} - 1\right) \quad (D > 4).$$

In that way, we come to the relative variances:

$$\frac{\text{var}(\hat{N})}{N} = \frac{T^{D/2}}{N\gamma_D} \left[\frac{2}{(4-D)\Gamma(D/2-1)} + \frac{1}{\Gamma(D/2)} \right] \left(\frac{T}{\varepsilon_0}\right)^{2-D/2} \quad (D < 4),$$

$$\frac{\text{var}(\hat{N})}{N} = \frac{T^2}{N\gamma_4} \ln\left(\frac{T}{\varepsilon_0}\right) \quad (D = 4),$$

$$\frac{\text{var}(\hat{N})}{N} = \frac{T^{D/2}}{N\gamma_D} \zeta\left(\frac{D}{2} - 1\right) \quad (D > 4). \tag{76}$$

Keeping in mind that $\varepsilon_0 \propto N^{-2/D}$ results in the scaling:

$$\frac{\text{var}(\hat{N})}{N} \propto N^{(4-D)/D} \quad (D < 4),$$

$$\frac{\text{var}(\hat{N})}{N} \propto \ln N \quad (D = 4),$$

$$\frac{\text{var}(\hat{N})}{N} \propto \text{const} \quad (D > 4). \tag{77}$$

This tells us that the system is stable only for $D > 4$. Therefore, the stability condition is:

$$d + \sum_{\alpha=1}^d \frac{2}{n_\alpha} > 4 \quad (T < T_c). \tag{78}$$

Notice that in the case of the often considered harmonic potential, when $n_\alpha = 2$, we have $D = 2d$ and $\gamma_D = \omega_0^d$. Then, the stability condition (78) reduces to the condition $d > 2$. The relative particle variance reads as:

$$\frac{\text{var}(\hat{N})}{N} = \frac{\zeta(d-1)}{\zeta(d)} \left(\frac{T}{T_c}\right)^d \quad (n_\alpha = 2, d > 2).$$

5. Interacting Bose System above the Condensation Temperature

The grand Hamiltonian for a system of interacting Bose particles is:

$$H = \int \psi^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} - \mu \right) \psi(\mathbf{r}) d\mathbf{r} + \frac{1}{2} \int \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') \Phi(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) d\mathbf{r} d\mathbf{r}' . \quad (79)$$

For generality, we consider a nonlocal isotropic interaction potential $\Phi(\mathbf{r}) = \Phi(r)$, where $r \equiv |\mathbf{r}|$. The integration is assumed to be over a rectangular box of volume V confining the system.

In the Hartree–Fock approximation, the Hamiltonian takes the form:

$$H_{HF} = E_{HF} + \int \psi^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} - \mu \right) \psi(\mathbf{r}) d\mathbf{r} + \int \Phi(\mathbf{r} - \mathbf{r}') \left[\rho(\mathbf{r}') \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) + \rho(\mathbf{r}, \mathbf{r}') \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}) \right] d\mathbf{r} d\mathbf{r}' , \quad (80)$$

where:

$$E_{HF} = -\frac{1}{2} \int \Phi(\mathbf{r} - \mathbf{r}') \left[\rho(\mathbf{r}) \rho(\mathbf{r}') + |\rho(\mathbf{r}, \mathbf{r}')|^2 \right] d\mathbf{r} d\mathbf{r}'$$

and the notations are used for the single-particle density matrix:

$$\rho(\mathbf{r}, \mathbf{r}') = \langle \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}) \rangle \quad (81)$$

and the particle density:

$$\rho(\mathbf{r}) = \rho(\mathbf{r}, \mathbf{r}) = \langle \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \rangle . \quad (82)$$

Employing the expansion of the field operators over plane waves, as in Equation (13), we get the Hamiltonian:

$$H_{HF} = E_{HF} + \sum_k \omega_k a_k^\dagger a_k , \quad (83)$$

in which:

$$E_{HF} = -\frac{1}{2} \rho \Phi_0 N - \frac{1}{2V} \sum_{kp} n_k n_p \Phi_{k+p} ,$$

Φ_k is a Fourier transform of $\Phi(\mathbf{r})$, and:

$$\Phi_0 = \int \Phi(\mathbf{r}) d\mathbf{r} . \quad (84)$$

The momentum distribution is given by the expression (16), with the spectrum:

$$\omega_k = \frac{k^2}{2m} + \rho \Phi_0 + \frac{1}{V} \sum_p n_p \Phi_{k+p} - \mu . \quad (85)$$

The function n_p possesses a maximum at $p \rightarrow 0$, because of which it is possible to use the approximation [32,33]:

$$\sum_p n_p \Phi_{k+p} \cong \Phi_k \sum_p n_p \quad (86)$$

giving:

$$\omega_k = \frac{k^2}{2m} + \rho(\Phi_0 + \Phi_k) - \mu . \quad (87)$$

Introducing the effective interaction radius by the relation:

$$r_{eff}^2 \equiv \frac{\int \Phi(\mathbf{r}) r^2 d\mathbf{r}}{\int \Phi(\mathbf{r}) d\mathbf{r}} = \frac{4\pi}{\Phi_0} \int_0^\infty \Phi(\mathbf{r}) r^4 d\mathbf{r} \quad (88)$$

shows that the long-wave limit of Φ_k is:

$$\Phi_k \simeq \left(1 - \frac{1}{6} k^2 r_{eff}^2\right) \Phi_0. \quad (89)$$

Then, the spectrum (87) can be represented as:

$$\omega_k \simeq \frac{k^2}{2m^*} - \mu_{eff} \quad (k \rightarrow 0), \quad (90)$$

with the effective mass:

$$m^* \equiv \frac{m}{1 - \rho \Phi_0 r_{eff}^2 / 3} \quad (91)$$

and effective chemical potential:

$$\mu_{eff} \equiv \mu - 2\rho\Phi_0. \quad (92)$$

In this approximation, the number of particles acquires the same form (23), however with the notations:

$$\lambda_T \equiv \sqrt{\frac{2\pi}{m^* T}}, \quad z \equiv \exp(\beta\mu_{eff}). \quad (93)$$

Using again the modified Bose function (19) and following the same analysis as in Section 3, we come to the conclusion that the system is stable for $d > 2$, when $T > T_c$. The difference is that now instead of mass m , there is the effective mass m^* , and at the critical temperature, we have:

$$\mu_{eff} = 0, \quad \mu = 2\rho\Phi_0 \quad (T = T_c). \quad (94)$$

Therefore, the critical temperature becomes:

$$T_c = \frac{2\pi}{m^*} \left[\frac{\rho}{g_{d/2}(1)} \right]^{2/d}. \quad (95)$$

As an example, let us consider the realistic three-dimensional case. Using the Robinson representation (see the details in review [18]), we can find the behavior of the effective chemical potential at high temperatures:

$$\mu_{eff} = T \ln(\rho \lambda_T^3) \quad (T \gg T_c) \quad (96)$$

and at the temperature approaching the critical point from above,

$$\mu_{eff} \simeq -T \frac{\zeta^2(3/2)}{4\pi} \left[1 - \left(\frac{T_c}{T} \right)^{3/2} \right]^2. \quad (97)$$

Then, the isothermal compressibility:

$$\kappa_T = \frac{g_{1/2}(z)}{\rho^2 T \lambda_T^3} \quad (98)$$

at high temperatures is:

$$\kappa_T \simeq \frac{1}{\rho T} \quad (T \gg T_c), \quad (99)$$

while close to the critical point, it is:

$$\kappa_T \simeq \frac{0.921}{\rho T} \left[1 - \left(\frac{T_c}{T} \right)^{3/2} \right]^{-1}; \quad (99)$$

respectively, particle fluctuations, described by the relative variance:

$$\frac{\text{var}(\hat{N})}{N} = \rho T \kappa_T = \frac{g_{1/2}(z)}{\rho \lambda_T^3}, \quad (100)$$

at high temperatures behave as:

$$\frac{\text{var}(\hat{N})}{N} \simeq 1 \quad (T \gg T_c), \quad (101)$$

and close to the critical point, we get:

$$\frac{\text{var}(\hat{N})}{N} \simeq 0.921 \left[1 - \left(\frac{T_c}{T} \right)^{3/2} \right]^{-1}. \quad (102)$$

Outside of the critical temperature itself, particle fluctuations are thermodynamically normal. The divergence of the compressibility at the critical point signifies a second-order phase transition. At the point of the phase transition, the system is not stable, and the fluctuations do not need to be finite.

6. Interacting Bose System below the Condensation Temperature

In Section 3, it is proven that the ideal Bose gas, confined in a box, is stable below the condensation temperature only for $d > 4$. In the present section, we show that interactions stabilize the system, making it stable already for $d = 3$.

6.1. Self-Consistent Approach

For describing a Bose system with the Bose–Einstein condensate, we employ the self-consistent approach [16–18,24,34,35], providing a gapless spectrum, correct thermodynamics, the validity of all conservation laws, and good agreement with Monte Carlo simulations and experimental data.

The energy Hamiltonian has the form:

$$\begin{aligned} \hat{H} = & \int \hat{\psi}^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} \right) \hat{\psi}(\mathbf{r}) d\mathbf{r} \\ & + \frac{1}{2} \int \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \Phi(\mathbf{r} - \mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}) d\mathbf{r} d\mathbf{r}'. \end{aligned} \quad (103)$$

The genuine Bose–Einstein condensation necessarily requires global gauge symmetry breaking [6,9,17,18]. Finite systems, strictly speaking, do not exhibit this symmetry breaking. However, a system with a large number of particles $N \gg 1$ enjoys asymptotic symmetry breaking [36] in the sense that the system properties asymptotically, with respect to N , are close to the system with broken symmetry. The global gauge symmetry can be broken by the Bogolubov shift [28–30]:

$$\hat{\psi}(\mathbf{r}) = \eta(\mathbf{r}) + \psi_1(\mathbf{r}), \quad (104)$$

in which the condensate function $\eta(\mathbf{r})$ and the operator of uncondensed particles $\psi_1(\mathbf{r})$ are mutually orthogonal,

$$\int \eta^*(\mathbf{r}) \psi_1(\mathbf{r}) d\mathbf{r} = 0 \quad (105)$$

and the operator of uncondensed particles satisfies the condition:

$$\langle \psi_1(\mathbf{r}) \rangle = 0. \quad (106)$$

The number of condensed particles is:

$$N_0 = \int |\eta(\mathbf{r})|^2 d\mathbf{r}, \quad (107)$$

while the number of uncondensed particles is given by the average:

$$N_1 = \langle \hat{N}_1 \rangle, \quad \hat{N}_1 = \int \psi_1^\dagger(\mathbf{r})\psi_1(\mathbf{r}) d\mathbf{r}. \quad (108)$$

The grand Hamiltonian reads as:

$$H = \hat{H} - \mu_0 N_0 - \mu_1 \hat{N}_1 - \hat{\Lambda}, \quad (109)$$

where:

$$\hat{\Lambda} = \int [\lambda(\mathbf{r})\psi_1^\dagger(\mathbf{r}) + \lambda^*(\mathbf{r})\psi_1(\mathbf{r})] d\mathbf{r}$$

and μ_0 , μ_1 , and $\lambda(\mathbf{r})$ are Lagrange multipliers guaranteeing the validity of the normalizations (107) and (108), as well as condition (106).

The evolution equation for the condensate function can be written as:

$$i \frac{\partial}{\partial t} \eta(\mathbf{r}, t) = \left\langle \frac{\delta H}{\delta \eta^*(\mathbf{r}, t)} \right\rangle \quad (110)$$

and the equation for the operator of uncondensed particles as:

$$i \frac{\partial}{\partial t} \psi_1(\mathbf{r}, t) = \frac{\delta H}{\delta \psi_1^\dagger(\mathbf{r}, t)}. \quad (111)$$

Keeping in mind, as usual, the periodic continuation of the box, we expand the field operators in plane waves, as in (13), and assume the existence of the Fourier representation for the interaction potential:

$$\Phi_k = \int \Phi(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}, \quad \Phi(\mathbf{r}) = \frac{1}{V} \sum_k \Phi_k e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (112)$$

Then, we get the normal density matrix:

$$\rho_1(\mathbf{r}, \mathbf{r}') = \langle \psi_1^\dagger(\mathbf{r}')\psi_1(\mathbf{r}) \rangle = \frac{1}{V} \sum_{k \neq 0} n_k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \quad (113)$$

and the anomalous matrix:

$$\sigma_1(\mathbf{r}, \mathbf{r}') = \langle \psi_1(\mathbf{r}')\psi_1(\mathbf{r}) \rangle = \frac{1}{V} \sum_{k \neq 0} \sigma_k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}, \quad (114)$$

in which:

$$n_k \equiv \langle a_k^\dagger a_k \rangle, \quad \sigma_k \equiv \langle a_k a_{-k} \rangle. \quad (115)$$

The condensate function $\eta(\mathbf{r}) = \eta$ defines the condensate density:

$$\rho_0 \equiv \frac{N_0}{V} = |\eta|^2. \quad (116)$$

The density of uncondensed particles is:

$$\rho_1 \equiv \frac{N_1}{V} = \rho_1(\mathbf{r}, \mathbf{r}) = \frac{1}{V} \sum_k n_k. \quad (117)$$

The diagonal anomalous matrix gives the anomalous average:

$$\sigma_1 \equiv \sigma_1(\mathbf{r}, \mathbf{r}) = \frac{1}{V} \sum_k \sigma_k. \quad (118)$$

The average density of particles is the sum:

$$\rho \equiv \frac{N}{V} = \rho_0 + \rho_1. \quad (119)$$

Then, we use the Hartree–Fock–Bogolubov approximation and accomplish the Bogolubov canonical transformation:

$$a_k = u_k b_k + v_{-k}^* b_{-k}^\dagger, \quad b_k = u_k^* a_k - v_k^* a_{-k}^\dagger,$$

where u_k and v_k are chosen so as to diagonalize the Hamiltonian. In that way, we obtain the diagonalized Hamiltonian:

$$H_B = E_B + \sum_k \varepsilon_k b_k^\dagger b_k, \quad (120)$$

in which:

$$E_B = -\frac{1}{2} N \rho \Phi_0 - \rho_0 \sum_p (n_p + \sigma_p) \Phi_p - \frac{1}{2V} \sum_{kp} (n_k n_p + \sigma_k \sigma_p) \Phi_{k+p} + \frac{1}{2} \sum_k (\varepsilon_k - \omega_k),$$

the particle spectrum is:

$$\varepsilon_k = \sqrt{\omega_k^2 - \Delta_k^2}, \quad (121)$$

and where:

$$\begin{aligned} \omega_k &= \frac{k^2}{2m} + \Delta + \rho_0 (\Phi_k - \Phi_0) + \frac{1}{V} \sum_p n_p (\Phi_{k+p} - \Phi_p), \\ \Delta_k &= \rho_0 \Phi_k + \frac{1}{V} \sum_p \sigma_p \Phi_{k+p}, \quad \Delta \equiv \lim_{k \rightarrow 0} \Delta_k = \rho_0 \Phi_0 + \frac{1}{V} \sum_p \sigma_p \Phi_p. \end{aligned} \quad (122)$$

For the expressions in (115), we find:

$$n_k = \frac{\omega_k}{2\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right) - \frac{1}{2}, \quad \sigma_k = -\frac{\Delta_k}{2\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right). \quad (123)$$

The chemical potentials are:

$$\mu_0 = \rho \Phi_0 + \frac{1}{V} \sum_k (n_k + \sigma_k) \Phi_k, \quad \mu_1 = \rho \Phi_0 + \frac{1}{V} \sum_k (n_k - \sigma_k) \Phi_k. \quad (124)$$

In the long-wave limit, we can use the expansion:

$$\Phi_{k+p} \simeq \Phi_p + \frac{k^2}{2} \Phi_p'' \quad (k \rightarrow 0),$$

where:

$$\Phi_p'' \equiv \frac{\partial^2 \Phi_p}{\partial p^2}.$$

Then, the spectrum (121) becomes of the phonon type:

$$\varepsilon_k \simeq ck \quad (k \rightarrow 0), \quad (125)$$

with the sound velocity:

$$c = \sqrt{\frac{\Delta}{m_{eff}}} \quad (126)$$

and with the notation for the effective mass:

$$m_{eff} \equiv \frac{m}{1 + \frac{m}{V} \sum_p (n_p - \sigma_p) \Phi_p''}. \quad (127)$$

Actually, Expression (126), which can be written as:

$$m_{eff} c^2 = \Delta,$$

is the equation:

$$\frac{mc^2}{1 + \frac{m}{V} \sum_p (n_p - \sigma_p) \Phi_p''} = \rho_0 \Phi_0 + \frac{1}{V} \sum_p \sigma_p \Phi_p, \quad (128)$$

defining the sound velocity c .

To simplify the consideration, we can resort to the approximation (86), similarly, to which we can write:

$$\sum_p \sigma_p \Phi_{k+p} \simeq \Phi_k \sum_p \sigma_p. \quad (129)$$

This gives:

$$\frac{1}{V} \sum_p (n_p - \sigma_p) \Phi_p'' = (\rho_1 - \sigma_1) \Phi_0'',$$

where:

$$\Phi_0'' = \lim_{p \rightarrow 0} \Phi_p'' = -\frac{4\pi}{3} \int_0^\infty \Phi(\mathbf{r}) r^4 d\mathbf{r}.$$

In view of the notation for the effective interaction radius (88), we get:

$$\Phi_0'' = -\frac{1}{3} \Phi_0 r_{eff}^2.$$

Then, the effective mass (127) acquires the form:

$$m_{eff} = \frac{m}{1 + (\sigma_1 - \rho_1) \Phi_0 m r_{eff}^2 / 3}. \quad (130)$$

In the approximations (86) and (129), the chemical potentials (124) become:

$$\mu_0 = \rho \Phi_0 + (\rho_1 + \sigma_1) \Phi_0, \quad \mu_1 = \rho \Phi_0 + (\rho_1 - \sigma_1) \Phi_0. \quad (131)$$

Furthermore, we have:

$$\omega_k = \frac{k^2}{2m} + \Delta + \rho(\Phi_k - \Phi_0) \quad \Delta_k = (\rho_0 + \sigma_1) \Phi_k, \quad \Delta = (\rho_0 + \sigma_1) \Phi_0. \quad (132)$$

The spectrum (121) can be written as:

$$\varepsilon_k^2 = \left[\frac{k^2}{2m} + (\rho_1 - \sigma_1)(\Phi_k - \Phi_0) \right] \left[\frac{k^2}{2m} + \rho(\Phi_k - \Phi_0) + (\rho_0 + \sigma_1)(\Phi_k + \Phi_0) \right]. \quad (133)$$

The density of uncondensed particles is:

$$\rho_1 = \int \left[\frac{\omega_k}{2\varepsilon_k} \coth\left(\frac{\varepsilon_k}{2T}\right) - \frac{1}{2} \right] \frac{d\mathbf{k}}{(2\pi)^3}. \quad (134)$$

The anomalous average (118) can be represented in the form:

$$\sigma_1 = - \int \frac{\Delta_k}{2\varepsilon_k} \frac{d\mathbf{k}}{(2\pi)^3} - \int \frac{\Delta_k}{2\varepsilon_k} \left[\coth\left(\frac{\varepsilon_k}{2T}\right) - 1 \right] \frac{d\mathbf{k}}{(2\pi)^3}. \quad (135)$$

When the first term here diverges, which happens for the local interaction, we can use dimensional regularization [18].

6.2. Particle Fluctuations

The number-of-particles variance can be found by involving the formula:

$$\frac{\text{var}(\hat{N})}{N} = 1 + \rho \int [g(\mathbf{r}) - 1] d\mathbf{r}, \quad (136)$$

in which:

$$g(\mathbf{r}_{12}) = \frac{1}{g^2} \langle \hat{\psi}^\dagger(\mathbf{r}_1) \hat{\psi}^\dagger(\mathbf{r}_2) \hat{\psi}(\mathbf{r}_2) \hat{\psi}(\mathbf{r}_1) \rangle \quad (137)$$

is the pair correlation function, with $\mathbf{r}_{12} \equiv \mathbf{r}_1 - \mathbf{r}_2$.

Accomplishing the Bogolubov shift (104), we use the Hartree–Fock–Bogolubov (HFB) decoupling for the expressions containing the operators ψ_1 . Since, mathematically, the HFB approximation is of second order with respect to the products of the operators ψ_1 , it is necessary to leave in the pair correlation function only the terms of second order with respect to these operators [3,16–18,23,24]. As a result, we obtain:

$$\int [g(\mathbf{r}) - 1] d\mathbf{r} = \frac{2}{\rho} \lim_{k \rightarrow 0} (n_k + \sigma_k). \quad (138)$$

In this way, for the relative variance, we find:

$$\frac{\text{var}(\hat{N})}{N} = 1 + 2 \lim_{k \rightarrow 0} (n_k + \sigma_k). \quad (139)$$

For small k , when ε_k tends to zero, we have:

$$\begin{aligned} n_k &\simeq \frac{T\Delta_k}{\varepsilon_k^2} + \frac{\Delta_k}{12T} + \frac{T}{2\Delta_k} - \frac{1}{2} + \left(\frac{\Delta_k}{3T} - \frac{T}{\Delta_k} - \frac{\Delta_k^3}{90T^3} \right) \frac{\varepsilon_k^2}{8\Delta_k^2}, \\ \sigma_k &\simeq - \frac{T\Delta_k}{\varepsilon_k^2} - \frac{\Delta_k}{12T} + \frac{\Delta_k \varepsilon_k^2}{720T^3} \quad (\varepsilon_k \rightarrow 0). \end{aligned} \quad (140)$$

Therefore:

$$\lim_{k \rightarrow 0} (n_k + \sigma_k) = \frac{1}{2} \left(\frac{T}{\Delta} - 1 \right),$$

with:

$$\Delta = m_{eff}c^2 = (\rho_0 + \sigma_1)\Phi_0.$$

Thus, we come to the expression:

$$\frac{\text{var}(\hat{N})}{N} = \frac{T}{m_{eff}c^2}; \quad (141)$$

respectively, the compressibility is:

$$\kappa_T = \frac{\text{var}(\hat{N})}{N\rho T} = \frac{1}{\rho m_{eff} c^2}. \quad (142)$$

Taking into account Formula (126) leads to the variance:

$$\frac{\text{var}(\hat{N})}{N} = \frac{T}{(\rho_0 + \sigma_1)\Phi_0}. \quad (143)$$

Note that Expression (143) is valid at zero temperature, as well. This is easy to check considering the quantities (123) at zero temperature,

$$n_k = \frac{\sqrt{\varepsilon_k^2 + \Delta_k^2}}{2\varepsilon_k} - \frac{1}{2}, \quad \sigma_k = -\frac{\Delta_k}{2\varepsilon_k} \quad (T = 0).$$

From here, in the long-wave limit, we have:

$$n_k \simeq \frac{\Delta_k}{2\varepsilon_k} + \frac{\varepsilon_k}{4\Delta_k} - \frac{1}{2} \quad (\varepsilon_k \rightarrow 0, T = 0).$$

Hence:

$$\lim_{k \rightarrow 0} (n_k + \sigma_k) = -\frac{1}{2} \quad (T = 0)$$

and:

$$\frac{\text{var}(\hat{N})}{N} = 0 \quad (T = 0).$$

The above result for the relative variance (143) can be generalized for nonuniform systems [37] by involving the local-density approximation, which yields:

$$\frac{\text{var}(\hat{N})}{N} = \frac{T}{N} \int \frac{\rho(\mathbf{r})}{\Delta(\mathbf{r})} d\mathbf{r}, \quad (144)$$

where:

$$\Delta(\mathbf{r}) = [\rho_0(\mathbf{r}) + \sigma_1(\mathbf{r})]\Phi_0. \quad (145)$$

Particle fluctuations in a three-dimensional Bose-condensed system of interacting particles are thermodynamically normal in both cases, when particles are in a box or in a nonuniform external potential.

7. Conclusions

Particle fluctuations in Bose systems were studied. Investigating the behavior of these fluctuations is important because they are directly connected with isothermal compressibility and define the system stability with respect to pressure variations. Thermodynamically-anomalous fluctuations signify system instability; while thermodynamically-normal fluctuations mean that the equilibrium system is stable. The obtained results are as follows.

The ideal Bose gas confined in a rectangular box is stable, depending on the temperature, in spatial dimensions:

$$\begin{aligned} d &> 2 & (T > T_c), \\ d &> 4 & (T < T_c). \end{aligned}$$

The stability of the ideal Bose gas in a power-law trap depends on the confining dimension:

$$D \equiv d + \sum_{\alpha=1}^d \frac{2}{n_{\alpha}} .$$

This gas is stable for the confining dimensions:

$$D > 2 \quad (T > T_c) ,$$

$$D > 4 \quad (T < T_c) .$$

Interactions stabilize Bose-condensed systems, so that an interacting system with Bose–Einstein condensate becomes stable at $d = 3$ for either a system in a box or in an external potential.

Nonlocal interactions with a stronger strength or with a larger interaction radius increase the effective mass, hence diminishing the condensation temperature.

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References

1. Courteille, P.W.; Bagnato, V.S.; Yukalov, V.I. Bose-Einstein condensation of trapped atomic gases. *Laser Phys.* **2001**, *11*, 659–800.
2. Andersen, J.O. Theory of the weakly interacting Bose gas. *Rev. Mod. Phys.* **2004**, *76*, 599–639. [[CrossRef](#)]
3. Yukalov, V.I. Principal problems in Bose-Einstein condensation of dilute gases. *Laser Phys. Lett.* **2004**, *1*, 435–461. [[CrossRef](#)]
4. Bongs, K.; Sengstock, K. Physics with coherent matter waves. *Rep. Prog. Phys.* **2004**, *67*, 907–964. [[CrossRef](#)]
5. Yukalov, V.I.; Girardeau, M.D. Fermi-Bose mapping for one-dimensional Bose gases. *Laser Phys. Lett.* **2005**, *2*, 375–382. [[CrossRef](#)]
6. Lieb, E.H.; Seiringer, R.; Solovej, J.P.; Yngvason, J. *The Mathematics of the Bose Gas and its Condensation*; Birkhauser: Basel, Switzerland, 2005.
7. Posazhennikova, A. Weakly interacting dilute Bose gases in 2D. *Rev. Mod. Phys.* **2006**, *78*, 1111–1134. [[CrossRef](#)]
8. Morsch, O.; Oberthaler, M. Dynamics of Bose-Einstein condensates in optical lattices. *Rev. Mod. Phys.* **2006**, *78*, 179–216. [[CrossRef](#)]
9. Yukalov, V.I. Bose-Einstein condensation and gauge symmetry breaking. *Laser Phys. Lett.* **2007**, *4*, 632–647. [[CrossRef](#)]
10. Letokhov, V. *Laser Control of Atoms and Molecules*; Oxford University Press: Oxford, UK, 2007.
11. Moseley, C.; Fialko, O.; Ziegler, K. Interacting bosons in an optical lattice. *Ann. Phys.* **2008**, *17*, 561–608. [[CrossRef](#)]
12. Bloch, I.; Dalibard, J.; Zwirger, W. Many-body physics with ultracold gases. *Rev. Mod. Phys.* **2008**, *80*, 885–964. [[CrossRef](#)]
13. Proukakis, N.P.; Jackson, B. Finite-temperature models of Bose-Einstein condensation. *J. Phys. B* **2008**, *41*, 203002. [[CrossRef](#)]
14. Yurovsky, V.A.; Olshanii, M.; Weiss, D.S. Collisions, correlations, and integrability in atom waveguides. *Adv. At. Mol. Opt. Phys.* **2008**, *55*, 61–138.
15. Pethick, C.J.; Smith, H. *Bose-Einstein Condensation in Dilute Gases*; Cambridge University: Cambridge, UK, 2008.
16. Yukalov, V.I. Cold bosons in optical lattices. *Laser Phys.* **2009**, *19*, 1–110. [[CrossRef](#)]
17. Yukalov, V.I. Basics of Bose-Einstein condensation. *Phys. Part. Nucl.* **2011**, *42*, 460–513. [[CrossRef](#)]
18. Yukalov, V.I. Theory of cold atoms: Bose-Einstein statistics. *Laser Phys.* **2016**, *26*, 062001. [[CrossRef](#)]

19. Yukalov, V.I.; Yukalova, E.P. Bose-Einstein condensation temperature of weakly interacting atoms. *Laser Phys. Lett.* **2017**, *14*, 073001. [[CrossRef](#)]
20. Kocharovskiy, V.V.; Kocharovskiy, V.V.; Holthaus, M.; Ooi, C.R.; Svidzinsky, A.; Ketterle, W.; Scully, M.O. Fluctuations in ideal and interacting Bose–Einstein condensates: From the laser phase transition analogy to squeezed states and Bogolubov quasiparticles. *Adv. At. Mol. Opt. Phys.* **2006**, *53*, 291–411.
21. Patashinskii, A.Z.; Pokrovskii, V.L. *Fluctuation Theory of Phase Transitions*; Pergamon: Oxford, UK, 1979.
22. Yukalov, V.I. Theory of cold atoms: Basics of quantum statistics. *Laser Phys.* **2013**, *23*, 062001. [[CrossRef](#)]
23. Yukalov, V.I. No anomalous fluctuations exist in stable equilibrium systems. *Phys. Lett. A* **2005**, *340*, 369–374. [[CrossRef](#)]
24. Yukalov, V.I. Fluctuations of composite observables and stability of statistical systems. *Phys. Rev. E* **2005**, *72*, 066119. [[CrossRef](#)]
25. Gaunt, A.L.; Tobias, F.; Schmidutz, T.F.; Gotlibovych, I.; Smith, R.P.; Hadzibabic, Z. Bose-Einstein condensation of atoms in a uniform potential. *Phys. Rev. Lett.* **2013**, *110*, 200406. [[CrossRef](#)]
26. Navon, N.; Gaunt, A.L.; Smith, R.P.; Hadzibabic, Z. Critical dynamics of spontaneous symmetry breaking in a homogeneous Bose gas. *Science* **2015**, *347*, 167–170. [[CrossRef](#)] [[PubMed](#)]
27. Lopes, R.; Eigen, C.; Navon, N.; Clement, D.; Smith, R.P.; Hadzibabic, Z. Quantum depletion of a homogeneous Bose-Einstein condensate. *Phys. Rev. Lett.* **2017**, *119*, 190404. [[CrossRef](#)]
28. Bogolubov, N.N. *Lectures on Quantum Statistics*; Gordon and Breach: New York, NY, USA, 1967; Volume 1.
29. Bogolubov, N.N. *Lectures on Quantum Statistics*; Gordon and Breach: New York, NY, USA, 1970; Volume 2.
30. Bogolubov, N.N. *Quantum Statistical Mechanics*; World Scientific: Singapore, 2015.
31. Yukalov, V.I. Modified semiclassical approximation for trapped Bose gases. *Phys. Rev. A* **2005**, *72*, 033608. [[CrossRef](#)]
32. Yukalov, V.I.; Yukalova, E.P. Bose-condensed atomic systems with nonlocal interaction potentials. *Laser Phys.* **2016**, *26*, 045501. [[CrossRef](#)]
33. Yukalov, V.I. Dipolar and spinor bosonic systems. *Laser Phys.* **2018**, *28*, 053001. [[CrossRef](#)]
34. Yukalov, V.I. Self-consistent theory of Bose-condensed systems. *Phys. Lett. A* **2006**, *359*, 712–717. [[CrossRef](#)]
35. Yukalov, V.I. Representative statistical ensembles for Bose systems with broken gauge symmetry. *Ann. Phys.* **2008**, *323*, 461–499. [[CrossRef](#)]
36. Birman, J.L.; Nazmitdinov, R.G.; Yukalov, V.I. Effects of symmetry breaking in finite quantum systems. *Phys. Rep.* **2013**, *526*, 1–91. [[CrossRef](#)]
37. Yukalov, V.I. Particle fluctuations in nonuniform and trapped Bose gases. *Laser Phys. Lett.* **2009**, *6*, 688–695. [[CrossRef](#)]



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