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Disturbance Decoupling Problem: Logic-Dynamic Approach-Based Solution

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Abstract: This paper considers the disturbance decoupling problem by the dynamic measurement feedback for discrete-time nonlinear control systems. To solve this problem, the algebraic approach, called the logic-dynamic approach, is used. Such an approach assumes that the system description may contain non-smooth functions. Necessary and sufficient conditions are obtained in terms of matrices similar to controlled and (h, f)-invariant functions. Furthermore, procedures are developed to determine the corresponding matrices and the dynamic measurement feedback.

Keywords: nonlinear control systems; disturbance decoupling; measurement feedback control; discrete-time systems; algebraic approaches

1. Introduction

The dynamic disturbance decoupling problem (DDDP) for nonlinear dynamic systems has been addressed in a few papers [1–7], while hybrid systems and finite automata have been considered in [8,9]. Different from [6], the papers [1–5] consider the continuous-time case and the papers [1–3] provide the solvability conditions within differential geometric framework. In the earliest paper [3], the feedback considered to be the dynamic measurement feedback is restricted, whereas [1,2] consider the general case but provide either only necessary conditions [1] or make additional assumptions [2]. In [5], a sufficient algorithm-based condition for a single-input single-output system with a single measurement is suggested, applying the results (in terms of differential 1-forms) of the input–output linearization by dynamic output feedback [10]. Moreover, [11] addresses the case where the measured output is the same as the output-to-be-controlled. To summarize, the DDDP is old, but to date has had no full solution for different classes of systems.

In the present paper, we consider the DDDP for discrete-time nonlinear control systems, and the problem statement is similar to that of [3]. In particular, note that the controller is designed to be a suitable subsystem of the original system and the initial state of the compensator has to be chosen in accordance with that of the system. This type of controller reduces the dimension of the closed-loop system compared, for example, with those in [1,2,5] and has contact points with the 'regular interconnection' as addressed in [12]. Note that in the solutions of [1,2,5], the dimension of the closed-loop system is the sum of those of the plant and the controller whereas in this paper (and in [3]) it is equal to the state of the plant.

It is known that the extensions of the differential geometric tools for discrete-time systems are not as well developed and universally accepted as those for continuous-time systems. To overcome this difficulty, it is suggested to solve the DDDP on the basis of the so-called logic-dynamic approach (LDA). The LDA was developed in [13,14] to solve different problems of system theory. The advantages of the LDA are that it uses methods of linear algebra only by imposing some restrictions on the initial system and on a class of the obtained solutions. Furthermore, the LDA can be applied to systems with non-smooth nonlinearities for continuous-time as well as for discrete-time systems; finally, the problem of probabilistic decoupling [15] can also be solved based on the LDA.

2. Preliminaries

Consider a discrete-time nonlinear control system described by the equations

$$\begin{aligned} x(k+1) &= f(x(k), u(k), w(k)), \\ y(k) &= h(x(k)), \\ y_*(k) &= h_*(x(k)), \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, and $y \in R^l$ are vectors of the state, control, and measured output; $y_* \in R^L$ is the output-to-be-controlled; f, h, and h_* are nonlinear functions; $w(k) \in R^p$ is the unmeasurable disturbance. Note that f may be a non-smooth function.

The DDDP under a dynamic feedback can be formulated as follows: Find a vector function $x_0 = \alpha(x)$, $x_0 \in R^{n_0}$, $n_0 \leq n$, and a feedback of the form

$$\begin{aligned} x_0(k+1) &= f_0(x_0(k), y(k), u_0(k)), \\ u(k) &= g(x_0(k), y(k), u_0(k)), \end{aligned} \tag{2}$$

where $u_0 \in U_0 \subseteq R^m$ such that the values of the outputs $y_*(k)$, for $k \geq 0$, of the closed-loop system are invariant with respect to the disturbance $w(k)$.

Consider the main results from [6].

To solve the DDDP, a vector function α_0 with the maximal number of independent components is found at first, such that the function $\alpha_0(f(x(k), u(k), w(k)))$ is invariant with respect to the unknown function $w(k)$.

The function α is said to be (h, f)-invariant (or f-invariant) if $\alpha(f(x, u, w)) = f_*(\alpha(x), h(x), u, w)$ (or $\alpha(f(x, u, w)) = f_*(\alpha(x), u, w)$) for some function f_*. The function χ is a controlled invariant if a static state feedback $u = g'(x, u_0)$ exists such that the function χ in the closed-loop system is f-invariant.

Theorem 1 [6]**.** *The output $y_* = h_*(x)$ can be decoupled from the unknown function $w(k)$ by compensator (2) if and only if there exist (h, f)-invariant function α and a controlled invariant function χ such that*

$$\alpha_0 \leq \alpha \leq \chi \leq h_*. \tag{3}$$

Here, $\beta \leq \gamma$ means that the function δ exists such that $\delta(\beta(x)) = \gamma(x)$ for all x [6–8].

Our goal is to find a solution of the DDDP similar to (3) in a class of linear functions using only methods of linear algebra by imposing some limitations on system (1). Such a solution is based on the LDA.

3. Logic-Dynamic Approach

To implement the LDA, system (1) should be presented in the form

$$\begin{aligned} x(k+1) &= Fx(k) + Gu(k) + \Psi(x(k), u(k)) + Dw(k), \\ y(k) &= Hx(k), \\ y_*(k) &= H_*x(k), \end{aligned} \tag{4}$$

where

$$\Psi(x(k), u(k)) = C \begin{pmatrix} \varphi_1(A_1 x(k), u(k)) \\ \cdots \\ \varphi_q(A_q x(k), u(k)) \end{pmatrix}; \tag{5}$$

matrices F and G describe the linear dynamic part of the system; H, H_*, C, and D are constant matrices; the functions $\varphi_1, \ldots, \varphi_q$ may be non-smooth; $A_1, \ldots, A_q$ are row matrices. Model (4) can be derived from the initial system (1) by some transformations [13,14]. Specifically, we separate the linear part, described by the matrices F and G, from the nonlinear addend (5) which contains the nonlinear functions $\varphi_1, \ldots, \varphi_q$ and matrices $C, A_1, \ldots, A_q$.

By analogy with (2), a dynamic measurement feedback (compensator) S_0 is described by

$$\begin{aligned} x_0^+ &= F_0 x_0 + G_0 u + J_0 y + C_0 \begin{pmatrix} \varphi_1(A_{01} z_0, u) \\ \cdots \\ \varphi_q(A_{0q} z_0, u) \end{pmatrix}, \\ u &= g(x_0 y, u_0), \end{aligned} \tag{6}$$

where the vector u_0 is a new control, $x_0 \in R^{n_0}$, $n_0 \le n$, F_0, G_0, J_0, C_0, $A_{01}, \ldots, A_{0q}$ are matrices to be determined, and $z_0 = (x_0^{\mathrm{T}} y^{\mathrm{T}})^{\mathrm{T}}$. For simplicity, the notation x_0^+ is used for $x_0(k+1)$.

We assume initially that $q = 1$ and thus we can construct the compensator (6). The LDA, used to solve this problem, contains three steps [13,14].

Step 1. The nonlinear term is removed from the initial nonlinear system (4).

Step 2. The problem under consideration is solved for the linear part, obtained in Step 1, under some linear limitation. Such a limitation is used to find out whether or not the nonlinear term is designed on the basis of the linear solution obtained in this step.

Step 3. The solution, obtained in Step 2, is supplemented by the transformed nonlinear term.

Recall [6] that the function α in (3) has the maximal number of independent components and satisfies the condition $\alpha_0 \le \alpha$. To obtain a linear method-based solution, we assume that $x_0(k) = \alpha(x(k)) = \Phi x(k)$ for some matrix Φ of maximal rank, which satisfies the following conditions [14]:

$$\Phi F = F_0 \Phi + J_0 H, G_0 = \Phi G, \Phi D = 0. \tag{7}$$

It can be shown that the relations $C_0 = \Phi C$ and

$$A = A_0 \begin{pmatrix} \Phi \\ H \end{pmatrix}, \tag{8}$$

describing the nonlinear term, are true [14].

An analogue of the function α_0 is the matrix D^0 of maximal rank such that $D^0 D = 0$. Clearly, the condition $\Phi D = 0$ is equivalent to the relation $\Phi = R D^0$ for some matrix R; this relation is an analogue of the condition $\alpha_0 \le \alpha$.

The relation (8) holds if and only if the matrix A linearly depends on the matrices Φ and H. This implies that (8) is equivalent to

$$\operatorname{rank}(\Phi^{\mathrm{T}} H^{\mathrm{T}}) = \operatorname{rank}(\Phi^{\mathrm{T}} H^{\mathrm{T}} A^{\mathrm{T}}). \tag{9}$$

If $q > 1$, the matrix A in (8) and (9) is replaced with A_i, $i = 1, \ldots, q$.

We assume that the matrices F_0 and H_0 take the canonical form

$$F_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, H_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Here, the equation $\Phi F = F_0 \Phi + J_0 H$ is replaced by k equations:

$$\Phi_i F = \Phi_{i+1} + J_{0i} H, \ i = 1, \ldots, k-1, \ \Phi_k F = J_{0k} H, \tag{10}$$

where Φ_i and J_{0i} are the i-th rows of the matrices Φ and J_0, respectively; $i = 1, \ldots, k$; k is the number of the matrix Φ rows.

4. Problem Solution

4.1. Disturbance Decoupling for the Linear Part of a System

Find the matrix Φ of maximal rank such that $\Phi D = 0$. It was shown in [14] that (10) and the condition $\Phi D = 0$ can be changed to the single equation

$$(\Phi_1 - J_{01} - J_{02} \ldots - J_{0k})(W^{(k)} B^{(k)}) = 0, \tag{11}$$

where

$$W^{(k)} = \begin{pmatrix} F^k \\ HF^{k-1} \\ \cdots \\ H \end{pmatrix}, \; B^{(k)} = \begin{pmatrix} D & FD & \cdots & F^{k-1}D \\ 0 & HD & \cdots & HF^{k-1}D \\ \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

To obtain the system S_0 of maximal dimension, take $k := n - p$ and check the condition

$$\operatorname{rank}(W^{(k)} H^{(k)}) < lk + n. \tag{12}$$

When (12) is satisfied, then the row $(\Phi_1 - J_{01} \ldots - J_{0k})$ exists such that (11) is solvable. Then we can construct the matrix Φ based on (10) and set $G_0 := \Phi G$. Thus, the linear part of the system S_0 independent of the unknown function $w(k)$ is constructed; set $n_0 := k$.

If (12) is not satisfied, take $k := k - 1$ and continue checking (12). If (12) is not satisfied for all k, then the system S_0, independent of the disturbance, does not exist and the DDDP is not solvable. Since the dimension n_0 is maximal, the best choice for the function α in (3) is $\alpha(x) = \Phi x$.

4.2. Dynamic Part of the Compensator Design

Clearly, if (9) is true for the matrix Φ found in Step 2, then the problem of constructing the nonlinear system S_0 reduces to that for a linear system. When (9) is not true, find the maximal k for which (11) has several solutions in the form

$$(\Phi_1^{(1)} - J_{01}^{(1)} \ldots - J_{0k}^{(1)}), \ldots, (\Phi_1^{(N)} - J_{01}^{(N)} \ldots - J_{0k}^{(N)}), \tag{13}$$

where N is the number of all solutions.

Theorem 2 [14]. *Let $\Phi^{(1)}, \ldots, \Phi^{(N)}$ be matrices calculated on the basis of (10) and (11) and satisfying the condition (7). Then the linear combination of rows (13) with some coefficients $v_1, \ldots, v_N$ yields the matrix $\Phi = v_1 \Phi^{(1)} + \ldots + v_N \Phi^{(N)}$, satisfying the condition (7) as well.*

Let k be as maximal as possible, and solutions of (11) are presented in the form (13). To find the vector $v = (v_1 \ldots v_N)$, rewrite (8) in the form

$$A = A_{01}\Phi + A_{02}H, \tag{14}$$

where $A_0 = (A_{01} A_{02})$, $A_{01} = (a_1 \ldots a_k)$. Denote

$$\Phi_1^{\Sigma} = \begin{pmatrix} \Phi_1^{(1)} \\ \cdots \\ \Phi_1^{(N)} \end{pmatrix}, \quad \ldots, \quad \Phi_k^{\Sigma} = \begin{pmatrix} \Phi_k^{(1)} \\ \cdots \\ \Phi_k^{(N)} \end{pmatrix}, \quad \Phi_{\Sigma} = \begin{pmatrix} \Phi_1^{\Sigma} \\ \cdots \\ \Phi_k^{\Sigma} \end{pmatrix},$$

and present (14) in the form

$$A = A_{01} \begin{pmatrix} v\Phi_1^{\Sigma} \\ \dots \\ v\Phi_k^{\Sigma} \end{pmatrix} + A_{02}H. \tag{15}$$

Similar to (8), Equation (15) is solvable if

$$\mathrm{rank}((\Phi_\Sigma)^{\mathrm{T}} H^{\mathrm{T}}) = \mathrm{rank}((\Phi_\Sigma)^{\mathrm{T}} H^{\mathrm{T}} A^{\mathrm{T}}). \tag{16}$$

We propose that (16) is true and assume firstly that the matrix A has the only row. Here, (15) can be presented in the form $A = (a_1 v \dots a_k v)\Phi_\Sigma + A_{02}H$, or

$$A = A_v \Phi_\Sigma + A_{02}H, \tag{17}$$

where A_v is assumed to be an unknown matrix. Solve (17) and find the matrices A_v and A_{02}. If A_v can be rewritten in the form $(a_1 v \dots a_k v)$ for some coefficients $a_1, \dots, a_k$ and the vector $v = (v_1 \dots v_N)$, then stop—the matrices A_{01} and A_{02} and the vector $v = (v_1 \dots v_N)$ are obtained. Then, find the rows of J_0 and Φ by

$$J_{0j} = \sum\nolimits_{i=1}^{N} v_i J_{0j}^{(i)}, \;\; \Phi_j = \sum\nolimits_{i=1}^{N} v_i \Phi_j^{(i)}, \;\; j = 1, \dots, k;$$

set $G_0 := \Phi G$, $C_0 := \Phi C$. As a result, a dynamic part of the compensator (2) is built.

If (16) is not satisfied or the matrix A_v cannot be presented in the form $(a_1 v \dots a_k v)$, the dimension k must be decreased and the described procedure repeated.

If the matrix A has several rows, Equation (17) is solved for each row with coefficients $a_1, \dots, a_k$ particular to the considered row; note that the vector v is identical for all rows.

4.3. Function χ Design

Let $h_* = (h_{*1} \dots h_{*L})^{\mathrm{T}}$ and r_i, w_i be relative degrees of $y_{*i} = h_{*i}(x)$ with respect to $u(k)$ and $w(k)$, respectively [6]. Moreover, denote $y_{*i}(k) = h_{*i}(x(k)) =: h_{*i,1}(x(k)), \dots, y_{*i}(k + r_i - 1) =: h_{*i,r_i}(x(k))$, $i = 1, \dots, L$. When $h_*(x) = H_* x$, these relations are transformed as follows.

Define the matrix $C*$: If $C(i,k) \neq 0$ and φ_k contains some components of the input u, set $C*(i,k) := 1$, otherwise $C*(i,k) := 0$.

Denote by r_i' the minimal integer p such that $H_{*i}F^{p-1}G \neq 0$, by w_i the minimal integer p such that $H_{*i}F^{p-1}D \neq 0$, and by r_{i*} the minimal integer p such that $H_{*i}F^{p-1}C* \neq 0$, $i = 1, \dots, L$. It can be shown that r_i' and r_{i*} are the relative degrees of $y_{*i}(k)$ with respect to $u(k)$; clearly, they correspond to the linear and nonlinear terms of system (4), respectively. Set $r_i := \min(r_i', r_{i*})$, $i = 1, \dots, L$.

Assumption 1 [6]**.** *$w_i > r_i$ and $w_i > r_i'$ for all $i = 1, \dots, L$, otherwise the DDDP is not solved.*

It follows from the definition of r_i and Assumption 1 that $y_{*i}(k + r_i) = \hat{f}_i(x(k), u(k))$ for some function $\hat{f}_i$, and the function $\hat{f}_i(x(k), u(k))$ is invariant with respect to $w(k)$. Assume that $L \leq m$ and set $\hat{f}(x,u) := (\hat{f}_1(x,u), \dots, \hat{f}_L(x,u))^{\mathrm{T}}$.

Vector $(r_1, \dots, r_L)$ is said to be the vector relative degree of $y_*(k)$ if the condition $\mathrm{rank}(\partial \hat{f}_1(x,u)/\partial u = L$ is satisfied for all (x,u) except on a set of zero measure.

Assumption 2 [6]**.** *The output $y_*(k)$ has a vector relative degree $(r_1, \dots, r_L)$.*

Theorem 3 [6]**.** *Set*

$$\chi := \begin{pmatrix} h_{*1}^0 \\ \dots \\ h_{*L}^0 \end{pmatrix}, \tag{18}$$

where $h_{*i}^0 = (h_{*i,1} \dots h_{*i,r_i})^{\mathrm{T}}$, $i = 1, \dots, L$. Then, under Assumptions 1 and 2, χ is the controlled invariant function; it satisfies the inequality $\chi \leq h_*$ and has a minimal number of components.

To determine the function χ to be linear, the additional assumption is formulated.

Assumption 3. *$r_i = r_i'$ for all $i = 1, \ldots, L$. This means that all relative degrees correspond to the linear terms of system (4).*

Set $y_{*1} = H_{*1}x, \ldots, y_{*1}^{r_1} = H_{*1}F^{r_1-1}x$; clearly, the expression

$$y_{*1}^{r_1+} = H_{*1}F^{r_1-1}x^+ = H_{*1}F^{r_1-1}(Fx + Gu + \Psi(x,u)) = H_{*1}F^{r_1}x + H_{*1}F^{r_1-1}Gu + \psi_1(x)$$

contains the control $u(k)$. Here $\psi_1(x) = H_{*1}F^{r_1-1}\Psi(x,u)$; clearly, $\psi_1(x)$ is invariant with respect to $u(k)$ due to Assumption 3. It can be shown that $H_{*1}F^{r_1}x + H_{*1}F^{r_1-1}Gu + \psi_1(x)$ corresponds to the function $\hat{f}_1(x,u)$. Based on these expressions, produce the set of equations as follows:

$$\begin{gathered} H_{*1}F^{r_1}x + H_{*1}F^{r_1-1}Gu + \psi_1(x) = u_{01}, \\ \cdots \\ H_{*L}F^{r_L}x + H_{*L}F^{r_L-1}Gu + \psi_L(x) = u_{0L}. \end{gathered} \tag{19}$$

Set

$$H_*^{(i)} := \begin{pmatrix} H_{*i} \\ \cdots \\ H_{*i}F^{r_i-1} \end{pmatrix}, \quad i = 1, \ldots, L, \quad \hat{H}_* := \begin{pmatrix} H_{*1}F^{r_1-1}G \\ \cdots \\ H_{*L}F^{r_L-1}G \end{pmatrix}.$$

For the sake of simplicity, assume that $\operatorname{rank}(\hat{H}_*) = L$, which is equivalent to Assumption 2. Here, Equations (19) are solvable for the control u.

Set $\Phi_* := \left((H_*^{(1)})^{\mathrm{T}} \ldots (H_*^{(L)})^{\mathrm{T}}\right)^{\mathrm{T}}$. If the condition

$$\operatorname{rank}(\Phi_*) = \operatorname{rank}\begin{pmatrix} \Phi_*^{\mathrm{T}} & A^{\mathrm{T}} \end{pmatrix} \tag{20}$$

is true, then the nonlinear term (5) can be obtained starting from the linear part. Note that the matrix Φ_* corresponds to the function χ from (18). Thus, this matrix can be treated as a controlled invariant one for the linear terms of systems (4) and (6). Furthermore, $H_* = Q_*\Phi_*$ for some matrix Q_*; that is, the equality $H_* = Q_*\Phi_*$ is analogue to the condition $\chi \leq h_*$ in (3). It follows from the definition of relation $\leq$ that the condition $\alpha \leq \chi$ corresponds to the equality

$$\operatorname{rank}(\Phi) = \operatorname{rank}\begin{pmatrix} \Phi_*^{\mathrm{T}} & \Phi^{\mathrm{T}} \end{pmatrix}. \tag{21}$$

If (20) and (21) are true, then the DDDP can be solved; otherwise, a solution does not exist. Assume that (20) and (21) are satisfied, therefore $\Phi_* = Q\Phi$ for some matrix Q.

The solution of (19) is of the form $u = g'(x, u_0)$, which corresponds to the feedback in a static state form. Since $\Phi_* = Q\Phi$ and the matrix Φ corresponds to the (h, f)-invariant function, then x in $u = g'(x, u_0)$ can be replaced by the pair (x_0, y), where $x_0 = \Phi x$. Consequently, a static state form $u = g'(x, u_0)$ is transformed into a dynamic measurement form $u = g(x_0, y, u_0)$ for some function g.

If the condition $r_i = r_i'$ is not satisfied for some i, then the function ψ_i in (19) contains the variable $u(k)$. In this case the expressions for the functions g' and g take more complex forms.

4.4. Discussion

Thus, we have established some analogues: The function α_0 corresponds to the matrix D^0, the condition $\alpha_0 \leq \alpha$ in (3) to the equality $\Phi = RD^0$, the condition $\chi \leq h_*$ to $H_* = Q_*\Phi_*$, and the inequality $\alpha \leq \chi$ to (21). Note that the condition of the DDDP for continuous-time systems described by

$$\dot{x}(t) = f(x(t)) + g(x)u(t) + p(x)w(t) \tag{22}$$

is of the form $p \in \Delta \subset H_0^{\perp}$, where Δ is the controlled invariant distribution [4] for system (22), $H_0 = \mathrm{span}\{dh_1, \ldots, dh_m\}$. The solution for finite automata is of the form $\pi_0 \leq \pi_\alpha \leq \pi_\chi \leq \pi_*$ [9], where the partitions $\pi_0, \pi_\alpha, \pi_\chi, \pi_*$ correspond to the functions $\alpha_0, \alpha, \chi, h_*$, respectively. Thus, one can see that there are some correspondences between solutions for different classes of systems.

5. Example

Consider the control system

$$\begin{array}{ll} x_1^+ = x_3 + x_6 + x_4 + u_3 + d_1, & x_2^+ = \mathrm{sign}(x_3) + x_6 + u_1, \\ x_3^+ = -x_3 x_4, & x_4^+ = x_4 + x_5 + u_1, \\ x_5^+ = x_3 + x_4 + d_2, & x_6^+ = x_2^2 + x_1 + u_2, \\ y_1 = x_1, y_2 = x_5. & \end{array}$$

According to [14], these equations should be corrected by adding formal terms as follows: The term $x_3 - x_3$ is added in the second equation, $x_3 + x_4 - x_3 - x_4$ in the third, and $x_2 - x_2$ in the fifth. As a result, the matrices and nonlinearities describing the system are as follows:

$$F = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, G = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}^{\mathrm{T}}, C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\begin{array}{rll} \varphi_1(x,u) = \mathrm{sign}(A_1 x) - A_1 x, & A_1 = (0\,0\,1\,0\,0\,0), \\ \varphi_2(x,u) = A_1 x A_2 x + A_1 x + A_2 x, & A_2 = (0\,0\,0\,1\,0\,0), \\ \varphi_3(x,u) = (A_3 x)^2 - A_3 x, & A_3 = (0\,1\,0\,0\,0\,0). \end{array}$$

Calculate

$$D^0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can show that $\Phi = D^0$. Clearly, condition (9) is met; therefore, the nonlinearities in (6) can be obtained on the basis of the matrix Φ. Clearly, $C* = 0$, $H_* = \Phi$, and $L = 4$.

Next, find $r_1' = r_3' = r_4' = 1$, $r_2' = 2$, $r_{1*} = \ldots = r_{4*} = \infty$, $w_1 = w_2 = 3$, and $w_3 = w_4 = 2$; clearly, Assumptions 1 and 3 are met.

One can check that conditions (20) and (21) are satisfied; therefore, the DDDP is solvable. Compute

$$\hat{H}_* = \begin{pmatrix} H_{*1} F^{r_1 - 1} G \\ \cdots \\ H_{*L} F^{r_L - 1} G \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since $\mathrm{rank}(\hat{H}_*) = 2 < 4$, Assumption 2 is not met. Here, it is recommended to find the matrix P such that $\mathrm{rank}(P\hat{H}_*) = \mathrm{rank}(\hat{H}_*) = 2$; as a result,

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \hat{H}_* := P\hat{H}_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Clearly, Equations (19) with $\hat{H}_*$ are solvable for u_1, u_2. Set $u_{*1} := x_4 + x_5 + u_1$ and $u_{*2} := x_2^2 + x_1 + u_2$. Since $x_0 := \Phi x$, set $(x_{01}, x_{02}, x_{03}, x_{04})^{\mathrm{T}} := (x_2, x_3, x_4, x_6)^{\mathrm{T}}$ and find the system S_0:

$$x_{01}^+ = \mathrm{sign}(x_{02}) + x_{04} + u_1, \qquad x_{02}^+ = -x_{02}x_{03},$$
$$x_{03}^+ = x_{03} + y_2 + u_1, \qquad x_{04}^+ = x_{02}^2 + y_1 + u_2.$$

Then, replace (x_2, x_3, x_4, x_6) by $(x_{01}, x_{02}, x_{03}, x_{04})$ and obtain

$$u_{01} := x_{03} + y_2 + u_1, \quad u_{02} := x_{02}^2 + y_1 + u_2.$$

As a result, the function $u = g(x_0, y, u_0)$ in (6) is as follows:

$$u_1 = u_{01} - x_{03} - y_2, u_2 = u_{02} - x_{02}^2 - y_1, u_3 = u_{03}.$$

6. Conclusions

This paper deals with the DDDP for dynamic systems. The so-called logic-dynamic approach is used to solve the problem. The advantage of the LDA is that the system under consideration may contain non-smooth nonlinearities such as Coulomb friction, backlash, and saturation. Moreover, the LDA can be applied both for continuous-time and discrete-time systems. The DDDP solution can be used as a basis to solve the problem of faulty plant reconfiguration [7].

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