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Lie Symmetry Analysis and Exact Solutions of Generalized Fractional Zakharov-Kuznetsov Equations

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Abstract: This paper considers the Lie symmetry analysis of a class of fractional Zakharov-Kuznetsov equations. We systematically show the procedure to obtain the Lie point symmetries for the equation. Accordingly, we study the vector fields of this equation. Meantime, the symmetry reductions of this equation are performed. Finally, by employing the obtained symmetry properties, we can get some new exact solutions to this fractional Zakharov-Kuznetsov equation.

Keywords: lie symmetry analysis; fractional Zakharov-Kuznetsov equation; symmetry groups; exact solutions

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1. Introduction

As one of the important high dimensional nonlinear evolution equations, the Zakharov-Kuznetsov (ZK) equation was first used to discuss the evolution of the propagation of plane waves in magnetized plasma containing cold ions and isothermal electrons [1]. The dimensionless form in renormalized variables of the ZK equation is as follows:

$$u_t + auu_x + bu_{xxx} + cu_{xyy} = 0, \tag{1}$$

where $u = u(t, x, y)$ is the normalized electric potential. a, b, c are all normalized constants with respect to different physical meanings. For details of these coefficients, see [1,2]. Due to its wide application in maths and physics, the study of this equation was a wide range of theoretical and practical significance. There are various results about the ZK equation, for more details, one can see [3–11].

In paper [12], Blaha et al. considered the following modified version of ZK equation.

$$u_t + au^2u_x + u_{xxx} + u_{xyy} = 0, \tag{2}$$

where the signs of a represents different physical phenomenon. In addition, Wazwaz [13] did a further study about a nonlinear dispersive modified ZK equation(mZK) as follows:

$$u_t + a(u^{\frac{n}{2}})_x + bu_{xxx} + ku_{xyy} = 0, \tag{3}$$

where $n \geq 3$ is odd and the sign is either positive or negative.

To include as many physical applications as possible, lots of papers discuss the generalized Zakharov-Kuznetsov equation of the following form [14,15].

$$u_t + a(u^p)_x + b(u^q)_{xxx} + c(u^r)_{xyy} = 0. \tag{4}$$

Please note that the paper [15] systematically illustrated the detailed group classification algorithm and the process of reduction by discussing Equation (4).

Recently, many important phenomena in various areas of science were well described by fractional order differential equations [16]. Due to the realistic senses, a lot of attention has been given to seek solutions of fractional differential equations (FDEs). Various methods such as the homotopy perturbation method (HPM) [17], the variational iteration method (VIM) [18] and the homotopy analysis method (HAM) [19] have been applied for fractional PDEs. Unfortunately, up to now, there are no general methods effective enough to solve the fractional order systems.

Using a new extended trial equation method, the authors of [20] considered the exact solutions of the generalized fractional Zakharov-Kuznetsov equations(FZK(p, q, r)) as follows:

$$D_t^\alpha u + a(u^p)_x + b(u^q)_{xxx} + c(u^r)_{xyy} = 0, \tag{5}$$

where $u = u(t, x, y)$ represents the electrostatic wave potential in plasmas, $0 < \alpha \leq 1$ is the order of fractional derivative. a, b, c are arbitrary constants and the coefficient of a is the nonlinear term, the coefficients of b and c characterize the spatial dispersions in multi-dimensions. $p, q, r \neq 0$ are integers. They successfully constructed some new exact solutions, i.e., the elliptic integral function F, Π solutions to (5). In particular, it is important to note that the symmetry analysis of (5) is not considered yet in [20].

As one of the most efficient and important methods of studying differential equations, the symmetry group theory has been extensively used to consider the symmetry properties of Zakharov-Kuznetsov equations, see [21–27] for example. Furthermore, the authors of [28] provided an interesting *Appendix* of how to proceed in the symmetry analysis of PDEs. Moreover, there have been excellent books about the symmetry analysis, one can be referred to [29–31].

However, not the same as it has been done in PDEs, the symmetry Lie group method is not so efficiently used in fractional differential equations (FDEs). To the best of our knowledge, there are lots of studies on the group properties of FDEs. In [32], the authors considered the Lie symmetries of a class of fractional order differential equations with an arbitrary number of independent variables. In [33], analysis of Lie symmetries with conservation laws of a (3+1)-dimensional fractional KdV-Zakharov-Kuznetsov (mKdV-ZK) equation has been considered. Some other results can be found in [34–37].

When it comes to fractional ZK equations, Lie symmetry analysis, conservation laws and exact solutions for a modified fractional (2+1)-mZK equation was considered in paper [38]. The equation reads as

$$D_t^\alpha u + u^2 u_x + u_{xxx} + u_{xyy} = 0. \tag{6}$$

Please note that Equation (6) is the special case of (5).

In view of the above discussion, we meant to study the group invariant properties of the fractional ZK Equation (5) studied in [20]. For the convenience of discussion, system (5) can be rewritten as follows

$$\begin{aligned} &D_t^\alpha u + pau^{p-1}u_x + b\left[\left(q^3 - 3q^2 + 2q\right) u^{q-3}u_x^3 + \left(3q^2 - 3q\right) u^{q-2}u_x u_{xx} + qu^{q-1}u_{xxx}\right] \\ &\quad + c\left[\left(r^3 - 3r^2 + 2r\right) u^{r-3}u_x u_y^2 + \left(r^2 - r\right) u^{r-2}u_x u_{yy} + \left(2r^2 - 2r\right) u^{r-2}u_y u_{xy} + ru^{r-1}u_{xyy}\right] \\ &\quad = 0 \end{aligned} \tag{7}$$

The organization of the rest of this paper is as follows: In Section 2, some preliminary results needed in later sections and some notations, definitions are stated. In Section 3, we establish our main results about Lie symmetries for the fractional ZK Equation (5). In Section 4, examples of group transformations of solutions are considered and new exact solutions are constructed. In Section 5, we study symmetry reductions to the time fractional ZK equation. Finally, we present discussions and conclusions.

2. Preliminaries

From the viewpoint of Brown motion, the modified Riemann-Liouville(RL) derivative is defined as

$$D_t^{\mu} f(t) = \begin{cases} \frac{\mathrm{d}^n f}{\mathrm{d}t^n}, & \mu = n, \\ \frac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\mu} f(t), & 0 \le n-1 < \mu < n, \end{cases} \tag{8}$$

where $n \in \mathbf{N}$, $I^{\nu} f(t)$ is the RL fractional integral of order ν, i.e.,

$$I^{\nu} f(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s) \mathrm{d}s, & \nu > 0, \\ f(t), & \nu = 0, \end{cases} \tag{9}$$

where $\Gamma(\bullet)$ is the Gamma function.

For the function $u(t,x)$, a Riemann-Liouville time fractional partial derivative of order μ can be defined as below [35]:

$$D_t^{\mu} u(t,x) = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \mu = n, \\ \frac{1}{\Gamma(n-\mu)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\mu-1} u(s,x) \mathrm{d}s, & 0 \le n-1 < \mu < n. \end{cases} \tag{10}$$

In this paper, we also need the following generalized definitions.

According to [32,39], the generalized *Erdélyi* − *Kober* fractional differential operator is defined as follows:

$$\mathcal{D}^{\delta,\alpha}_{\{\beta_1,\beta_2,\ldots,\beta_m\}} g := \prod_{j=0}^{n-1} \left(\delta + j + \sum_{k=1}^{m} -\frac{1}{\beta_k} \xi_k \frac{\partial}{\partial \xi_k} \right) \left(\mathcal{K}^{\delta+\alpha, n-\alpha}_{\{\beta_1,\beta_2,\ldots,\beta_m\}} g \right) (\xi_1, \xi_2, \ldots, \xi_m), \tag{11}$$

where

$$n = \begin{cases} [\alpha] + 1, & \alpha \notin \mathbf{N}, \\ \alpha, & \alpha \in \mathbf{N}, \end{cases}$$

and the generalized *Erdélyi* − *Kober* fractional integral operator reads

$$\begin{aligned} &\left(\mathcal{K}^{\delta,\alpha}_{\{\beta_1,\beta_2,\ldots,\beta_m\}} g \right) \quad (\xi_1, \xi_2, \ldots, \xi_m) \\ &= \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (\nu-1)^{\alpha-1} \nu^{-(\delta+\alpha)} g(\xi_1 \nu^{\frac{1}{\beta_1}}, \ldots, \xi_m \nu^{\frac{1}{\beta_m}}) \, \mathrm{d}\nu, & \alpha > 0, \\ g(\xi_1, \ldots, \xi_m), & \alpha = 0. \end{cases} \end{aligned} \tag{12}$$

In the following, we will give a short introduction about how to find Lie point symmetries of fractional order systems. For further information, one can refer to [30,31].

We now consider the following FDE

$$D_t^{\alpha} u(t,x,y) = F(t,x,y,u,u_x,u_y,u_{xx},u_{yy},u_{xy},u_{xxx},u_{xyy}), \quad 0 < \alpha < 1, \tag{13}$$

with t, x, y independent variables, u dependent variables.

According to the Lie theory, for some group parameter ε, we need to determine a one parameter Lie group of infinitesimal transformations

$$\begin{aligned} t^\star &= t + \varepsilon\tau(t,x,y,u) + O(\varepsilon^2), \\ x^\star &= x + \varepsilon\xi(t,x,y,u) + O(\varepsilon^2), \\ y^\star &= y + \varepsilon\eta(t,x,y,u) + O(\varepsilon^2), \\ u^\star &= u + \varepsilon\phi(t,x,y,u) + O(\varepsilon^2). \end{aligned} \tag{14}$$

The associated infinitesimal generator is defined by

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial u},$$

where

$$\tau = \left.\frac{\mathrm{d}t^\star}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}, \xi = \left.\frac{\mathrm{d}x^\star}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}, \eta = \left.\frac{\mathrm{d}y^\star}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}, \phi = \left.\frac{\mathrm{d}u^\star}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}.$$

According to the infinitesimal invariance criterion [31], Equation (13) admits transformation group (14) if and only if the following equation holds

$$\mathrm{pr}^{(\alpha,3)}X(\Delta)|_{\Delta=0} = 0, \Delta = D_t^\alpha u - F. \tag{15}$$

By keeping the essential terms, the operator $\mathrm{pr}^{(\alpha,3)}X$ takes the following form

$$\begin{aligned} \mathrm{pr}^{(\alpha,3)}X &= X + \phi^{\alpha,t}\frac{\partial}{\partial_{D_t^\alpha u}} + \phi^x\frac{\partial}{\partial u_x} + \phi^y\frac{\partial}{\partial u_y} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} \\ &\quad + \phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{yyy}\frac{\partial}{\partial u_{yyy}} + \phi^{xyy}\frac{\partial}{\partial u_{xyy}}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \phi^{\alpha,t} &= D_t^\alpha(\phi) + \xi D_t^\alpha(u_x) + \eta D_t^\alpha(u_y) - D_t^\alpha(\xi u_x) - D_t^\alpha(\eta u_y) \\ &\quad + D_t^\alpha(uD_t(\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u), \\ \phi^x &= D_x(\phi) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\eta), \\ \phi^y &= D_y(\phi) - u_t D_y(\tau) - u_x D_y(\xi) - u_y D_y(\eta), \\ \phi^{xx} &= D_x(\phi^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\eta), \\ \phi^{yy} &= D_y(\phi^y) - u_{yt} D_y(\tau) - u_{xy} D_y(\xi) - u_{yy} D_y(\eta), \\ \phi^{xy} &= D_y(\phi^x) - u_{xt} D_y(\tau) - u_{xx} D_y(\xi) - u_{xy} D_y(\eta), \\ \phi^{xxx} &= D_x(\phi^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi) - u_{xxy} D_x(\eta), \\ \phi^{xyy} &= D_x(\phi^{yy}) - u_{yyt} D_x(\tau) - u_{yxy} D_x(\xi) - u_{yyy} D_x(\eta), \\ \phi^{yyy} &= D_y(\phi^{yy}) - u_{yyt} D_y(\tau) - u_{yxy} D_y(\xi) - u_{yyy} D_y(\eta) \end{aligned} \tag{17}$$

Here, the total derivative operator D_i is defined by

$$D_i = \frac{\partial}{\partial x^i} + u_i^p\frac{\partial}{\partial u^p} + u_{ij}^p\frac{\partial}{\partial u_j^p} + \cdots, i,j = 1,2,3, p = 1, \tag{18}$$

and $(x^1, x^2, x^3) = (t, x, y), (u^1) = (u)$. We see that the explicit expression for the above ones can be obtained in a standard procedure [31]. In addition, according to [40], after similar calculation as [35], we can have the explicit expression for $\phi^{\alpha,t}$:

$$\begin{aligned}\phi^{\alpha,t} &= \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^n\phi_u - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]\partial_t^{\alpha-n}u - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)\partial_t^{\alpha-n}(u_x) \\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\eta)\partial_t^{\alpha-n}(u_y) + \partial_t^\alpha\phi + (\phi_u - \alpha D_t(\tau))\partial_t^\alpha u - u\partial_t^\alpha\phi_u + \mu_t^\alpha \end{aligned} \tag{19}$$

where

$$\mu_t^\alpha = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}(u^{k-r})\frac{\partial^{n-m+k}\phi}{\partial t^{n-m}\partial u^k}. \tag{20}$$

Because we will use *FracSym* [40] in our paper, as request, hereafter, we only consider symmetries where ϕ is linear in u (assume $\mu_t^\alpha = 0$), i.e.,

$$\phi(t, x, y, u) = uG(t, x, y) + H(t, x, y). \tag{21}$$

According to (15), by applying the operators $\mathrm{pr}^{(\alpha,3)}X$ to Equation (13), after splitting the obtained relations by independent variables, we obtain a system of linear PDEs and FDEs by equating these coefficients to zero. Finally, by solving this over-determined system, we can obtain the vector fields X admitted by FDE (13).

3. Lie Symmetries for the Generalized Fractional Zakharov-Kuznetsov Equation

Applying the third prolongation $\mathrm{pr}^{(\alpha,3)}X$ to (7), we obtain

$$\begin{aligned}\phi^{\alpha,t} &+ ap(p-1)\phi u^{p-2}u_x + apu^{p-1}\phi^x + b\Big[(q^3 - 3q^2 + 2q)[(q-3)\phi u^{q-4}u_x^3 + u^{q-3}(\phi^x)^3] \\ &+(3q^2 - 3q)[(q-2)\phi u^{q-3}u_x u_{xx} + u^{q-2}\phi^x u_{xx} + u^{q-2}u_x\phi^{xx}] \\ &+q[(q-1)\phi u^{q-2}u_{xxx} + u^{q-1}\phi^{xxx}]\Big] \\ &+c\Big[(r^3 - 3r^2 + 2r)[(r-3)\phi u^{r-4}u_x u_y^2 + u^{r-3}\phi^x u_y^2 + u^{r-3}u_x(\phi^y)^2] \\ &+(r^2 - r)[(r-2)\phi u^{r-3}u_x u_{yy} + u^{r-2}\phi^x u_{yy} + u^{r-2}u_x\phi^{yy}] \\ &+(2r^2 - 2r)[(r-2)\phi u^{r-3}u_y u_{xy} + u^{r-2}\phi^y u_{xy} + u^{r-2}u_y\phi^{xy}] \\ &+r[(r-1)\phi u^{r-2}u_{xyy} + u^{r-1}\phi^{xyy}]\Big] = 0. \end{aligned} \tag{22}$$

Substituting (17) into (22), by the Maple package [40,41], we can obtain the determining equations for the symmetry group. For simplicity, we omit the long expressions of the determining equations. In addition, by the DESOLVII PDE solver *pdesolv* [42], we obtain the general solution of determining equations with respect to τ, ξ, η, ϕ:

$$\begin{aligned}\tau(t, x, y, u) &= (3p - q - 2)c_4 t + c_3 \\ \eta(t, x, y, u) &= (p - r)\alpha c_4 y + c_2 \\ \xi(t, x, y, u) &= (p - q)\alpha c_4 x + c_1 \\ \phi(t, x, y, u) &= -2\alpha c_4 u \end{aligned} \tag{23}$$

where c_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

Furthermore, due to transformation (14), to preserve the invariance of the RL fractional derivative operator, we need

$$\tau(0,x,y,u) = 0 \Rightarrow c_3 = 0. \tag{24}$$

In fact, transformation (14) is required to leave the lower limit of the integral in the expression of (8), and therefore the equation $t = 0$ should keep the invariant form under this transformation.

Hence, the final symmetry for the time fractional ZK equation is:

$$\begin{aligned} \tau(t,x,y,u) &= (3p-q-2)c_4 t \\ \eta(t,x,y,u) &= (p-r)\alpha c_4 y + c_2 \\ \xi(t,x,y,u) &= (p-q)\alpha c_4 x + c_1 \\ \phi(t,x,y,u) &= -2\alpha c_4 u \end{aligned} \tag{25}$$

Finally, the symmetry group of the time fractional ZK equation is given by the follwing vector fields

$$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}, X_3 = (3p-q-2)t\frac{\partial}{\partial t} - 2u\alpha\frac{\partial}{\partial u} + (p-q)\alpha x\frac{\partial}{\partial x} + (p-r)\alpha y\frac{\partial}{\partial y}. \tag{26}$$

From (26), we can find the symmetry generators. They form a closed Lie algebra as shown in Table 1.

Table 1. Commutator table of Lie algebra (26).

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$(p-q)\alpha X_1$
X_2	0	0	$(p-r)\alpha X_2$
X_3	$-(p-q)\alpha X_1$	$-(p-r)\alpha X_2$	0

Here, the entry in row i and column j means $[X_i, X_j]$. It is the commutator for the Lie algebra defined by

$$[X_i, X_j] = X_i X_j - X_j X_i.$$

4. Examples of Group Transformations of Solutions

In this part, by solving the following initial problems, we can get the Lie symmetry group from the related symmetries to get some new exact solutions from the known ones.

$$\begin{aligned} \frac{dt^\star}{d\varepsilon} &= \tau, \quad t^\star|_{\varepsilon=0} = t, \\ \frac{dx^\star}{d\varepsilon} &= \xi, \quad x^\star|_{\varepsilon=0} = x, \\ \frac{dy^\star}{d\varepsilon} &= \eta, \quad y^\star|_{\varepsilon=0} = y, \\ \frac{du^\star}{d\varepsilon} &= \phi, \quad u^\star|_{\varepsilon=0} = u. \end{aligned} \tag{27}$$

Therefore, for the infinitesimal generator $X_1 = \frac{\partial}{\partial x}$, the corresponding Lie symmetry group are translation along the x-axis

$$g_1 : (t^\star, x^\star, y^\star, u^\star) \to (t, x+\varepsilon_1, y, u), \tag{28}$$

where ε_1 is an arbitrary real number. The group g_1 shows the space-invariance of the equation along the x-axis. Hence, if $u = f(t,x,y)$ is a solution of (5), by group g_1, we can obtain the corresponding new solutions of (5), i.e.,

$$u_1 = f(t, x-\varepsilon_1, y). \tag{29}$$

For $X_2 = \frac{\partial}{\partial y}$, the corresponding Lie symmetry group are translation along the y-axis

$$g_2 : (t^\star, x^\star, y^\star, u^\star) \rightarrow (t, x, y + \varepsilon_2, u), \tag{30}$$

where ε_2 is an arbitrary real number. The group g_2 shows the space-invariance of the equation along the y-axis. Hence, if $u = f(t, x, y)$ is a solution of (5), by group g_2, we can obtain the corresponding new solutions of (5), i.e.,

$$u_2 = f(t, x, y - \varepsilon_2). \tag{31}$$

In addition, $X_3 = (3p - q - 2)t\frac{\partial}{\partial t} - 2u\alpha\frac{\partial}{\partial u} + (p - q)\alpha x\frac{\partial}{\partial x} + (p - r)\alpha y\frac{\partial}{\partial y}$ corresponds to the nonhomogeneous scaling group

$$g_3 : (t^\star, x^\star, y^\star, u^\star) \rightarrow (te^{(3p-q-2)\varepsilon_3}, xe^{\alpha(p-q)\varepsilon_3}, ye^{\alpha(p-r)\varepsilon_3}, ue^{-2\alpha\varepsilon_3}), \tag{32}$$

where ε_3 is an arbitrary real number. The group g_3 is the well-known scaling transformations. Hence, if $u = f(t, x, y)$ is any solution of (5), by group g_3, we can obtain the corresponding new solutions of (5), that is,

$$u_3 = e^{2\alpha\varepsilon_3} f(te^{-(3p-q-2)\varepsilon_3}, xe^{-\alpha(p-q)\varepsilon_3}, ye^{-\alpha(p-r)\varepsilon_3}). \tag{33}$$

Moreover, the above scaling transformations either expand or contract the size of not only the independent variables but also the dependent ones. In addition, scaling transformation can provide a way to associate the behavior of the solution from different perspectives, for example, a short time solution with large initial values can be rescaled to a longer time solution with small initial values.

To illustrate this, we consider the following example discussed in [20]:

Consider $p = q = r = n$ in the Equation (5), i.e.,

$$D_t^\alpha u + a(u^n)_x + b(u^n)_{xxx} + c(u^n)_{xyy} = 0, \quad 0 < \alpha \leq 1. \tag{34}$$

The author of [20] obtained two exact solutions of (34):

$$\begin{aligned} u_1(t,x,y) &= \left[\exp\left[\tfrac{1}{A}\left(x + y - \tfrac{\lambda t^\alpha}{\Gamma(1+\alpha)} - \eta_0\right)\right] + \alpha_1\right]^{\frac{1}{n-1}}, \\ u_2(t,x,y) &= \left[\frac{\exp\left[\frac{1}{A}\left(x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} - \eta_0\right)\right] + (\alpha_1 - \alpha_2)^2 \exp\left[\frac{-1}{A}\left(x + y - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} - \eta_0\right)\right] + 2(\alpha_1 + \alpha_2)}{4}\right]^{\frac{1}{n-1}}. \end{aligned} \tag{35}$$

Then, one can obtain the following two new exact solutions of (34) by applying the group (32).

$$\begin{aligned} u_{13}(t,x,y) &= e^{2\alpha\varepsilon_3}\left[\exp(\Theta) + \alpha_1\right]^{\frac{1}{n-1}}, \\ u_{23}(t,x,y) &= e^{2\alpha\varepsilon_3}\left[\frac{\exp(\Theta) + (\alpha_1 - \alpha_2)^2 \exp(-\Theta) + 2(\alpha_1 + \alpha_2)}{4}\right]^{\frac{1}{n-1}}, \end{aligned} \tag{36}$$

where $\Theta = \frac{1}{A}\left(xe^{-\alpha(p-q)\varepsilon_3} + ye^{-\alpha(p-r)\varepsilon_3} - \frac{\lambda(te^{-(3p-q-2)\varepsilon_3})^\alpha}{\Gamma(1+\alpha)} - \eta_0\right)$. Compared with the existed solutions, we see that these new solutions are the size dilations of not only the independent variables but also the dependent ones.

For the other two symmetry groups, new invariant solutions can be found through existed solutions for the time fractional ZK equation. Maybe more interesting solutions from the physics point of view can be found by applying the full group to the equation. Thus we enrich the former results in [20].

5. Symmetry Reductions and Exact Solutions to the Time Fractional ZK Equation

In this section, we mainly consider the symmetry reductions to the time fractional ZK Equation.

(i) For the generator $X_1 = \frac{\partial}{\partial x}$, we have the invariant

$$u = g(\tau, \eta), \tag{37}$$

where $\tau = t, \eta = y$ are the group-invariant. Substituting it into (5) yields the following reduced fractional ODE

$$D^{\alpha}_{\tau} g(\tau, \eta) = 0. \tag{38}$$

To solve (38), we need the Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \mathrm{d}t \equiv F(s). \tag{39}$$

After taking the inverse transform to $F(s)$, we have the solution, which is

$$g(\tau, \eta) = f_4(\eta) \frac{C_0}{\Gamma(\alpha)} \tau^{\alpha-1}, \tag{40}$$

where $f_4(\eta)$ is an arbitrary function about η, C_0 is a constant.

Hence, for Equation (5), we give the following group-invariant solution:

$$u = f_4(y) \frac{C_0}{\Gamma(\alpha)} t^{\alpha-1}. \tag{41}$$

Please note that this solution can be viewed as a kind of standing wave solution and it is independent of the space variable x. In addition, due to $0 < \alpha < 1$, it decays in time. This solution has not appeared in previous papers.

(ii) For the generator $X_2 = \frac{\partial}{\partial y}$, we have the invariant

$$u = g(\tau, \xi), \tag{42}$$

where $\tau = t, \xi = x$ are the group-invariant. Substituting it into (5), we obtain the following reduced fractional differential equation

$$D^{\alpha}_{\tau} g + apg^{p-1} g_{\xi} + b\Big((q^3 - 3q^2 + 2q) g^{q-3} g_{\xi}^3 + (3q^2 - 3q) g^{q-2} g_{\xi} g_{\xi\xi} + q g^{q-1} g_{\xi\xi\xi} \Big) = 0. \tag{43}$$

(iii) For the generator $X_3 = (3p - q - 2)t\frac{\partial}{\partial t} - 2u\alpha\frac{\partial}{\partial u} + (p-q)\alpha x\frac{\partial}{\partial x} + (p-r)\alpha y\frac{\partial}{\partial y}$, by integrating the invariant condition

$$\frac{\mathrm{d}t}{(3p-q-2)t} = \frac{\mathrm{d}x}{(p-q)\alpha x} = \frac{\mathrm{d}y}{(p-r)\alpha y} = \frac{\mathrm{d}u}{-2u\alpha},$$

we obtain the invariant

$$u = g(\xi, \eta) t^{\frac{2\alpha}{3p-q-2}}, \tag{44}$$

where $\xi = xt^{-\frac{(p-q)\alpha}{3p-q-2}}, \eta = yt^{-\frac{(p-r)\alpha}{3p-q-2}}$ are the group-invariant.

Hereafter, for simplicity, we note $a_0 = -\frac{(p-q)\alpha}{3p-q-2}, b_0 = -\frac{(p-r)\alpha}{3p-q-2}, c_0 = \frac{2\alpha}{3p-q-2}$.

Substituting (44) into (5), for $0 < \alpha < 1$, according to the definition of RL fractional derivative, we obtain

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} [\int_0^t (t-s)^{-\alpha} s^{c_0} g(xs^{a_0}, ys^{b_0}) \, \mathrm{d}s]. \tag{45}$$

Let $\tau = \frac{t}{s}$, then $\mathrm{d}s = -\frac{t}{\tau^2}\mathrm{d}\tau$, then (45) can be rewritten as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}[t^{1+c_0-\alpha}\int_1^\infty (\tau-1)^{-\alpha}\tau^{-((1+c_0)+(1-\alpha))}g(\xi\tau^{-a_0},\eta\tau^{-b_0})\ \mathrm{d}\tau]. \tag{46}$$

By the generalized *Erdélyi* − *Kober* fractional integration operator, from (46) we have

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{\partial}{\partial t}\left[t^{1+c_0-\alpha}\left(\mathcal{K}^{1+c_0,1-\alpha}_{\{-\frac{1}{a_0},-\frac{1}{b_0}\}}g\right)(\xi,\eta)\right]\\ &= t^{c_0-\alpha}\left(1+c_0-\alpha+a_0\xi\frac{\partial}{\partial\xi}+b_0\eta\frac{\partial}{\partial\eta}\right)\left(\mathcal{K}^{1+c_0,1-\alpha}_{\{-\frac{1}{a_0},-\frac{1}{b_0}\}}g\right)(\xi,\eta).\end{aligned} \tag{47}$$

In addition, by the generalized *Erdélyi* − *Kober* fractional differential operator, (47) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{c_0-\alpha}\left(\mathcal{D}^{1+c_0-\alpha,\alpha}_{\{-\frac{1}{a_0},-\frac{1}{b_0}\}}g\right)(\xi,\eta). \tag{48}$$

Then, by (48), Equation (5) is reduced into the following nonlinear fractional partial differential equation

$$\begin{aligned}&\left(\mathcal{D}^{1+c_0-\alpha,\alpha}_{\{-\frac{1}{a_0},-\frac{1}{b_0}\}}g\right)(\xi,\eta)+apt^{c_0(p-1)+a_0+\alpha}g^{p-1}g_\xi\\ &+bqt^{c_0(q-1)+3a_0+\alpha}\left[(q-1)g^{q-3}((q-2)g_\xi+3gg_\xi g_{\xi\xi})+g^{q-1}g_{\xi\xi\xi}\right]\\ &+crt^{c_0(r-1)+a_0+2b_0+\alpha}\left[(r-1)g^{r-3}(gg_\xi g_{\eta\eta}+2gg_\eta g_{\xi\eta}+(r-2)g_\xi g_\eta^2)+g^{r-1}g_{\xi\eta\eta}\right]\\ &=0.\end{aligned} \tag{49}$$

For X_1, we have discussed the reduced equation (38) and obtained its solution (41). For X_3, we also have obtained the reduced equation (49). However, this equation is a nonlinear fractional partial differential equation with generalized Erdelyi-Kober fractional differential operator and it is difficult to discuss this FPDE.

In the following, we will do further study on symmetry reductions and exact solutions of (43) in detail.

If Equation (43) is invariant under the point transformations

$$\begin{aligned}\tau^* &= \tau+\epsilon\bar{\tau}(\xi,\tau,g)+O(\epsilon^2),\\ \xi^* &= \xi+\epsilon\bar{\xi}(\xi,\tau,g)+O(\epsilon^2),\\ g^* &= g+\epsilon\bar{g}(\xi,\tau,g)+O(\epsilon^2),\end{aligned} \tag{50}$$

with the group parameter ϵ, the associated Lie algebra is spanned by

$$V = \bar{\tau}(\xi,\tau,g)\frac{\partial}{\partial\tau}+\bar{\xi}(\xi,\tau,g)\frac{\partial}{\partial\xi}+\bar{g}(\xi,\tau,g)\frac{\partial}{\partial g}, \tag{51}$$

in which $\bar{\tau}(\xi,\tau,g), \bar{\xi}(\xi,\tau,g), \bar{g}(\xi,\tau,g)$ are to be determined.

If the vector fields above generates a symmetry of (43), we obtain the following Lie symmetry condition

$$\mathrm{pr}^{(\alpha,3)}V(\Delta_1)|_{\Delta_1=0} = 0, \tag{52}$$

where

$$\Delta_1 = D^\alpha_\tau g+apg^{p-1}g_\xi+b\Big((q^3-3q^2+2q)g^{q-3}g_\xi^3+(3q^2-3q)g^{q-2}g_\xi g_{\xi\xi}+qg^{q-1}g_{\xi\xi\xi}\Big).$$

As with the similar discussion given in previous sections, again by using the Maple package [31,40,41], we can have the symmetry algebra of (43), which is spanned by the following vector fields

$$V_1 = \frac{\partial}{\partial \xi}, V_2 = (3p - q - 2)\tau\frac{\partial}{\partial \tau} - 2g\alpha\frac{\partial}{\partial g} + (p - q)\alpha\xi\frac{\partial}{\partial \xi}. \tag{53}$$

For V_1, we obtain the group invariant

$$g = f_5(\tau). \tag{54}$$

Inserting it into (43), we have the following reduced fractional ODE

$$D_\tau^\alpha f_5(\tau) = 0, \tag{55}$$

which indicates that $f_5(\tau) = C_1\tau^{\alpha-1}$, where C_1 is a constant. As a result, we obtain a group-invariant solution of (5) of the form

$$u = C_1 t^{\alpha-1}. \tag{56}$$

This solution is only related to the time variable. In addition, it also decays in time.

For V_2, we have the group invariant

$$g = h(\tilde{\xi})\tau^{c_0} \tag{57}$$

where $\tilde{\xi} = \xi\tau^{a_0}, \tau = t, \xi = x$.

As the similar discussion in (iii), we can see that (43) can be reduced into the following fractional differential equation

$$\begin{aligned} &\left(\mathcal{D}^{1+c_0-\alpha,\alpha}_{\{-\frac{1}{a_0}\}} h\right)(\tilde{\xi}) + apt^{c_0(p-1)+a_0+\alpha} h^{p-1} h_{\tilde{\xi}} \\ &+bqt^{c_0(q-1)+3a_0+\alpha}\left[(q-1)h^{q-3}\left((q-2)h_{\tilde{\xi}} + 3hh_{\tilde{\xi}}h_{\tilde{\xi}\tilde{\xi}}\right) + h^{q-1}h_{\tilde{\xi}\tilde{\xi}\tilde{\xi}}\right] = 0, \end{aligned} \tag{58}$$

where $\mathcal{D}$ is the *Erdélyi* $-$ *Kober* fractional differential operator.

6. Conclusions

In this paper, by using the Lie symmetry analysis method, we have considered the invariance properties of a class of generalized fractional Zakharov-Kuznetsov equations. Lie point symmetries to this equation is performed. The Lie algebra and the symmetry reductions of this fractional Zakharov-Kuznetsov equation are obtained. Finally, some new exact solutions were constructed to the fractional Zakharov-Kuznetsov equation. Although there have been lots of symmetry results for the time fractional Zakharov-Kuznetsov equations, all those models considered can be viewed as the special cases of the one we considered in this paper. Hence, we have extended some existing results. However, as we see, the coefficients a, b and c are all normalized constants according to different physical meanings. To stay closely to the former research and to keep its specific physical meaning of parameters, therefore, we only consider the fractional Zakharov-Kuznetsov equation with constant coefficients. Furthermore, note that in [24], a class of generalized Zakharov-Kuznetsov equations with variable coefficients was considered. Perhaps we will pay our attention to this more generalized model for future studies.

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