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On Ω Class of Mappings in a p -Cyclic Complete Metric Space

Erdal Karapinar ^{1,*}, Saravanan Karpagam ^{2,*}, Prabavathy Magadevan ³ and Boyan Zlatanov ^{4,*}

¹ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

² Department of Science and Humanities, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602 105, India

³ P. G and Research Department of Mathematics, A.D.M College for Women, Bharathidasan University, Nagapattinam 611 001, India; prabavathy09@gmail.com

⁴ University of Plovdiv, "Paisii Hilendarskiq", 24 Tzar Assen str., 4000 Plovdiv, Bulgaria

* Correspondence: erdalkarapinar@yahoo.com or karapinar@mail.cmuh.org.tw (E.K.); karpagam.saravanan@gmail.com (S.K.); bzlatanov@gmail.com (B.Z.)

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Abstract: In this manuscript, we introduce the concept of Ω -class of self mappings on a metric space and a notion of p -cyclic complete metric space for a natural number ($p \geq 2$). We not only give sufficient conditions for the existence of best proximity points for the Ω -class self-mappings that are defined on p -cyclic complete metric space, but also discuss the convergence of best proximity points for those mappings.

Keywords: p -cyclic maps; p -cyclic contractions; strict contractions; best proximity points

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1. Introduction

In the classical Banach fixed point theorem, the undertaking operator is necessarily continuous due to contraction inequality. This simple observation brings a natural question: Does a discontinuous contraction mapping possess a fixed point? The answer to this question is affirmative. Indeed, there are various approaches to overcome weakness of the discontinuous mapping for guaranteeing a fixed point. One of the significant results was constructed by Bryant [1] who proved the following result: In a complete metric space, if, for some positive integer $n \geq 2$, the n th iteration of the given mapping forms a contraction, then it possess a unique fixed point. Another outstanding approach was proposed by Kirk, Srinivasan and Veeramani [2] by introducing the notion of cyclic contraction. More precisely, every cyclic contraction in a complete metric space possess a unique fixed point. This statement is plain but significant when we compare with the results of Bryant. Attendantly, the concept of the cyclic contractions has been investigated densely by a considerable number of authors who bring several variants of the notion and derive a number of interesting results (see, e.g., [3–16] and the references therein).

Let there be a self-mapping on a metric space (X, d) . Suppose that A and B are non-empty subsets of X such that $X = A \cup B$. A self-mapping T on $A \cup B$ is called cyclic [2]

$$T(A) \subset B \text{ and } T(B) \subset A.$$

Further, a mapping T is called cyclic contraction [2] if there is a $k \in [0, 1)$ such that the following inequality is satisfied:

$$d(Tx, Ty) \leq kd(x, y), \text{ for all } x \in X_i, y \in X_{i+1}, i \in \{1, 2, \dots, m\}.$$

After this initial construction, several extension of cyclic mappings and cyclic contractions have been introduced. In this paper, we mainly follow the notations defined in [9].

2. Motivation

In [9], a notion of p -cyclic map is introduced. Let $B_1, B_2, \dots, B_p$ ($p \geq 2$) be non-empty sets. A p -cyclic map $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is defined such that $T(B_i) \subseteq B_{i+1}$, $\forall i \in \{1, 2, \dots, p\}$. $x = x_0 \in B_i$ defines a sequence $\{x_n\} \subseteq \cup_{i=1}^p B_i$ as $x_n = T(x_{n-1})$. Then, $\{x_{pn}\}$ is a subsequence in B_i , $\{x_{pn+1}\}$ is a subsequence in B_{i+1} and so on. The arrangement of such a sequence formed by a p -cyclic map motivated us to introduce a notion of p -cyclic sequence (Definition 6(1)). If B_i s are subsets of a metric space (M, ρ) , then we observe that, to obtain a best proximity point of T under various contractive conditions (some of them given in the literature), it is enough to prove that: given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\rho(x_{pn}, x_{pm+1}) < \text{dist}(B_i, B_{i+1}) + \epsilon, \quad \forall n, m \geq N_0.$$

This observation motivated us to introduce a concept of p -cyclic Cauchy sequence and p -cyclic complete metric space (Definition 6). In addition, while investigating the behavior of such p -cyclic maps, it is often the case that, if $\rho(x, y) > \text{dist}(B_i, B_{i+1})$, then $\rho(Tx, Ty) < \rho(x, y)$ and, if $\rho(x, y) = \text{dist}(B_i, B_{i+1})$, then $\rho(Tx, Ty) = \rho(x, y)$, $x \in B_i$, $y \in B_{i+1}$. This motivated us to call a p -cyclic map with this property as p -cyclic strict contraction map (Definition 7). Note that, if the distances between the adjacent sets are zero, then a p -cyclic strict contraction map is a strict contraction map in the usual sense. All such maps invariably satisfy the condition: $x, y \in B_i$, $\rho(T^{pn}x, T^{pn+1}y) \rightarrow \text{dist}(B_i, B_{i+1})$ as $n \rightarrow \infty$. In this paper, all p -cyclic maps which satisfy the above two properties are said to belong to class Ω (Definition 8). Finally, we prove the existence and convergence of best proximity points of Ω class of mappings in a p -cyclic complete metric space.

3. Preliminaries

In what follows, we give some definitions and fundamental results that are essential to understand and prove the main results.

Definition 1 ([9], Definitions 3.1). For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Define $B_{p+i} := B_i$, for all $i \in \{1, 2, \dots, p\}$. A map $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is called a **p -cyclic map** if $T(B_i) \subseteq B_{i+1}$, $\forall i \in \{1, 2, \dots, p\}$. If $p = 2$, the map T is called cyclic. A point $x \in B_i$ is said to be a **best proximity point** of T in B_i , if $\rho(x, Tx) = \text{dist}(B_i, B_{i+1})$, where $\text{dist}(B_i, B_{i+1}) := \inf\{\rho(x, y) : x \in B_i, y \in B_{i+1}\}$.

In this paper, we give the conditions for the underlying space and for the subsets of the space, to have a unique best proximity point under a p -cyclic map, if it exists, irrespective of the contraction condition imposed on the map.

Proposition 1. Let $B_1, B_2, \dots, B_p$, ($p \geq 2$) be non-empty convex subsets of a strictly convex norm linear space M such that $\text{dist}(B_i, B_{i+1}) > 0$, $i \in \{1, 2, \dots, p\}$. Let $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ be a p -cyclic map. Then, T has at most one best proximity point in B_i , $1 \leq i \leq p$.

Proof. Let $x, y \in B_i$ be such that $\|x - Tx\| = \text{dist}(B_i, B_{i+1}) = \|y - Ty\|$, $1 \leq i \leq p$. If $x \neq y$, then $\|\frac{x-Tx}{\text{dist}(B_i, B_{i+1})}\| = 1$ and $\|\frac{y-Ty}{\text{dist}(B_i, B_{i+1})}\| = 1$. Since M is strictly convex, $\|\frac{(x-Tx)+(y-Ty)}{2\text{dist}(B_i, B_{i+1})}\| < 1$. Thus, we get $\|(\frac{x+y}{2}) - (\frac{Tx+Ty}{2})\| < \text{dist}(B_i, B_{i+1})$, which is a contradiction. Hence, $x = y$. $\square$

Let T be a p -cyclic map as given in Definition 1.

T is said to be p -cyclic non expansive map if for all $x \in B_i, y \in B_{i+1}$, the following holds:

$$\rho(Tx, Ty) \leq \rho(x, y), \forall i \in \{1, 2, \dots, p\}.$$

The Lemma given below naturally follows for a p -cyclic non expansive map.

Lemma 1 ([9], Lemma 3.3). For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . If $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is a p -cyclic non-expansive map, then

$$\text{dist}(B_i, B_{i+1}) = \text{dist}(B_{i+1}, B_{i+2}) = \text{dist}(B_1, B_2), \forall i \in \{1, 2, \dots, p\}. \quad (1)$$

In addition, if $\xi \in B_i \cap \mathcal{B}(T)_i \neq \emptyset$, then $T^j \xi \in B_{i+j} \cap \mathcal{B}(T)_{i+j} \neq \emptyset$, for all $j = 1, 2, \dots, (p-1)$, where $\mathcal{B}(T)_k$ is the set of best proximity point of the mapping T in B_k .

In [5], the following lemma is proved, which is again proved here. This lemma is crucial to prove that a given sequence is Cauchy.

Lemma 2 ([5], Lemma 3.7). For a uniformly convex Banach space $(X, \|\cdot\|)$, we suppose that C, D are non-empty closed subsets of X and $\{a_n\}, \{b_n\} \subset C$ and $\{d_n\} \subset D$. If C is convex such that

- (i) $\|a_n - d_n\| \rightarrow \text{dist}(C, D)$; and
- (ii) for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$, $\|a_m - d_n\| \leq \text{dist}(C, D) + \epsilon$,

then for all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $m > n \geq N_1$, $\|a_m - b_n\| \leq \epsilon$.

Proof. If $\text{dist}(C, D) = 0$, then, for a given $\epsilon > 0$, we can find positive integers N_2 and N_0 such that $\|b_n - d_n\| \leq \frac{\epsilon}{2}$ for all $n \geq N_2$ and $\|a_m - d_n\| \leq \frac{\epsilon}{2}$ for all $m > n > N_0$. Now, choosing $N_1 = \max(N_0, N_2)$, then

$$\|a_m - d_n\| \leq \|a_m - d_n\| + \|b_n - d_n\| \leq \epsilon, \text{ for all } m > n > N_1.$$

Suppose $\text{dist}(C, D) > 0$; assuming the contrary, there exists $\epsilon_0 > 0$ such that, for every $k \in \mathbb{N}$, there exists $m_k > n_k \geq k$, for which $\|a_{m_k} - d_{n_k}\| \geq \epsilon_0$. Choose $0 < \gamma < 1$ such that $\epsilon_0/\gamma > \text{dist}(C, D)$ and choose ϵ such that $0 < \epsilon \min\left(\frac{\epsilon_0}{\gamma} - \text{dist}(C, D), \frac{\text{dist}(C, D)\delta(\gamma)}{1-\delta(\gamma)}\right)$, where $\delta(\gamma)$ is the modulus of convexity. For this $\epsilon > 0$, there exists N_0 such that for all $m_k > n_k \geq N_0$, $\|a_{m_k} - d_{n_k}\| < \text{dist}(C, D) + \epsilon$. In addition, there exists N_2 such that $\|b_{n_k} - d_{n_k}\| < \text{dist}(C, D) + \epsilon$ for all $n_k \geq N_2$. Choose $N_1 = \max(N_0, N_2)$. By uniform convexity, for all $m_k > n_k \geq N_1$,

$$\|\frac{a_{m_k} + d_{n_k}}{2} - d_{n_k}\| \leq \left(1 - \delta\left(\frac{\epsilon_0}{\text{dist}(C, D) + \epsilon}\right)\right) (\text{dist}(C, D) + \epsilon).$$

Since

$$\epsilon < \frac{\epsilon_0}{\gamma} - \text{dist}(C, D) \Rightarrow \frac{\epsilon_0}{\text{dist}(C, D) + \epsilon} > \gamma,$$

which implies $1 - \delta \left(\frac{\epsilon_0}{\text{dist}(C, D) + \epsilon} \right) < 1 - \delta(\gamma)$,

$$\begin{aligned} \text{III } \left\| \frac{a_{m_k} + d_{n_k}}{2} - d_{n_k} \right\| &\leq (1 - \delta(\gamma))(\text{dist}(C, D)) + \epsilon \\ &\leq (1 - \delta(\gamma)) \left(\text{dist}(C, D) + \frac{\text{dist}(C, D)}{\delta}(\gamma)1 - \delta(\gamma) \right) \\ &< \text{dist}(C, D), \end{aligned}$$

which is a contradiction. Hence, the lemma holds. $\square$

Next, we recall few p -cyclic maps with some contraction conditions imposed on them, which are defined in [3,9,10].

Definition 2 ([10], Definition 3.1). For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Let $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ be a p -cyclic map, T is said to be p -cyclic contraction, if there exists $k \in (0, 1)$ such that for all $x \in B_i$ and $y \in B_{i+1}$, we have

$$\rho(Tx, Ty) \leq k\rho(x, y) + (1 - k)\text{dist}(B_i, B_{i+1}), \quad \forall i \in \{1, 2, \dots, p\}.$$

Definition 3 ([9], Definition 3.5). Let $B_1, B_2, \dots, B_p$ ($p \geq 2$) be non-empty subsets of a metric space (M, ρ) . A p -cyclic map $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is said to be p -cyclic MK-contraction, if for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\rho(x, y) < \text{dist}(B_i, B_{i+1}) + \epsilon + \delta$, we have

$$\rho(Tx, Ty) < \text{dist}(B_i, B_{i+1}) + \epsilon,$$

where $x \in B_i, y \in B_{i+1}$ and $1 \leq i \leq p$.

Definition 4 ([17], Definition 2). Let $\psi : [0, \infty) \rightarrow [0, \infty)$ be a map such that $\psi(0) = 0$ and $\psi(\vartheta) > 0$ if $\vartheta > 0$. We say that ψ is an L -function if for all $\vartheta > 0$ there exists $\delta > 0$ such that $\psi(t) \leq \vartheta$ for all $t \in [\vartheta, \vartheta + \delta]$.

Definition 5 ([3], Definition 2.1). For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and C and D are non-empty subsets of M . A cyclic map $T : C \cup D \rightarrow C \cup D$ is said to be cyclic φ -contraction if

$$\rho(Tx, Ty) \leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\text{dist}(C, D)) \quad \text{for all } x \in C, y \in D,$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a strictly increasing map.

4. p -Cyclic Sequence and p -Cyclic Complete Metric Space

In this article, $\mathbb{N}_0$ refers to $\mathbb{N} \cup \{0\}$. The notion of p -cyclic sequence is given as follows:

Definition 6. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M .

1. A sequence $\{x_n\}_{n=1}^\infty \subseteq \cup_{i=1}^p B_i$ is called a p -cyclic sequence if $x_{pn+i} \in B_i$, for all $n \in \mathbb{N}_0$ and $i = 1, 2, \dots, p$.
2. We say that $\{x_n\}_{n=1}^\infty$ is a p -cyclic Cauchy sequence, if for given $\epsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that for some $i \in \{1, 2, \dots, p\}$, we have

$$\rho(x_{pn+i}, x_{pm+i+1}) < \text{dist}(B_i, B_{i+1}) + \epsilon, \quad \forall m, n \geq N_0. \quad (2)$$

3. A p -cyclic sequence $\{x_n\}_{n=1}^\infty$ in $\cup_{i=1}^p B_i$ is said to be p -cyclic bounded, if $\{x_{pn+i}\}_{n=0}^\infty$ is bounded in B_i for some $i \in \{1, 2, \dots, p\}$.
4. Let $\{x_n\}_{n=1}^\infty$ be a p -cyclic sequence in $\cup_{i=1}^p B_i$. If for some $j \in \{1, 2, \dots, p\}$ the subsequence $\{x_{pn+j}\}$ of $\{x_n\}_{n=1}^\infty$ converges in B_j , then we say that $\{x_n\}_{n=1}^\infty$ is p -cyclic convergent.
5. Under the assumption that $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty closed subsets of a metric space (M, ρ) , we say that $\cup_{i=1}^p B_i$ is **p -cyclic complete** if every p -cyclic Cauchy sequence in $\cup_{i=1}^p B_i$ is p -cyclic convergent.
6. If there are subsets $B_1, B_2, \dots, B_p$ ($p \geq 2$) of (M, ρ) such that $M = \cup_{i=1}^p B_i$ and $\cup_{i=1}^p B_i$ is p -cyclic complete, then we call (M, ρ) is p -cyclic complete.

Remark 1. Note that a p -cyclic sequence that is a Cauchy sequence in the usual sense is a p -cyclic Cauchy sequence. On the other hand, p -cyclic Cauchy sequences need not be Cauchy sequences in the usual sense, even if $\text{dist}(B_i, B_{i+1}) = 0 \ \forall i \in \{1, 2, \dots, p\}$.

The following examples illustrate the notion of p -cyclic sequence and p -cyclic Cauchy sequence.

Example 1. Consider $\mathbb{R}$ with the usual metric. Let $I_1 = [0, 1]$, $I_2 = [1, 2]$ and $I_3 = [2, 3]$. The sequence $\{x_n\}_{n=1}^\infty$ defined by $x_n = n - 1 \pmod{3} + \frac{1}{n}$, $n = 1, 2, 3, \dots$ is a three-cyclic sequence in $\cup_{i=1}^3 I_i$ but not a three-cyclic Cauchy sequence.

Example 2. Let $X = \mathbb{R}^2$ be a Euclidean space. Let the subsets B_i , $i = 1, 2, 3, 4$ be as follows:

$$B_1 = \{(0, 1+x) : 0 \leq x \leq 1\}, \quad B_2 = \{(1+x, 0) : 0 \leq x \leq 1\},$$

$$B_3 = \{(0, -(1+x)) : 0 \leq x \leq 1\} \quad \text{and} \quad B_4 = \{(-(1+x), 0) : 0 \leq x \leq 1\}.$$

Then, $\text{dist}(B_i, B_{i+1}) = \sqrt{2}$, for $i = 1, 2, 3, 4$, where $B_5 = B_1$.

Let us define a sequence $\{(x_n, y_n)\}_{n=1}^\infty$ in $\cup_{i=1}^4 B_i$ as follows:

$$(x_n, y_n) = \begin{cases} (0, 1 + \frac{1}{t^n}), & n = 4k + 1, \ k = 0, 1, 2, 3, \dots; \\ (1 + \frac{1}{t^n}, 0), & n = 4k + 2, \ k = 0, 1, 2, 3, \dots; \\ (0, -(1 + \frac{1}{t^n})), & n = 4k + 3, \ k = 0, 1, 2, 3, \dots; \\ (-(1 + \frac{1}{t^n}), 0), & n = 4k + 4, \ k = 0, 1, 2, 3, \dots \end{cases}$$

where $t \in \mathbb{N}$ and $t > 1$.

Then, $\{(x_n, y_n)\}_{n=1}^\infty$ is a four-cyclic Cauchy sequence in $\cup_{i=1}^4 B_i$.

The following Proposition shows that a p -cyclic Cauchy sequence is p -cyclic bounded.

Proposition 2. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Then, every p -cyclic Cauchy sequence in $\cup_{i=1}^p B_i$ is p -cyclic bounded.

Proof. Let $\{x_n\}_{n=1}^\infty$ be a p -cyclic Cauchy sequence in $\cup_{i=1}^p B_i$. Then, for some $i \in \{1, 2, \dots, p\}$, there exists an $N \in \mathbb{N}$ such that

$$\rho(x_{pn+i}, x_{pN+i+1}) < 1 + \text{dist}(B_i, B_{i+1}) \quad \text{for all } n \geq N.$$

Therefore, for all $n \in \mathbb{N}$, $x_{pn+i} \in B(x_{pN+i+1}, r)$ where

$$r = \max\{1 + \text{dist}(B_i, B_{i+1}), \rho(x_{pn+1}, x_{pN+i+1}) : n = 1, 2, 3, \dots, N\}.$$

Thus, $\{x_{pn+i}\}_{n=0}^\infty$ is bounded for some $i \in \{1, 2, \dots, p\}$. Hence, $\{x_n\}_{n=1}^\infty$ is p -cyclic bounded. $\square$

Remark 2. A complete metric space need not be p -cyclic complete. For example, let us consider $(l^\infty, \|\cdot\|_{sup})$ and let $X = \cup_{i=1}^p B_i$, where

$$B_i = \{e_{pn+i}, n \in \mathbb{N}_0\}, i = 1, 2, 3, \dots, p,$$

and $\{e_n\}$ is a sequence whose n th term is 1 and all the other terms are zero. Then, $B_1, B_2, \dots, B_p$ are closed subsets of l^∞ and hence (M, ρ) is complete. Further, $\text{dist}(B_i, B_{i+1}) = 1$ for all $i = 1, 2, \dots, p$. Since

$$\|e_{pn+i} - e_{pm+i+1}\| = 1 = \text{dist}(B_i, B_{i+1}), \quad n, m \in \mathbb{N}_0, i = 1, 2, \dots, p.$$

Then, the sequence $\{e_n\}_{n=1}^\infty$ is a p -cyclic Cauchy sequence in X . However, none of the subsequence $\{e_{pn+i}\}_{n=0}^\infty$ of $\{e_n\}_{n=1}^\infty$ converges in B_i for all $i = 1, 2, \dots, p$. Hence, (M, ρ) is not p -cyclic complete.

The following Proposition is an example of two-cyclic complete metric space.

Proposition 3. Let A_1 and A_2 be subsets of a uniformly convex Banach space X , which are non-empty and closed. If either A_1 or A_2 is convex, then $A_1 \cup A_2$ is two-cyclic complete.

Proof. Let $\{x_n\}$ be a two-cyclic Cauchy sequence in $A_1 \cup A_2$. Then, $\{x_{2n+1}\}_{n=0}^\infty \subseteq A_1$ and $\{x_{2n}\}_{n=1}^\infty \subseteq A_2$. Assume that A_1 is convex. Since $\{x_n\}$ is a two-cyclic Cauchy sequence in $A_1 \cup A_2$, for $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\|x_{2m+1} - x_{2n+2}\| < \text{dist}(A_1, A_2) + \epsilon, \quad m > n \geq n_0. \quad (3)$$

In addition, since $\|x_{2n+1} - x_{2n+2}\| < \text{dist}(A_1, A_2) + \epsilon$, $n \geq n_0$, we have $\|x_{2n+1} - x_{2n+2}\| \rightarrow \text{dist}(A_1, A_2)$ as $n \rightarrow \infty$. Let $n = n_1 = n_0 + 1$ and $m = m_1 = n_0 + s + 1$ where $s \in \mathbb{N}$ be the first element satisfying Equation (3). Then, $\{x_{2n_i+1}\}_{i=1}^\infty$ and $\{x_{2m_i+1}\}_{i=1}^\infty$ are two sequences in A_1 and $\{x_{2n_i+2}\}_{i=1}^\infty$ is a sequence in A_2 satisfying the following:

(i) For given $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that

$$\|x_{2m_i+1} - x_{2n_i+2}\| < \text{dist}(A_1, A_2) + \epsilon, \quad m_i > n_i \geq N_1.$$

(ii) $\|x_{2n_i+1} - x_{2n_i+2}\| \rightarrow \text{dist}(A_1, A_2)$ as $i \rightarrow \infty$.

Thus, Conditions (i) and (ii) of Lemma 2 are satisfied. Since A_1 is convex, by Lemma 2, there exists $N_2 \in \mathbb{N}$ such that

$$\|x_{2n_i+1} - x_{2m_i+1}\| \leq \epsilon, \quad m_i > n_i \geq N_2.$$

Choosing $N_2 > n_0 + s$, we have

$$\|x_{2n+1} - x_{2m+1}\| \leq \epsilon, \quad m > n \geq N_2.$$

Hence, $\{x_{2n+1}\}_{n=1}^\infty$ is a Cauchy sequence in A_1 . Therefore, $\{x_{2n+1}\}_{n=1}^\infty$ converges in A_1 . It yields that $\{x_n\}_{n=1}^\infty$ is two-cyclic convergent and hence $A_1 \cup A_2$ is two-cyclic complete.

Similarly, we can prove that $A_1 \cup A_2$ is two-cyclic complete if A_2 is convex. $\square$

5. p -Cyclic Strict Contraction Maps

We introduce a notion of p -cyclic strict contraction, which is a generalization of strict contraction in the usual sense.

Definition 7. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . A p -cyclic map T is said to be **p -cyclic strict contraction** if, for all $x \in B_i, y \in B_{i+1}, 1 \leq i \leq p$:

- (i) $\rho(x, y) > \text{dist}(B_i, B_{i+1}) \Rightarrow \rho(Tx, Ty) < \rho(x, y)$; and
- (ii) $\rho(x, y) = \text{dist}(B_i, B_{i+1}) \Rightarrow \rho(Tx, Ty) = \rho(x, y)$.

Remark 3. Note that, if $B_i = A$, for all $i = 1, 2, \dots, p$, then p -cyclic strict contraction is a strict contraction in the usual sense. It is clear that the p -cyclic strict contraction also forms a p -cyclic non-expansive map.

The following Proposition proves an important property of p -cyclic strict contraction map.

Proposition 4. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Let $x \in B_i$ ($1 \leq i \leq p$). Suppose that $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is a p -cyclic strict contraction map and if for all $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$d(T^{pn}x, T^{pm+1}x) < \text{dist}(B_i, B_{i+1}) + \epsilon, \quad n, m \geq n_0, \quad (4)$$

then for a given $\epsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$d(T^{pn+k}x, T^{pm+k+1}x) < \text{dist}(B_{i+k}, B_{i+k+1}) + \epsilon, \quad n, m \geq n_1, \quad k \in \{1, 2, \dots, p\}.$$

Proof. Let $x \in B_i$ be such that $\{T^{pn}x\}$ satisfies Equation (4) ($1 \leq i \leq p$) and let $\epsilon > 0$ be given. Since T is p -cyclic non-expansive, for any $m, n \in \mathbb{N}$ with $m > n, k \in \{1, 2, \dots, p\}$, we have

$$\begin{aligned} d(T^{pn+k}x, T^{pm+k+1}x) &\leq d(T^{pn}x, T^{pm+1}x) \\ &< \text{dist}(B_i, B_{i+1}) + \epsilon, \quad n, m \geq n_0 \quad (\text{by Equation (4)}) \\ &= \text{dist}(A_{i+k}, A_{i+k+1}) + \epsilon, \quad n, m \geq n_0 \quad (\text{by Equation (1)}). \end{aligned}$$

□

6. Ω Class of Mappings

Many p -cyclic maps with various contractive conditions possess some common properties listed in the following definition. Thus, we introduce a notion of class Ω .

Definition 8. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . A p -cyclic map $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is said to belong to the class Ω if

1. T is p -cyclic strict contraction.
2. If $x, y \in B_i$, then $\lim_{n \rightarrow \infty} \rho(T^{pn}x, T^{pn+1}y) = \text{dist}(B_i, B_{i+1}), 1 \leq i \leq p$.

We list some p -cyclic maps that belong to the class Ω . First, we prove that a p -cyclic contraction map, which is defined in [10] (Definition 2), belongs to the class Ω .

Example 3. Let $B_1, B_2, \dots, B_p$ be non-empty subsets of a metric space (M, ρ) . Let $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is a p -cyclic contraction map. Then, $T \in \Omega$.

Proof. Because the map T is a p -cyclic contraction, we have

$$\rho(Tx, Ty) \leq k\rho(x, y) + (1 - k)\text{dist}(B_i, B_{i+1}), \quad x \in B_i, y \in B_{i+1}, i = 1, 2, \dots, p,$$

for some $k \in (0, 1)$. If $\rho(x, y) = \text{dist}(B_i, B_{i+1})$, then $\rho(Tx, Ty) = \rho(x, y)$. In addition, if $\rho(x, y) > \text{dist}(B_i, B_{i+1})$, then

$$\begin{aligned} \rho(Tx, Ty) &< k\rho(x, y) + (1 - k)\rho(x, y) \\ &= \rho(x, y). \end{aligned}$$

Therefore, T is p -cyclic strict contraction. The second condition of Definition 8 follows from Lemma 3.3 in [10]. Hence, $T \in \Omega$. $\square$

Next, we prove that the p -cyclic Meir–Keeler map (p -cyclic MK-map) introduced in [9] (Definition 3) belongs to the class Ω .

Example 4. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Let $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ be a p -cyclic MK-contraction map. Then, $T \in \Omega$.

Proof. From Remark 3.6 in [9], we know that T is a p -cyclic MK-contraction if and only if there is an L -function ψ such that for all $x \in B_i, y \in B_{i+1}$ and $1 \leq i \leq p$, the following conditions hold:

(i) If $\rho(x, y) - \text{dist}(B_i, B_{i+1}) > 0$, then

$$\rho(Tx, Ty) - \text{dist}(B_i, B_{i+1}) < \psi(\rho(x, y) - \text{dist}(B_i, B_{i+1})) \leq \rho(x, y) - \text{dist}(B_i, B_{i+1}).$$

(ii) If $\rho(x, y) - \text{dist}(B_i, B_{i+1}) = 0$, then $\rho(Tx, Ty) - \text{dist}(B_i, B_{i+1}) = 0$.

Therefore, T is p -cyclic strict contraction and Condition (2) of Definition 8 follows from Lemma 3.8 in [9]. Thus, $T \in \Omega$. $\square$

Now, we prove that cyclic φ -contraction introduced in [3] (Definition 5) belongs to the class Ω .

Example 5. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and C, D are non-empty subsets of M . Let $T : C \cup D \rightarrow C \cup D$ be a cyclic φ -contraction map. Then, $T \in \Omega$.

Proof. Let $x \in C$ and $y \in D$. If $\rho(x, y) > \text{dist}(C, D)$, since φ is strictly increasing,

$$\begin{aligned} \rho(Tx, Ty) &\leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\text{dist}(C, D)) \\ &< \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\rho(x, y)) \\ &= \rho(x, y). \end{aligned}$$

If $\rho(x, y) = \text{dist}(C, D)$, then $\rho(Tx, Ty) \leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\rho(x, y)) = \rho(x, y)$. Therefore, we get $\rho(x, y) = \text{dist}(C, D) \leq \rho(Tx, Ty) \leq \rho(x, y)$. It yields that $\rho(Tx, Ty) = \rho(x, y)$. Thus, T is a two-cyclic strict contraction and Condition (2) of Definition 8 follows from Theorem 3 in [3] (by putting $d_n = \rho(T^{2n}x, T^{2n+1}y)$, $x, y \in C$ or $x, y \in D$). Hence, $T \in \Omega$. $\square$

Next, we establish an example of p -cyclic map satisfying a contraction condition of Geraghty's type [18] and show that it belongs to the class Ω . Here, we use a class of functions $\mathbb{S}$ introduced by

Geraghty [18], where, if $\mathbb{S}$ is the class of all functions $\vartheta : [0, \infty) \rightarrow [0, 1)$ that satisfies $\vartheta(t_n) \rightarrow 1$, then $t_n \rightarrow 0$, $t_n \in [0, \infty)$ for $n \in \mathbb{N}$.

Example 6. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Let $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ be a p -cyclic map such that for some $\vartheta \in \mathbb{S}$,

$$\rho(Tx, Ty) \leq \vartheta(\rho(x, y))\rho(x, y) + (1 - \vartheta(\rho(x, y)))\text{dist}(B_i, B_{i+1}), \quad x \in B_i, y \in B_{i+1}.$$

Then,

- (a) T is a p -cyclic strict contraction.
- (b) $\lim_{n \rightarrow \infty} \rho(T^{pn}x, T^{p(n+1)}y) = \text{dist}(B_i, B_{i+1})$, $x, y \in B_i$.

Proof. (a) Let $x \in B_i$, $y \in B_{i+1}$.

Case (1): If $\rho(x, y) > \text{dist}(B_i, B_{i+1})$, we have

$$\begin{aligned} \rho(Tx, Ty) &\leq \vartheta(\rho(x, y))\rho(x, y) + (1 - \vartheta(\rho(x, y)))\text{dist}(B_i, B_{i+1}) \\ &= \vartheta(\rho(x, y))\rho(x, y) + \text{dist}(B_i, B_{i+1}) - \vartheta(\rho(x, y))\text{dist}(B_i, B_{i+1}) \\ &= \vartheta(\rho(x, y))[\rho(x, y) - \text{dist}(B_i, B_{i+1})] + \text{dist}(B_i, B_{i+1}), (*) \\ &< \rho(x, y) - \text{dist}(B_i, B_{i+1}) + \text{dist}(B_i, B_{i+1}) \quad (\text{because } \vartheta(\rho(x, y)) < 1) \\ &= \rho(x, y). \end{aligned}$$

Case (2): If $\rho(x, y) = \text{dist}(B_i, B_{i+1})$, then from (*),

$$\rho(Tx, Ty) \leq \rho(x, y).$$

By Equation (1), $\rho(x, y) = \text{dist}(B_i, B_{i+1}) = \text{dist}(B_{i+1}, B_{i+2}) \leq \rho(Tx, Ty) \leq \rho(x, y)$, therefore

$$\rho(Tx, Ty) = \rho(x, y).$$

Hence, T is p -cyclic strict contraction.

(b) Let $x, y \in B_i$. Since T is p -cyclic non-expansive, $\{\rho(T^{pn}x, T^{p(n+1)}y)\}$ is a decreasing sequence and is bounded below by $\text{dist}(B_i, B_{i+1})$. Therefore,

$$\rho(T^{pn}x, T^{p(n+1)}y) \rightarrow r \text{ as } n \rightarrow \infty \text{ and } r \geq \text{dist}(B_i, B_{i+1}),$$

where $r = \inf_{n \geq 1} \rho(T^{pn}x, T^{p(n+1)}y)$.

Claim: $r = \text{dist}(B_i, B_{i+1})$.

If $\rho(T^{pn}x, T^{p(n+1)}y) = \text{dist}(B_i, B_{i+1})$ for some n , then by the p -cyclic non-expansiveness of T ,

$$\rho(T^{pn+k}x, T^{p(n+k+1)}y) = \rho(T^{pn}x, T^{p(n+1)}y), \quad k = 1, 2, \dots$$

Hence, we have

$$\rho(T^{pn}x, T^{p(n+1)}y) \rightarrow \text{dist}(B_i, B_{i+1}) \text{ as } n \rightarrow \infty.$$

Let us assume that $\rho(T^{pn}x, T^{p(n+1)}y) > \text{dist}(B_i, B_{i+1})$, $n \in \mathbb{N}$.

Suppose that $r > \text{dist}(B_i, B_{i+1})$. Since T is p -cyclic non expansive,

$$\begin{aligned} \rho(T^{p(n+1)}x, T^{p(n+1)+1}y) &\leq \rho(T^{pn+1}x, T^{pn+2}y) \\ &\leq \vartheta(\rho(T^{pn}x, T^{pn+1}y))\rho(T^{pn}x, T^{pn+1}y) \\ &\quad + (1 - \vartheta(\rho(T^{pn}x, T^{pn+1}y)))\text{dist}(B_i, B_{i+1}), \\ \rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \text{dist}(B_i, B_{i+1}) &\leq \vartheta(\rho(T^{pn}x, T^{pn+1}y)) \\ &\quad [\rho(T^{pn}x, T^{pn+1}y) - \text{dist}(B_i, B_{i+1})], \\ \text{Since } \vartheta \in \mathbb{S}, \quad \frac{\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \text{dist}(B_i, B_{i+1})}{\rho(T^{pn}x, T^{pn+1}y) - \text{dist}(B_i, B_{i+1})} &\leq \vartheta(\rho(T^{pn}x, T^{pn+1}y)) < 1. \end{aligned} \quad (5)$$

Since $r = \lim_{n \rightarrow \infty} \rho(T^{p(n+1)}x, T^{p(n+1)+1}y) > \text{dist}(B_i, B_{i+1})$ by our assumption, letting $n \rightarrow \infty$ in Equation (5), we get

$$1 \leq \lim_{n \rightarrow \infty} \vartheta(\rho(T^{pn}x, T^{pn+1}y)) \leq 1,$$

that is, $\lim_{n \rightarrow \infty} \vartheta(\rho(T^{pn}x, T^{pn+1}y)) = 1$. However, $\lim_{n \rightarrow \infty} \rho(T^{pn}x, T^{pn+1}y) = r > 0$, which contradicts $\vartheta \in \mathbb{S}$. Hence, $r = \text{dist}(B_i, B_{i+1})$. This proves Part (b). $\square$

Finally, an example of a p -cyclic map that is of Boyd–Wong type [19] and belongs to the class Ω is given:

Example 7. Let M be a non-empty set equipped with a metric ρ . Suppose that $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Suppose that $\psi : [0, \infty) \rightarrow [0, \infty)$ is upper semi-continuous from the right and satisfies $\psi(t) < t$ for $t > 0$ and $\psi(0) = 0$. Suppose also that $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ is a p -cyclic map. Suppose

$$\rho(Tx, Ty) \leq \psi(\rho(x, y) - \text{dist}(B_i, B_{i+1})) + \text{dist}(B_i, B_{i+1}), \quad (6)$$

$x \in B_i, y \in B_{i+1}, 1 \leq i \leq p$.

Then, the following conditions hold:

- (a) T is p -cyclic strict contraction.
- (b) For $x, y \in B_i, \rho(T^{pn}x, T^{pn+1}y) \rightarrow \text{dist}(B_i, B_{i+1})$, as $n \rightarrow \infty$.

Proof. (a) Let $x \in B_i$ and $y \in B_{i+1}$.

Case (i): If $\rho(x, y) > \text{dist}(B_i, B_{i+1})$. Since $\psi(t) < t$ for $t > 0$, by Equation (6), we have

$$\begin{aligned} \rho(Tx, Ty) &< \rho(x, y) - \text{dist}(B_i, B_{i+1}) + \text{dist}(B_i, B_{i+1}) \\ &= \rho(x, y). \end{aligned}$$

Case (ii): If $\rho(x, y) = \text{dist}(B_i, B_{i+1})$, then $\psi(\rho(x, y) - \text{dist}(B_i, B_{i+1})) = 0$. Therefore, by Equation (6), $\rho(Tx, Ty) = \text{dist}(B_i, B_{i+1})$. That is, $\rho(Tx, Ty) = \rho(x, y)$.

(b) Let $x, y \in B_i$. Note that

$$\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) \leq \rho(T^{pn}x, T^{pn+1}y), \quad n \in \mathbb{N}.$$

Then, the sequence $\{\rho(T^{pn}x, T^{pn+1}y)\}_{n=1}^{\infty}$ is bounded below by $\text{dist}(B_i, B_{i+1})$ and non-increasing sequence. Hence, $\rho(T^{pn}x, T^{pn+1}y) \rightarrow r$ as $n \rightarrow \infty$ and $r \geq \text{dist}(B_i, B_{i+1})$, where $r = \inf_{n \geq 1} \{\rho(T^{pn}x, T^{pn+1}y)\}$.

Claim: $r = \text{dist}(B_i, B_{i+1})$.

Case (1): If $\rho(T^{pn}x, T^{pn+1}y) = \text{dist}(B_i, B_{i+1})$ for some $n \in \mathbb{N}$.

Then, by the p -cyclic non-expansiveness of T ,

$$\rho(T^{pn+k}x, T^{pn+k+1}y) = \text{dist}(B_i, B_{i+1}), \quad k = 1, 2, \dots$$

Thus, $\rho(T^{pn}x, T^{pn+1}y) \rightarrow \text{dist}(B_i, B_{i+1})$, as $n \rightarrow \infty$.

Case (2): If $\rho(T^{pn}x, T^{pn+1}y) > \text{dist}(B_i, B_{i+1})$ for all $n \in \mathbb{N}$.

Since T is p -cyclic non-expansive,

$$\begin{aligned} \rho(T^{p(n+1)}x, T^{p(n+1)+1}y) &\leq \rho(T^{pn+1}x, T^{pn+2}y) \\ &\leq \psi(\rho(T^{pn}x, T^{pn+1}y) - \text{dist}(B_i, B_{i+1})) \\ &\quad + \text{dist}(B_i, B_{i+1}), \\ \rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \text{dist}(B_i, B_{i+1}) &\leq \psi(\rho(T^{pn}x, T^{pn+1}y) - \text{dist}(B_i, B_{i+1})). \end{aligned}$$

Taking the lim sup on both sides,

$$\lim_{n \rightarrow \infty} \rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \text{dist}(B_i, B_{i+1}) \leq \limsup_{n \rightarrow \infty} \psi(\rho(T^{pn}x, T^{pn+1}y) - \text{dist}(B_i, B_{i+1})).$$

Since $\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \text{dist}(B_i, B_{i+1}) \downarrow r - \text{dist}(B_i, B_{i+1})$ and ψ is upper semi-continuous from the right,

$$r - \text{dist}(B_i, B_{i+1}) \leq \psi(r - \text{dist}(B_i, B_{i+1})).$$

Let $t = r - \text{dist}(B_i, B_{i+1})$. If $r > \text{dist}(B_i, B_{i+1})$, then $t > 0$ and also $t \leq \psi(t)$, which is a contradiction to the definition of ψ . Hence, $r = \text{dist}(B_i, B_{i+1})$. $\square$

In a similar way, one may check whether some other known/unknown p -cyclic maps belong to the class Ω .

7. Best Proximity Point Results of Ω Class of Mappings

Below, we give a convergence result for Ω class of mappings.

Theorem 1. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M .

Let $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ be a p -cyclic map that belongs to the class Ω . Assume for some $k \in \mathbb{N}$ and $x \in B_i$, ($1 \leq i, k \leq p$), $\{T^{pn+k}x\}$ converges to $\xi \in B_{i+k}$. Then, ξ is a best proximity point of T in B_{i+k} .

Proof. Let $x \in B_i$ be as given in the theorem. By Equation (1), for each $n \in \mathbb{N}$, we have,

$$\begin{aligned} \text{dist}(B_{i+k}, B_{i+k+1}) &= \text{dist}(B_{i+k-1}, B_{i+k}) \\ &\leq \rho(T^{pn+k-1}x, \xi) \\ &\leq \rho(T^{pn+k-1}x, T^{pn+k}x) + \rho(T^{pn+k}x, \xi). \end{aligned}$$

Since $T \in \Omega$,

$$\lim_{n \rightarrow \infty} (\rho(T^{pn+k-1}x, T^{pn+k}x) + \rho(T^{pn+k}x, \xi)) = \text{dist}(B_{i+k-1}, B_{i+k}).$$

Therefore,

$$\lim_{n \rightarrow \infty} \rho(T^{pn+k-1}x, \xi) = \text{dist}(B_{i+k-1}, B_{i+k}) = \text{dist}(B_{i+k}, B_{i+k+1}). \quad (7)$$

Now,

$$\begin{aligned} \text{dist}(B_{i+k}, B_{i+k+1}) &\leq \rho(\xi, T\xi) \\ &= \lim_{n \rightarrow \infty} \rho(T^{pn+k}x, T\xi) \\ &\leq \lim_{n \rightarrow \infty} \rho(T^{pn+k-1}x, \xi) \\ &= \text{dist}(B_{i+k}, B_{i+k+1}), \quad (\text{by (7)}). \end{aligned}$$

Hence, $\rho(\xi, T\xi) = \text{dist}(B_{i+k}, B_{i+k+1})$. $\square$

Now, we prove the existence of best proximity point for mappings which belong to class Ω defined on a p -cyclic complete metric space.

Theorem 2. For a non-empty set M , suppose $\rho : M \times M \rightarrow [0, \infty)$ forms a metric and $B_1, B_2, \dots, B_p$ ($p \geq 2$) are non-empty subsets of M . Suppose that $M = \bigcup_{i=1}^p B_i$ and $\bigcup_{i=1}^p B_i$ is p -cyclic complete. Let $T : \bigcup_{i=1}^p B_i \rightarrow \bigcup_{i=1}^p B_i$ be a p -cyclic map which belongs to the class Ω . Then, there exists a best proximity point of T in B_j for some $j \in \{1, 2, \dots, p\}$.

Proof. Let $x \in B_i$, $1 \leq i \leq p$.

Define a sequence $\{x_n\}_{n=1}^\infty$ in (M, ρ) by

$$x_n := T^n x \text{ for } n \in \mathbb{N}.$$

Claim: $\{T^n x\}_{n=1}^\infty$ is a p -cyclic Cauchy sequence.

Let $m, n \in \mathbb{N}$ be such that $m > n$,

$$\begin{aligned} \rho(T^{pm}x, T^{pn+1}x) &= \rho(T^{p(n+r)}x, T^{pn+1}x), \text{ where } m = n + r, r \in \mathbb{N} \\ &= \rho(T^{pn}y, T^{pn+1}x), \text{ where } y = T^{pr}x \in B_i \\ &\rightarrow \text{dist}(B_i, B_{i+1}), \text{ as } n \rightarrow \infty \text{ (because } T \in \Omega). \end{aligned}$$

This implies that, for all $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\rho(T^{pm}x, T^{pn+1}x) < \epsilon + \text{dist}\{(B_i, B_{i+1})\}, \quad m, n \geq n_0.$$

By Proposition 4, for a given $\epsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$d(T^{pm+k}x, T^{pn+k+1}x) < \epsilon + d(A_{i+k}, A_{i+k+1}), \quad m, n \geq n_1, \quad k \in \{1, 2, \dots, p\}.$$

Therefore, the sequence $\{T^n x\}$ is a p -cyclic Cauchy sequence in (M, ρ) . Since (M, ρ) is p -cyclic complete, there exists a $k \in \{1, 2, \dots, p\}$ such that $\{T^{pn+k}x\}$ converges to $z \in A_{i+k}$. By Theorem 1, z is best proximity point of T in B_j , where $j = i + k$. $\square$

Remark 4. In Theorem 2, suppose that $\text{dist}(B_i, B_{i+1}) = 0$, for some $i \in \{1, 2, \dots, p\}$. Then, by Equation (1), $\text{dist}(B_k, B_{k+1}) = 0$, $1 \leq k \leq p$. This implies that $Tz = z$, that is, z is a fixed point of T . Since T is p -cyclic, $z \in \bigcap_{i=1}^p B_i$ and hence $\bigcap_{i=1}^p B_i$ is non-empty.

To prove the uniqueness of fixed point of T , let $\alpha, \beta \in B_i$ be such that $\alpha = T\alpha$, $\beta = T\beta$ and $\alpha \neq \beta$. Then, $\rho(\alpha, \beta) > 0 = \text{dist}(B_i, B_{i+1})$. Since T is p -cyclic strict contraction,

$$\rho(\alpha, \beta) = \rho(T\alpha, T\beta) < \rho(\alpha, \beta),$$

which is a contradiction. Hence, $\alpha = \beta$. This shows that there exists a unique fixed point for Ω class of mappings in a p -cyclic complete metric space.

Theorem 3. Let X be a strictly convex normed linear space. Let $B_1, B_2, \dots, B_p$ ($p \geq 2$) be non-empty, closed, convex subsets of X such that $X = \cup_{i=1}^p B_i$ and $\cup_{i=1}^p B_i$ is p -cyclic complete. Let $T : \cup_{i=1}^p B_i \rightarrow \cup_{i=1}^p B_i$ be a p -cyclic map which belongs to the class Ω . Then, for each $j \in \{1, 2, \dots, p\}$, there exists a unique best proximity point $z_j \in A_j$. In addition, z_j is a unique periodic point of T in B_j and $T^k z_j$ is the unique best proximity point of T in B_{j+k} , $k = 0, 1, 2, \dots, (p-1)$.

Proof. From the proof of Theorem 2, for any $x \in B_i$, $i \in \{1, 2, \dots, p\}$, $\{T^{pn+q}x\}$ converges to $z \in B_j$, $j = i + q$, for some $q \in \{1, 2, \dots, p\}$ and by Theorem 1, z is a best proximity point of T in B_j . Since X is a strictly convex, by Proposition 1, z is unique.

By Lemma 1, since T is p -cyclic non expansive map, $T^k z$ is the best proximity point of T in B_{j+k} , for $k = 1, 2, \dots, (p-1)$.

To prove z is a periodic point of T in B_j , it is enough to prove $\|T^p z - Tz\| = \text{dist}(B_j, B_{j+1})$. Suppose $\|T^p z - Tz\| > \text{dist}(B_j, B_{j+1})$. Since T is p -cyclic strict contraction,

$$\begin{aligned} \|T^{p+1}z - T^2z\| &< \|T^p z - Tz\| \\ &\leq \|T^{p-1}z - z\| \\ &= \lim_n \|T^{p-1}z - T^{pn+q}x\| \\ &\leq \lim_n \|z - T^{(pn+q)-(p-1)}x\| \\ &= \lim_n \|z - T^{p(n-1)+q+1}x\| \\ &= \lim_n \|T^{p(n-1)+q}x - T^{p(n-1)+q+1}x\| \\ &= \text{dist}(B_{i+q}, B_{i+q+1}), \quad (\text{because } T \in \Omega) \\ &= \text{dist}(B_i, B_{i+1}), \quad (\text{by Equation (1)}) \\ &= \text{dist}(B_{j+1}, B_{j+2}) \end{aligned}$$

which is a contradiction. Hence, $\|T^p z - Tz\| = \text{dist}(B_j, B_{j+1})$. Since X is strictly convex, B_j is convex and $\|z - Tz\| = \text{dist}(B_j, B_{j+1})$, then $T^p z = z$, that is z is a periodic point of T in B_j . To prove the uniqueness of z , let $\xi \in B_j$ be such that $T^p \xi = \xi$, then $\{T^{pn}\xi\}$ converges to $\xi \in B_j$. By Theorem 1, ξ is a best proximity point of T in B_j . Since z is the unique best proximity point of T in A_j , we have $\xi = z$. Hence, the theorem holds. $\square$

8. Conclusions

In this paper, we have introduced a notion of p -cyclic Cauchy sequence, which is weaker than the notion of Cauchy sequence in the usual sense. If one subsequence of a p -cyclic Cauchy sequence converges, then we say that the p -cyclic sequence is p -cyclic convergent. If all p -cyclic Cauchy sequences converge, then we call the underlying metric space as p -cyclic complete metric space. We have shown that a complete metric space need not be p -cyclic complete. A class of mappings called Ω is introduced. The existence of

fixed point and best proximity point of mappings of class Ω is guaranteed in a p -cyclic complete metric space. Many p -cyclic maps with various contractive conditions introduced in the literature fall under class Ω , where the best proximity point for such maps were obtained in a uniformly convex Banach space, whereas we have obtained a unique best proximity point in a strictly convex norm linear space. Thus, our main result is a natural generalization of main results of Al-Thagafi [3], Boyd [19], Eldred [5], Geraghty [18], Karpagam [10], Karpagam [9], Kirk [2], Meir [20].

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