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Hyers-Ulam Stability for Linear Differences with Time Dependent and Periodic Coefficients

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Abstract: Let $q \geq 2$ be a positive integer and let (a_j), (b_j), and (c_j) (with j a non-negative integer) be three given $\mathbb{C}$- valued and q-periodic sequences. Let $A(q) := A_{q-1} \cdots A_0$, where A_j is as is given below. Assuming that the "monodromy matrix" $A(q)$ has at least one multiple eigenvalue, we prove that the linear scalar recurrence $x_{n+3} = a_n x_{n+2} + b_n x_{n+1} + c_n x_n, \quad n \in \mathbb{Z}_+$ is Hyers-Ulam stable if and only if the spectrum of $A(q)$ does not intersect the unit circle $\Gamma := \{w \in \mathbb{C} : |w| = 1\}$. Connecting this result with a recently obtained one it follows that the above linear recurrence is Hyers-Ulam stable if and only if the spectrum of $A(q)$ does not intersect the unit circle.

Keywords: difference equations; discrete dichotomy; Hyers-Ulam stability; stability theory

MSC: 34D09, 39B82, 34K20

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1. Introduction

An open problem, arising naturally in [1], is a problem referring to the relationship between the Hyers-Ulam stability of a certain linear recurrence of order n with periodic coefficients and the exponential dichotomy of the monodromy matrix associated to the recurrence. The corresponding problem for second-order recurrences was completed in [2], where second-order linear differential equations were also analyzed.

Here, we continue the analysis started in [3] and, finally, we complete the discussion raised in [1] for periodic linear recurrences of order three. Thus, this article can be seen as a new link in the chain of articles [1–5] which address the Hyers-Ulam stability of linear scalar recurrences. The connections of this topic to those existing in the literature was already presented in [3], so we do not present them again here.

It seems that the methods used here can be extended to recurrences of higher order in Banach spaces and, hopefully, this will be considered in the future; the autonomous case was analyzed in [6–9]. For developments concerning differential equations with impulses see, [10–13], and the references therein.

2. Definitions and Notations

We use the same notation as in [3]. Recall that the entry m_{ij} (of a matrix M) is denoted by $[M]_{ij}$, and the uniform norm of a $\mathbb{C}^m$-valued and bounded sequence $g = (g_n)$ is defined and denoted by $\|g\|_\infty := \sup_{j \in \mathbb{Z}_+} \|g_j\|$. Let $\varepsilon > 0$ be given. We recall (see also [3]):

Definition 1. *A scalar valued sequence* (y_j) *is called an* ε*-approximative solution of the linear recurrence*

$$x_{n+3} = a_n x_{n+2} + b_n x_{n+1} + c_n x_n, \quad n \in \mathbb{Z}_+ \tag{1}$$

if

$$|y_{n+3} - a_n y_{n+2} - b_n y_{n+1} - c_n y_n| \leq \varepsilon, \quad \forall n \in \mathbb{Z}_+. \tag{2}$$

Definition 2. *([3]) The recurrence (1) is Hyers-Ulam stable if there exists a positive constant L such that, for every* $\varepsilon > 0$ *and* ε*-approximative solution* $y = (y_j)$ *of (1), there exists an exact solution* $\theta = (\theta_j)$ *of (1) such that* $\|y - \theta\|_\infty \leq L\varepsilon$.

Obviously, any ε-approximative solution of the recurrence (1) can be seen as a solution of the non-homogeneous equation

$$x_{n+3} - a_n x_{n+2} - b_n x_{n+1} - c_n x_n = f_{n+1}, \quad n \in \mathbb{Z}_+, \tag{3}$$

for some scalar valued sequence (f_n) with $f_0 = 0$ and $\|(f_k)\|_\infty \leq \varepsilon$.

We denote the solution of the nonhomogeneous linear recurrence (3) initiated from Y_0 by $(\phi(n, Y_0, (f_k))$.

The solution of the system

$$X_{n+1} = A_n X_n + F_{n+1}, \quad n \in \mathbb{Z}_+, \tag{4}$$

initiated from Z_0, where $X_n := \begin{pmatrix} z_n & v_n & w_n \end{pmatrix}^T \in \mathbb{C}^3$, $F_n = \begin{pmatrix} 0 & 0 & f_n \end{pmatrix}^T$, and

$$A_n := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_n & b_n & a_n \end{pmatrix}, \tag{5}$$

is given by

$$\Phi_n := \Phi(n, Z_0, (F_k)) = U_{\mathcal{A}}(n, 0) Z_0 + \textstyle\sum_{k=1}^{n} U_{\mathcal{A}}(n, k) F_k. \tag{6}$$

Denoting by $(\varphi(n, Z_0, (f_k))$ the solution of (1), obviously we have

$$\varphi_n := \varphi(n, Z_0, (f_k)) = \left[U_{\mathcal{A}}(n, 0) Z_0 + \textstyle\sum_{k=1}^{n} U_{\mathcal{A}}(n, k) F_k \right]_{11} \tag{7}$$

and $\Phi_n = \begin{pmatrix} \varphi_n & \varphi_{n+1} & \varphi_{n+2} \end{pmatrix}^T$.

Here, $\mathcal{A}$ is the family of all matrices A_j (with $j \in \mathbb{Z}_+$, where A_j is given in (5)) and the matrix $U_{\mathcal{A}}(n, k)$ is given by $U_{\mathcal{A}}(n, k) = A_{n-1} \cdots A_k$, $n > k$ and $U_{\mathcal{A}}(n, n) = I_3$. The family $\mathcal{U}_{\mathcal{A}} := \{U_{\mathcal{A}}(n, k) : n \geq k \in \mathbb{Z}_+\}$ will be called the evolution family associated to $\mathcal{A}$.

3. Background and the Main Result

The next two propositions appear (in a slightly different form) in [14].

Proposition 1. *Suppose that the eigenvalues* x, y*, and* z *of the matrix* $A \in \mathcal{M}(3, \mathbb{C})$ *verify the condition*

$$x = y = z \neq 0. \tag{8}$$

Then,

$$A^n = x^n (n^2 B + nC + I_3) \text{ for all } n \in \mathbb{Z}_+, \tag{9}$$

where

$$B = \frac{1}{2x^2}(A - xI_3)^2 \tag{10}$$

and

$$C = -\frac{1}{2x^2}(A - xI_3)(A - 3xI_3). \tag{11}$$

Proposition 2. *If the characteristic polynomial of the matrix A is*

$$p_A(\lambda) = (\lambda - x)^2(\lambda - y), \text{ with } x \neq y \text{ and } x \neq 0, \tag{12}$$

then its natural powers are given by

$$A^n = x^n(nB + C) + y^n D, \quad n \in \mathbb{Z}_+, \tag{13}$$

where B, C, and D are given by

$$B = \frac{1}{x(x-y)}(A - xI_3)(A - yI_3), \tag{14}$$

$$C = -\frac{1}{(x-y)^2}[A - (2x - y)I_3](A - yI_3), \tag{15}$$

$$D = \frac{1}{(x-y)^2}(A - xI_3)^2. \tag{16}$$

Remark 1. *Taking into account that x and y are different roots of the minimal polynomial m_A (of A), the matrices C in (15) and D in (16) are not the zero matrix.*

Let q, (a_j), (b_j), and (c_j) be as above. Recall that

$$A(q) := A_{q-1} \cdots A_0, \text{ where } A_j := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_j & b_j & a_j \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{17}$$

Our main result reads as follows.

Theorem 1. *Assume that either of the conditions (8) or (12) (concerning the spectrum of $A(q)$) are fulfilled. The linear recurrence*

$$x_{n+3} = a_n x_{n+2} + b_n x_{n+1} + c_n x_n, \quad n \in \mathbb{Z}_+ \tag{18}$$

is Hyers-Ulam stable if and only if the spectrum of $A(q)$ does not intersect the unit circle.

Combining this result with ([3], Theorem 3.1) we get the following Corollary that completes an open problem, raised in [1] for the particular case $n = 3$.

Corollary 1. *The linear recurrence (18) is Hyers-Ulam stable if and only if the spectrum of $A(q)$ does not intersect the unit circle.*

Remark 2. *Motivated by the applications suggested in [15], we are also interested in studying the Hyers-Ulam stability of the linear recurrence in (18), but with $\mathbb{Z}$ instead of $\mathbb{Z}_+$. This can be seen as a symmetrization of the result in Corollary 1. Next, we summarize some ideas, but do not give all the details. For simplicity, we assume that $c_j \neq 0$ for all $j \in \mathbb{Z}$.*

It is well-known that the equivalent statements of Corollary 1 are also equivalent to the fact that the system of recurrences in $\mathbb{C}^3$

$$\begin{cases} z_{n+1} = & v_n \\ v_{n+1} = & w_n \\ w_{n+1} = & c_n z_n + b_n v_n + a_n w_n \quad n \in \mathbb{Z}_+ \end{cases}, \tag{19}$$

possesses a discrete dichotomy on $\mathbb{Z}_+$; see ([1], Proposition 1.2, Theorem 2.1) for a more general framework of this result.

A new challenge for us is to see if the following three statements (presented in formal terms) are equivalent.

1. The linear recurrence

$$x_{n+3} = a_n x_{n+2} + b_n x_{n+1} + c_n x_n, \quad n \in \mathbb{Z}_- := \{\cdots, -2, -1, 0\} \tag{20}$$

is Hyers-Ulam stable on $\mathbb{Z}_-$.

2. The "symmetric" linear system

$$\begin{cases} z_{n-1} = & -\frac{b_n}{c_n} z_n - \frac{a_n}{c_n} v_n + \frac{1}{c_n} w_n \\ v_{n-1} = & z_n \\ w_{n-1} = & v_n, \quad n \in \mathbb{Z}_- \end{cases} \tag{21}$$

possesses a discrete dichotomy on $\mathbb{Z}_-$.

3. The spectrum of the monodromy matrix associated with (21) does not intersect the unit circle.

It seems that all these statements are also equivalent to the fact that the spectrum of the monodromy matrix associated with (19) does not intersect the unit circle.

The main ingredient in the proof in Section 3 of the "if" part of the Theorem 1 is the following technical Lemma, whose proof is presented in the next section.

Lemma 1. *Assume that either of the conditions (8) or (12) (concerning the spectrum of $A(q)$) are fulfilled. If the spectrum of $A(q)$ intersects the unit circle then, for each $\varepsilon > 0$, there exists a $\mathbb{C}$-valued sequence $(f_j)_{j\in\mathbb{Z}_+}$ with $f_0 = 0$ and $\|(f_j)\|_\infty \le \varepsilon$ such that, for every initial condition $Z_0 = (x_0, y_0, z_0)^T \in \mathbb{C}^3$, the $\mathbb{C}$-valued sequence*

$$\left(\left[U_{\mathcal{A}}(n,0)Z_0 + \sum_{k=1}^{n} U_{\mathcal{A}}(n,k)F_k\right]_{11}\right)_{n\in\mathbb{Z}_+} \tag{22}$$

(with $F_k = (0, 0, f_k)^T$), is unbounded.

4. Proofs

Proof. Proof of Lemma 1. We use Propositions 1 and 2, with $A(q)$ instead of A. Denote by x, y, and z the eigenvalues of $A(q)$.

Case I. Let $x = y = z$ and $|x| = 1$. We use the notation of the previous sections.

I.1. When $B = (b_{rs})_{r,s\in\{1,2,3\}} \neq 0_3$, there exists a pair (i, j) with $i, j \in \{1, 2, 3\}$ such that $b_{ij} \neq 0$. We analyze three cases:

I.1.1. Let $j = 3$ and $b_{13} \neq 0$. Set

$$F_k = F_k^1 := \begin{cases} x^{k/q} u_0, \text{ if } k = nq, n \in \mathbb{Z}_+ \\ 0, \text{ if } k \text{ is not a multiple of } q, \end{cases} \tag{23}$$

where $u_0 := \begin{pmatrix} 0 & 0 & c_0 \end{pmatrix}^T$ and c_0 is a given nonzero complex scalar with $|c_0| < \varepsilon$. Successively, we have

$$\Phi_{nq} = U_{\mathcal{A}}(nq, 0)Z_0 + \sum_{k=1}^{nq} U_{\mathcal{A}}(nq, k)F_k$$

$$
\begin{aligned}
&= U_{\mathcal{A}}(nq,0)Z_0 + \textstyle\sum_{j=1}^{n} U_{\mathcal{A}}(nq,jq)F_{jq} \\
&= x^n\left(n^2B + nC + I_3\right)Z_0 + \textstyle\sum_{j=1}^{n} x^j x^{n-j}\left[(n-j)^2B + (n-j)C + I_3\right]u_0 \\
&= x^n\left(n^2B + nC + I_3\right)Z_0 + x^nBu_0\sum_{j=0}^{n-1} j^2 + x^nCu_0\sum_{j=0}^{n-1} j + \sum_{j=1}^{n} x^n u_0 \\
&= \frac{(n-1)n(2n-1)}{6}x^nBu_0 + x^n\left[\left(n^2B + nC + I_3\right)Z_0 + \frac{(n-1)n}{2}Cu_0 + nu_0\right].
\end{aligned} \tag{24}
$$

On the other hand,

$$
\left|\left[\frac{(n-1)n(2n-1)}{6}x^nBu_0\right]_{11}\right| = \frac{(n-1)n(2n-1)}{6}|b_{13}c_0| \to \infty \text{ as } n \to \infty. \tag{25}
$$

Thus, (24) and (25) yield the unboundedness of the sequence (ϕ_n). When $b_{23} \neq 0$ or $b_{33} \neq 0$, arguing as above, we can show that the sequences (φ_{n+1}) and (φ_{n+2}) are unbounded, and that then (φ_n) is unbounded as well.

I.1.2. Let $j = 2$ and $b_{12} \neq 0$. Set

$$
F_k = F_k^2 := \begin{cases} x^{k/q}A_{q-1}u_0, & \text{if } k = nq \\ 0, & \text{if } k \text{ is not a multiple of } q, \end{cases} \tag{26}
$$

where u_0 and c_0 are taken as above. We obtain

$$
\Phi_{nq} = A(q)^nZ_0 + \textstyle\sum_{j=1}^{n} x^jA(q)^{n-j}A_{q-1}u_0,
$$

which leads to

$$
\begin{aligned}
\varphi_{nq} &= \left[A(q)^nZ_0 + \textstyle\sum_{j=1}^{n} x^jA(q)^{n-j}A_{q-1}u_0\right]_{11} \\
&= \left[x^n\frac{(n-1)n(2n-1)}{6}BA_{q-1}u_0\right]_{11} \\
&+ \left[x^nn^2BZ_0 + x^nC\left(nZ_0 + \frac{(n-1)n}{2}A_{q-1}u_0\right) + x^n\left(Z_0 + nA_{q-1}u_0\right)\right]_{11}.
\end{aligned} \tag{27}
$$

For our purposes, it is enough to prove that the sequence (whose general term is given in (27)) is unbounded. Indeed, we have

$$
\left|\left[x^n\frac{(n-1)n(2n-1)}{6}BA_{q-1}u_0\right]_{11}\right| = \frac{(n-1)n(2n-1)}{6}|b_{12}c_0| \to \infty,
$$

as $n \to \infty$.

The cases when $b_{22} \neq 0$ and $b_{23} \neq 0$ can be treated in a similar manner, and we omit the details.

I.1.3. Let $j = 1$ and $b_{11} \neq 0$. Set

$$
F_k = F_k^3 := \begin{cases} x^{k/q}A_{q-2}A_{q-1}u_0, & \text{if } k = nq \\ 0, & \text{if } k \text{ is not a multiple of } q, \end{cases} \tag{28}
$$

with u_0 and c_0 as above.

As in the previous cases, we obtain that

$$
\phi_{nq} = \left[x^nB\left(n^2Z_0 + \frac{(n-1)n(2n-1)}{6}A_{q-2}A_{q-1}u_0\right)\right]_{11},
$$

and so (ϕ_{nq}) is unbounded, as

$$\left|\left[x^n\frac{(n-1)n(2n-1)}{6}BA_{q-2}A_{q-1}u_0\right]_{11}\right|$$

$$=\frac{(n-1)n(2n-1)}{6}\left|b_{11}c_0\right|\to\infty \text{ as } n\to\infty.$$

I.2. Let $B=0_3$ and $C\neq 0_3$ be of the form

$$C=\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

Let $c_{ij}\neq 0$, for some pair (i,j) with $i,j\in\{1,2,3\}$. Then,

$$A(q)^n = x^n\left(nC+I_3\right) \qquad \text{for all } n\in\mathbb{Z}_+.$$

We have to consider the following three steps:
I.2.1. Let $j=3$ and $c_{13}\neq 0$. Set $F_k=F_k^1$. As above, we have

$$\Phi_{nq}=x^nC\left(Z_0+\frac{(n-1)n}{2}x^nCu_0\right)+x^n\left(Z_0+nu_0\right).$$

Thus,

$$\varphi_n=\left[x^n\frac{(n-1)n}{2}Cu_0\right]_{11}+$$
$$\left[x^nCZ_0+x^n\left(Z_0+nu_0\right)\right]_{11}.$$

The sequence (ϕ_n) is unbounded, since

$$\left[x^n\frac{(n-1)n}{2}Cu_0\right]_{11}=\frac{(n-1)n}{2}\left|c_{13}c_0\right|\to\infty \text{ as } n\to\infty.$$

When $c_{23}\neq 0$ or $c_{33}\neq 0$, we can argue as in the previous cases.
I.2.2. Let $j=2$ and $c_{12}\neq 0$. Set $F_k=F_k^2$. Then, we obtain

$$\Phi_{nq} \quad = \quad A(q)^nZ_0+\textstyle\sum_{j=1}^{n}x^jA(q)^{n-j}A_{q-1}u_0\ ,$$

which leads to

$$\varphi_{nq}=\left[\frac{(n-1)n}{2}CA_{q-1}u_0\right]_{11}$$
$$+\left[x^nCZ_0+x^n\left(Z_0+nA_{q-1}u_0\right)\right]_{11},$$

and the sequence (ϕ_{nq}) is unbounded because

$$\left|\left[x^n\frac{(n-1)n}{2}CA_{q-1}u_0\right]_{11}\right|=$$

$$\frac{n(n-1)}{2}\left|c_{12}c_0\right|\to\infty \text{ as } n\to\infty.$$

When $c_{22}\neq 0$ or $c_{23}\neq 0$, we can proceed in a similar manner.

I.2.3. let $j = 1$ and $c_{11} \neq 0$. Set $F_k = F_k^3$. As in the previous cases, we obtain

$$\left| \left[x^n \frac{(n-1)n}{2} C A_{q-2} A_{q-1} u_0 \right]_{11} \right| =$$

$$\frac{n(n-1)}{2} |c_{11} c_0| \to \infty \text{ as } n \to \infty.$$

Therefore, (φ_n) is unbounded.

I.3. When $B = 0_3$ and $C = 0_3$, then $A(q)^n = x^n I_3$ for all $n \in \mathbb{Z}_+$. Set $F_k = F_k^1$. Then,

$$\Phi_{nq} = A(q)^n Z_0 + \sum_{j=1}^{n} x^j A(q)^{n-j} u_0$$

$$= x^n Z_0 + \sum_{j=1}^{n} x^j x^{n-j} u_0$$

$$= x^n Z_0 + n x^n u_0.$$

It is enough to prove that the sequence $\left([n x^n u_0]_{11}\right)_n$ is unbounded, and note that

$$|[n x^n u_0]_{11}| = n |c_0| \to \infty \text{ as } n \to \infty.$$

Case II. When the characteristic polynomial $p_{A(q)}(\lambda)$ is given by

$$p_{A(q)}(\lambda) = (\lambda - x)^2 (\lambda - y), x \neq y.$$

Let λ be an eigenvalue of $A(q)$ and let P_λ be the Riesz projection associated to $A(q)$ and λ; that is,

$$P_\lambda = \frac{1}{2\pi i} \int_{C(\lambda,r)} (w I_3 - A(q))^{-1} \, dw,$$

where $C(\lambda, r)$ is the circle centered at λ of radius r, and r is small enough such that all other eigenvalues of $A(q)$ are located outside of the circle. Using the Dunford integral calculus (see [16]) and the Cauchy formula (see, e.g., [17], Theorem 10.15) it is easy to show that $P_x^2 = P_x$ and $P_x P_y = P_y P_x = 0_3$.

On the other hand, by Proposition 2 and the Spectral Decomposition Theorem (see, e.g., [18], Theorem 1), for every $v \in \mathbb{C}^3$, one has

$$\begin{array}{rcl} A(q)^n v & = & x^n (nB + C) v + y^n D v \\ & = & x^n (nB + C) P_x v + y^n D P_y v \end{array},$$

and so $P_x A(q)^n = x^n (nB + C) P_x = x^n (nB + C)$ and $P_y A(q)^n = y^n D P_y = y^n D$ for every $n \in \mathbb{Z}_+$.

In the following, we will analyze three cases:

II.1. When $|x| = 1$ and $|y| < 1$.

II.1.1. When $B \neq 0_3$, let us first assume that $b_{13} \neq 0$ and set $F_k = F_k^1$. Then,

$$P_x \Phi_{nq} = P_x A(q)^n Z_0 + \sum_{j=1}^{n} x^j P_x A(q)^{n-j} u_0$$

$$= x^n (nB + C) Z_0 + \sum_{j=1}^{n} x^j x^{n-j} [(n-j)B + C] u_0$$

$$= x^n (nB + C) Z_0 + x^n B u_0 \sum_{j=0}^{n-1} j + \sum_{j=0}^{n-1} x^n C u_0$$

$$= \frac{(n-1)n}{2} x^n B u_0 + x^n (nB + C) Z_0 + n x^n C u_0.$$

It is enough to prove that $\left(\left[P_x \Phi_{nq}\right]_{11}\right)_n$ is unbounded; it follows because

$$\left|\left[\frac{(n-1)n}{2}x^n Bu_0\right]_{11}\right| = \frac{(n-1)n}{2}|b_{13}c_0| \to \infty \text{ as } n \to \infty.$$

The cases $b_{23} \neq 0$ and $b_{33} \neq 0$ can be treated in a similar manner, and we omit the details.

II.1.2. When $b_{12} \neq 0$, set $F_k = F_k^2$. Then,

$$\Phi_{nq} = A(q)^n Z_0 + \sum_{j=1}^{n} x^j A(q)^{n-j} A_{q-1} u_0,$$

which yields

$$P_x \Phi_{nq} = \frac{(n-1)n}{2}x^n BA_{q-1}u_0 + x^n(nB+C)Z_0 + nx^n CA_{q-1}u_0.$$

Therefore, the sequence $\left([P_x\Phi_{nq}]_{11}\right)_n$ is unbounded, as

$$\left|\left[\frac{(n-1)n}{2}x^n BA_{q-1}u_0\right]_{11}\right| = \frac{(n-1)n}{2}|b_{12}c_0| \to \infty \text{ as } n \to \infty.$$

The cases $b_{22} \neq 0$ and $b_{32} \neq 0$ can be treated in a similar manner, and we omit the details.

The case when $b_{11} \neq 0$ is similar to Case **I.1.3.**, so we omit the details.

II.1.2. Let $B = 0_3$. As $C \neq 0_3$ (see Remark 1), we can proceed in a similar manner as in Case **I.2.**.

II.2. When $|x| = |y| = 1$.

II.2.1. Let $|x| = |y| = 1$ and $B = 0_3$. Let $F_k = F_k^1$ and u_0 be as defined above. An easy calculation yields

$$\Phi_{nq} = x^n nCu_0 + x^n CZ_0 + y^n DZ_0 + \left[\frac{x^n - y^n}{x-y} + x^n\right] Du_0. \tag{29}$$

Note that the last three terms in (29) are bounded (as functions of n). Now, if $[C]_{13} \neq 0$ then $|[x^n nCu_0]_{11}| = n|[C]_{13}c_0| \to \infty$ as $n \to \infty$, and (29) yields the unboundedness of the sequence (φ_{nq}). As $C \neq 0_3$, at least one of its entries is nonzero and we arrive at the same conclusion by arguing in a similar manner (such arguments were given above a few times, so we omit the details).

II.2.2. When $|x| = |y| = 1$ and $B \neq 0_3$, it can be treated like in Case **II.1.1.**.

II. 3. When $|x| = 1$ and $|y| > 1$. Taking into account that D is not the zero matrix (of order 3), we can choose a sequence (F_k) and a pair (i, j) such that the sequence $([P_y\Phi(nq)]_{i,j})_n$ is unbounded (we omit the details).

The proof of Lemma 1 is now complete. □

Proof. Proof of Theorem 1. Necessity: We argue by contradiction. Suppose that $\sigma(A(q))$ intersects the unit circle. Let Y_0 and X_0 be as in [[3], Remark 1]. From Lemma 1, it follows that the sequence in (22) with $(Y_0 - X_0)$ instead of Z_0 is unbounded and this contradicts the Hyers-Ulam stability property of the recurrence given in (18).

Sufficiency: Can be done exactly as in the proof of the implication **2** ⇒ **1** in ([3], Theorem 1); we omit the details. □

5. Examples

The following example illustrates our theoretical result.

Example 1. *Let consider the linear recurrence of order* 3

$$x_{n+3} = \sin\frac{2n\pi}{3}x_{n+2} + \cos\frac{2n\pi}{3}x_{n+1} + a\tan\frac{2n\pi}{3}x_n, \quad n \in \mathbb{Z}_+. \tag{30}$$

We find the values of the real parameter a, such that the recurrence in (30) is Hyers-Ulam stable. With the above notation, we have:

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a\sqrt{3} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a\sqrt{3} & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Now, the monodromy matrix associated to (30) is

$$A(3) = A_2A_1A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \sqrt{3}\left(\frac{1}{2} - a\right) & -\frac{1}{2} \\ 0 & -\frac{5}{4} + \frac{3}{2}a & \sqrt{3}\left(a + \frac{1}{4}\right) \end{pmatrix},$$

and the characteristic equation associated to $A(3)$ *is*

$$\lambda^3 - 3\frac{\sqrt{3}}{4}\lambda^2 + \left(-3a^2 + \frac{3}{2}a - \frac{1}{4}\right)\lambda = 0. \tag{31}$$

Obviously, all roots of the equation (31) are real; one of them being 0*. An easy calculation (which is omitted) shows that the recurrence (30) is Hyers-Ulam stable if and only if*

$$a \neq \frac{1}{2}\left(\frac{1}{2} \mp \sqrt{\sqrt{3} - \frac{17}{12}}\right).$$

Remark 3. *We thank the anonymous reviewers for their comments and suggestions on the original version of the manuscript that allowed us to improve it. In particular, the suggestion to complement the results presented here with the appropriate variants for differential equations will be an important goal for us in future research.*

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