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Caristi, Nadler and $\mathcal{H}^+$-Type Contractive Mappings and Their Fixed Points in θ-Metric Spaces

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Abstract: A new proper generalization of metric called as θ-metric is introduced by Khojasteh et al. (Mathematical Problems in Engineering (2013) Article ID 504609). In this paper, first we prove the Caristi type fixed point theorem in an alternative and comparatively new way in the context of θ-metric. We also investigate two θ-metrics on $\mathcal{CB}(X)$ (family of nonempty closed and bounded subsets of a set X). Furthermore, using the obtained θ-metrics on $\mathcal{CB}(X)$, we prove two new fixed point results for multi-functions which generalize the results of Nadler and Lim type in the context of such spaces. In order to illustrate the usability of our results, we equipped them with competent examples.

Keywords: θ-metric; θ-Hausdorff distance; multivalued mapping; fixed point

1. Introduction

A wide range of pertinence made fixed point theory one of the most attractive areas of research in nonlinear analysis and hence mathematics. Fixed point results are the indispensable aid for showing the existence of solutions, not only in mathematical sciences, but also in game theory and economics. Kakutani [1] provided one such standard tool by means of a generalized form of Brouwer's fixed point theorem, which is used to prove the existence of Nash equilibrium in non-cooperative games. In order to study the applications of fixed point theorems and their equivalence to other results like intersection theorems, we refer the readers to a monograph of Border [2].

One of the most celebrated and applicable results in nonlinear analysis is Banach Contraction Principle (BCP), which inspired many mathematicians to work in fixed point theory. A number of generalizations of BCP have been obtained by many fixed point theorists in order to achieve the likelihood of more general fixed point results for mappings (both single and multivalued) in metric type spaces (cf. Boyd and Wong [3], Meir and Keeler [4], Geraghty [5], Lim [6], Khojasteh et al. [7], etc.). The famous version of BCP for multivalued mappings is obtained by Nadler [8] using the notion of the Pompeiu–Hausdorff metric. The fixed point theorem of Nadler is generalized by many authors in complete metric spaces, one of which is given by Pathak et al. in [9] using the notion of $\mathcal{H}^+$ metric.

An interesting and fruitful generalization of the Banach Contraction Principle (BCP) on a complete metric space is the Caristi fixed point theorem (Caristi's FPT) [10]. Caristi FPT is equivalent to the Ekeland's variational principle and Takahashi's nonconvex minimization theorem [11,12]. Weston [13] proved the equivalence of the conclusion of Caristi's fixed point theorem with metric completeness. The result of Caristi has been extended and generalized in various ways (cf. [14,15] and references therein). Caristi's [10] fixed point theorem can be stated as follows: for a mapping T on a complete metric space (X, d), if there exists a lower semicontinuous function ψ from X into $[0, \infty)$ such that

$$d(x, Tx) \leq \psi(x) - \psi(Tx)$$

for every $x \in X$, then T has a fixed point.

On the other hand, in an effort to generalize BCP, which holds in all complete metric spaces, to a wide class of spaces, Khojasteh et al. [16] coined the notion of θ-metric. This proper generalization of metric is accomplished by replacing the triangular inequality with a weaker assumption. The authors in [16] also investigated the topology induced by θ-metric and presented some topological properties of this space. In addition to this, they characterized the BCP and Caristi type fixed point theorems in the setting of θ-metric space.

In this article, we prove the Caristi fixed point theorem in the θ-metric setting with a novel approach of proof. In addition to this, we investigate $\mathcal{H}_\theta$ and $\mathcal{H}_\theta^+$ metrics along the lines of [9,17,18]. Moreover, we prove some fixed point results for multivalued mappings along with illustrative examples.

The flow of work in this article is as follows: Section 2 presents some of the basic concepts. First, in Section 3, we prove the Caristi type fixed point theorems in an alternative and comparatively new way in the context of θ-metric. We also investigate two θ-metrics on $\mathcal{CB}(X)$ in Section 4. Furthermore, fixed point theorems for multifunction in the context of θ-metric spaces are proved in Section 5, which generalize various metric fixed point results. We equipped this article with competent examples.

2. Preliminaries

Let $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ be the set of all natural numbers. Let us begin with the following definition.

Definition 1 ([16]). *Let mapping $\theta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be continuous in both variables and $Im(\theta) = \{\theta(u, v) : u \geq 0, v \geq 0\}$. Then, θ is said to be a B-action if and only if the following hold:*

(i) $\theta(0, 0) = 0$ *and* $\theta(v, u) = \theta(u, v)$ *for all* $v, u \geq 0$,

(ii)

$$\theta(u, v) < \theta(p, q) \quad \text{if} \quad \begin{cases} \text{either } u < p, \ v \leq q, \\ \\ \text{or } u \leq p, \ v < q, \end{cases}$$

(iii) *for each* $r \in Im(\theta)$ *and for each* $u \in [0, r]$, *there exists* $v \in [0, r]$ *such that* $\theta(v, u) = r$,

(iv) $\theta(u, 0) \leq u$, *for all* $u > 0$.

The set of all B-actions is denoted by Y.

Example 1 ([16]). *The following functions are examples of B-action:*

(i) $\theta(v, u) = k(v + u)$, *where* $k \in (0, 1]$,

(ii) $\theta(v, u) = k(v + u + vu)$, *where* $k \in (0, 1]$,

(iii) $\theta(v, u) = vu/(1 + vu)$,

(iv) $\theta(v, u) = v + u + \sqrt{vu}$.

In the following result, the notion of inverse B-action η is brought into focus.

Lemma 1 ([16]). *Let θ be a B-action. For each $r \in Im(\theta)$ and $s \in B = [0, r]$, there exist $t \in [0, r]$ and a function $\eta : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ such that $\eta(r, s) = t$. Then, one derives the following.*

(a_1) $\eta(0, 0) = 0$.

(a_2) $\theta(\eta(r,s),s) = r$ *and* $\theta(r,\eta(s,r)) = s$.
(a_3) η *is continuous with respect to the first variable.*
(a_4) *If* $\eta(r,s) \geq 0$, *then* $0 \leq s \leq r$.

In what follows, θ denotes B-actions. The authors of [16] formulated the concept of θ-metric spaces as follows:

Definition 2 ([16])**.** *A mapping* $d_\theta : X \times X \to \mathbb{R}^+$ *is said to be* θ*-metric on a nonempty set* X *with respect to* B*-action* $\theta \in \Upsilon$ *if the following hold true:*

(i) $d_\theta(a,b) = 0$ *if and only if* $a = b$,
(ii) $d_\theta(a,b) = d_\theta(b,a)$, *for all* $a, b \in X$,
(iii) $d_\theta(a,b) \leq \theta(d_\theta(a,c), d_\theta(c,b))$, *for all* $a, b, c \in X$.

A pair (X, d_θ) *is called* θ*-metric space. For examples of* θ*-metric, readers are referred to [16].*

In the following definition, the notions of convergence of a sequence, Cauchy sequence, completeness of θ-metric and continuity of mapping are discussed.

Definition 3 ([16])**.** *Let* (X, d_θ) *be a* θ*-metric space. Then,*

(i) *a sequence* $\{x_n\}$ *in* X *is said to be converged to* $x \in X$ *if* $d_\theta(x_n, x) \to 0$ *as* $n \to \infty$,
(ii) *a sequence* $\{x_n\}$ *is Cauchy if, for each* $\epsilon > 0$, *there exists* $N > 0$ *such that, for all* $m \geq n \geq N$, $d_\theta(x_n, x_m) < \epsilon$.
(iii) (X, d_θ) *is complete if every Cauchy sequence* $\{x_n\}$ *is convergent in* X.
(iv) *A self-mapping* T *on* (X, d_θ) *is said to be* θ*-continuous if* $d_\theta(Tx_n, Tx) \to 0$ *whenever* $d_\theta(x_n, x) \to 0$ *as* $n \to \infty$.

The authors of [16] observed that, in a θ-metric space (X, d_θ), every open ball is an open set and the topology is formed by the collection of open sets (denoted by τ_{d_θ}). A pair (X, τ_{d_θ}) is a Hausdorff topological space induced by a θ-metric on X. The set $\{B_{d_\theta}(u, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at u and the topology τ_{d_θ} is first countable.

Remark 1. *Recently, Brzdek et al. [19] introduced a notion of generalized* d_q *metric which can be defined as: a function* $d : X \times X \to \mathbb{R}^+$ *satisfying following axioms for all* $a, b, c \in X$,

(B_1) *if* $d(a,b) = 0$ *and* $d(b,a) = 0$, *then* $a = b$;
(B_2) *there exist a mapping* $\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *which is nondecreasing with respect to each variable such that* $d(a,c) \leq \mu(d(a,b), d(b,c))$.

If we compare the two functions θ *and* μ, *it is observed that* μ *enjoys more freedom over* θ, *since continuity and symmetry are relaxed in case of* μ. *Prima facie, it appears that the concept of generalized* d_q*-metric is more general than the* θ*-metric. It is also noteworthy here that, in order to generalize the notion of metric in the analogous form, the continuity and symmetry are necessary for* θ *function.*

We consider the following class of mappings which act as auxiliary functions in defining Caristi type contractive conditions.

Definition 4 ([16])**.** *Suppose that* (X, d_θ) *is a complete* θ*-metric space. Define* $\mathcal{P}_\theta$ *as the class of all maps* $\mu : X \times X \to [0, +\infty)$ *which satisfies the following conditions:*

(E1) *there exists* $\hat{x}$ *such that* $\mu(\hat{x}, .)$ *is bounded below and lower semicontinuous, and* $\mu(., y)$ *is upper semicontinuous for each* $y \in X$,

(E2) $\mu(x,y) = 0$ *if and only if* $x = y$,
(E3) $\theta(\mu(x,y),\mu(y,z)) \le \mu(x,z)$ *for each* $x,y,z \in X$.

By virtue of the above definition, the following results hold:

Lemma 2 ([16]). $\mu(x,y) \le \eta(\mu(x,z),\mu(x,y))$ *for each* $x,y,z \in X$.

Definition 5 ([16]). *Let* Γ_θ *denote the family of functionals* $\nu : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

(i) $\nu(\theta(x,y)) \le \theta(\nu(x),\nu(y))$ *for each* $x,y \in \mathbb{R}^+$,
(ii) ν *is nondecreasing map,*
(iii) ν *is continuous,*
(iv) $\nu(t) = 0$ *if and only if* $t = 0$.

Example 2. *(a)* *Let* $\theta(t,s) = \frac{st}{1+st}$; *thus,* $Im(\theta) = [0,1)$. *Now, let* $\mu_1 : X \times X \to [0,+\infty)$ *be defined by*

$$\mu_1(x,y) = \begin{cases} \exp\Big(\varphi(y) - \varphi(x)\Big) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

(b) *Let* $\theta(s,t) = s+t$; *thus,* $Im(\theta) = [0,+\infty)$. *Now, let* $\mu_2 : X \times X \to [0,+\infty)$ *be defined by*

$$\mu_2(x,y) = \varphi(y) - \varphi(x).$$

(c) *Let* $\theta(s,t) = \sqrt[2n+1]{s+t}$, $n \ge 1$; *thus,* $Im(\theta) = [0,+\infty)$. *Now, let* $\mu_3 : X \times X \to [0,+\infty)$ *be defined by*

$$\mu_3(x,y) = \sqrt[2n+1]{\varphi(y) - \varphi(x)}.$$

If $\varphi : X \to \mathbb{R}$ *is a lower bounded, lower semicontinuous function, then clearly* $\mu_i \in \mathcal{P}_\theta$, $i = 1,2,3$.

Definition 6. *Let* (X,d_θ) *be a* θ*-metric space.*

- *A point* $x \in X$ *is called a fixed point of a mapping* $f : X \to X$ *if and only if* $f(x) = x$.
- *A point* $x \in X$ *is called a periodic point of a mapping* $f : X \to X$ *if and only if there exists* $n \in \mathbb{N}$ *such that* $f^n(x) = x$.

3. Caristi Type Fixed Point Theorems

The following result is the restatement of Caristi type fixed point theorem presented by Khojasteh et al. in ([16], Theorem 34). We prove this theorem with a new and simple approach in the context of θ-metric space.

Theorem 1. *Let* (X,d_θ) *be a complete* θ*-metric space and* $\mu \in \mathcal{P}_\theta$ *and* $\nu \in \Gamma_\theta$. *Let* $T : X \to X$ *be a mapping satisfying*

$$\nu(d_\theta(x,Tx)) \le \mu(Tx,x) \tag{1}$$

for any $x \in X$. *Then,* T *has a fixed point in* X.

Proof. Let us define a multivalued map $S : X \to 2^X$ as

$$S(u) = \{v \in X : \nu(d_\theta(u,v)) \le \mu(u,v)\}, \text{ for any } u \in X.$$

Since $0 = \nu(0) = \nu(d_\theta(u,u)) \le \mu(u,u) = 0$, thus $u \in S(u)$. Hence, $S(u)$ is nonempty for every $u \in X$.

We now show that, for each $v \in S(u)$, $S(v) \subseteq S(u)$.

Let $v \in S(u)$. This gives us $\nu(d_\theta(u, v)) \leq \mu(u, v)$. As $S(v)$ is nonempty, let $w \in S(v)$. Then, $\nu(d_\theta(v, w)) \leq \mu(v, w)$. We show that $w \in S(u)$. Since ν is nondecreasing,

$$\begin{aligned} \nu(d_\theta(u, w)) &\leq \nu(\theta(d_\theta(u, v), d_\theta(v, w))) \\ &\leq \theta(\nu(d_\theta(u, v)), \nu(d_\theta(v, w))) \\ &\leq \theta(\mu(u, v), \mu(v, w)) \\ &\leq \mu(u, w). \end{aligned}$$

Therefore, $w \in S(u)$. Thus, $S(v) \subset S(u)$.

We define a sequence $\{u_n\}$ in X which starts from some arbitrary $u_1 \in X$. Suppose u_{n-1} is known and choose $u_{n+1} \in S(u_n)$ such that, for $\hat{u} \in X$,

$$\lim_{n\to\infty} \mu(\hat{u}, u_{n+1}) \leq \inf_{z \in S(u_n)} \mu(\hat{u}, z) = \tau. \tag{2}$$

For any $n \in \mathbb{N}$, since $u_{n+1} \in S(u_n)$, by using Lemma 2, we have

$$\begin{aligned} \nu(d_\theta(u_n, u_{n+1})) &\leq \mu(u_n, u_{n+1}) \\ &\leq \eta(\mu(\hat{u}, u_n), \mu(\hat{u}, u_{n+1})). \end{aligned}$$

Taking the limit as $n \to \infty$ and using continuity of ν, we have

$$\begin{aligned} \lim_{n\to\infty} \nu(d_\theta(u_n, u_{n+1})) &= \nu(\lim_{n\to\infty} d_\theta(u_n, u_{n+1})) \\ &\leq \eta(\lim_{n\to\infty} \mu(\hat{u}, u_n), \lim_{n\to\infty} \mu(\hat{u}, u_{n+1})) \\ &< \eta(\tau, \tau) = 0. \end{aligned}$$

This yields us

$$\nu(\lim_{n\to\infty} d_\theta(u_n, u_{n+1})) = 0. \tag{3}$$

From (iv) of Definition 5, we get

$$\lim_{n\to\infty} d_\theta(u_n, u_{n+1}) = 0.$$

Now, using (i) of Definition 5, we have

$$\begin{aligned} \nu(d_\theta(u_n, u_{n+k})) &\leq \nu(\theta(d_\theta(u_n, u_{n+1}), d_\theta(u_{n+1}, u_{n+k}))) \\ &\leq \theta(\nu(d_\theta(u_n, u_{n+1})), \nu(d_\theta(u_{n+1}, u_{n+k}))). \end{aligned}$$

Taking the limit as $n \to \infty$ on both sides of the above inequality, we get

$$\begin{aligned}
\lim_{n\to\infty} \nu(d_\theta(u_n, u_{n+k})) &\le \theta(0, \lim_{n\to\infty} \nu(d_\theta(u_{n+1}, u_{n+k}))) \\
&\le \lim_{n\to\infty} \nu(d_\theta(u_{n+1}, u_{n+k})) \\
&\le \lim_{n\to\infty} \theta(\nu(d_\theta(u_n, u_{n+1})), \nu(d_\theta(u_{n+1}, u_{n+k}))) \\
&\le \theta(0, \lim_{n\to\infty} \nu(d_\theta(u_{n+2}, u_{n+k}))) \\
&\le \lim_{n\to\infty} \nu(d_\theta(u_{n+2}, u_{n+k})) \\
&\le \cdots \\
&\le \lim_{n\to\infty} \nu(d_\theta(u_{n+k-1}, u_{n+k})) \\
&\le 0.
\end{aligned}$$

Therefore, by (iv) of Definition 5, we have

$$\lim_{n\to\infty} d_\theta(u_n, u_{n+k}) = 0.$$

Thus, $\{u_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists some $p \in X$ such that $u_n \to p$ as $n \to \infty$.

Since ν is continuous and μ is upper semicontinuous in the first variable,

$$\begin{aligned}
\nu(d_\theta(p, u_n)) &\le \limsup_{m\to\infty} \nu(d_\theta(u_m, u_n)) \\
&\le \limsup_{m\to\infty} \mu(u_m, u_n) \\
&\le \mu(p, u_n). \qquad (4)
\end{aligned}$$

Thus, $p \in \cap_{n=1}^{\infty} S(u_n)$. Hence, $\cap_{n=1}^{\infty} S(u_n)$ is nonempty and $S(p) \subseteq \cap_{n=1}^{\infty} S(u_n)$.
Now, for any $q \in \cap_{n=1}^{\infty} S(u_n)$ such that $u_n \neq q$, by Definition 4, we have

$$\mu(\hat{u}, q) \le \inf_{n\in\mathbb{N}} \mu(\hat{u}, u_n) = \lim_{n\to\infty} \mu(\hat{u}, u_n).$$

Therefore, by Inequality (1), we have

$$0 \le \nu(d_\theta(u_n, q)) \le \mu(u_n, q) \le \eta(\mu(\hat{u}, u_n), \mu(\hat{u}, q)) < \eta(\mu(\hat{u}, u_n), \lim_{n\to\infty} \mu(\hat{u}, u_n)).$$

Varying n over $\mathbb{N}$, we get

$$\lim_{n\to\infty} \nu(d_\theta(u_n, q)) = 0.$$

Therefore $\{u_n\} \to q$. The uniqueness of limit of a sequence ensures that $p = q$. Thus, we have, $S(p) \subseteq \cap_{n=1}^{\infty} S(u_n) = \{p\}$. Thus, $S(p) = \{p\}$.

In addition, from Inequality (1), we have $\nu(d_\theta(p, Tp)) \le \mu(p, Tp)$. This yields $Tp \in S(p) = \{p\}$. Thus, $p = Tp$. □

Remark 2. *The earlier proofs of Caristi fixed point theorem in metric space setting involve assigning a partial order on X. Then, they used Zorn's Lemma or the Brezis Browder order principle or transfinite induction. Even Khojasteh et al. [16] proved the above theorem using the same technique.*

In our proof, we do not assume any partial order on X, so Zorn's Lemma or the Brezis–Browder theorem can not be applied. Thus, our proof is different from earlier proofs and comparatively new in the context of space as well as technique.

As a consequence, we obtain the following theorems.

Theorem 2. *Let (X, d_θ) be a complete θ-metric space and $\mu \in \mathcal{P}_\theta$ and $\nu \in \Gamma_\theta$. Let $T : X \to P(X)$ be a mapping satisfying*

$$\nu(d_\theta(x, y)) \leq \mu(y, x), \tag{5}$$

for any $x \in X$ and $y \in Tx$. Then, T has a fixed point in X.

Proof. The proof follows in the same manner as proof of Theorem 1. □

Theorem 3. *Let (X, d_θ) be a complete θ-metric space and $\mu \in \mathcal{P}_\theta$ and $\nu \in \Gamma_\theta$. Let $f : X \to X$ be a mapping satisfying*

$$\nu(d_\theta(x, f^n x)) \leq \mu(f^n x, x), \tag{6}$$

for any $x \in X$. Then, T has a periodic point in X.

Proof. Let $T : X \to X$ be defined by $Tx = f^n x$. Then, from Inequality (6), we have

$$\nu(d_\theta(x, Tx)) \leq \mu(Tx, x)$$

for any $x \in X$. Then, by Theorem 1, $Tu = u$. Hence, $f^n u = u$, i.e., u is periodic point of f. □

Example 3. *Let $X = [0, \infty)$ with $d_\theta(x, y) = |x - y|$ and $\theta(s, t) = s + t$. We define a mapping $T : X \to X$ as*

$$Tx = \begin{cases} 0, & \text{if } x = 0, \\ \frac{3}{2}, & \text{if } x \in (0, 1], \\ \frac{x}{2}, & \text{if } x > 1. \end{cases}$$

Let $\nu : [0, \infty) \to [0, \infty)$ be defined by $\nu(t) = \ln(1 + t)$. We now verify that $\nu \in \Gamma$.

(i) *For every $s, t \in Im(\theta)$, we have*

$$\begin{aligned} \theta(\nu(s), \nu(t)) &= \nu(s) + \nu(t) \\ &= \ln(1 + s) + \ln(1 + t) \\ &= \ln[(1 + s)(1 + t)] \\ &= \ln(1 + s + t + st) \\ &> \ln(1 + s + t) \\ &= \nu(s + t) = \nu(\theta(s, t)). \end{aligned}$$

(ii) *Since $\ln(t)$ is nondecreasing for $t > 1$, $\nu(t)$ for $t > 0$ is too.*

(iii) *Continuity of log function implies continuity of ν.*

(iv) *Let $\nu(t) = 0 \Leftrightarrow \ln(1 + t) = 0 \Leftrightarrow 1 + t = 1 \Leftrightarrow t = 0$.*

Thus, $\nu(t) = \ln(1 + t) \in \Gamma_\theta$.

Let $\varphi : X \to [0, +\infty)$ be defined as $\varphi(x) = d_\theta(x, Tx)$. Then, one can see that φ is lower bounded and a lower semicontinuous function.

Consider $\mu(x, y) = \varphi(y) - \varphi(x)$. Then, clearly $\mu \in \mathcal{P}_\theta$.

For the above ν and μ, T satisfies

$$\nu(d_\theta(x, Tx)) \leq \mu(Tx, x)$$

for every $x \in X$. Thus, T satisfies all the conditions of Theorem 1. Consequently, T has a fixed point 0.

Here, it is worth mentioning that T does not satisfy $d(x, Tx) \leq \phi(x) - \phi(Tx)$ when $x = 1$. Thus, T does not obey the Caristi fixed point theorem.

4. $\mathcal{H}_\theta$ and $\mathcal{H}_\theta^+$ Metrics

Let (X, d_θ) be a θ-metric space. Let $\mathcal{CB}(X) = \{P \subset X : \phi \neq P \text{ is } \theta\text{-bounded and closed}\}$. For $P, Q \in \mathcal{CB}(X)$, define

$$\mathcal{H}_\theta(P, Q) = \max\left\{\sup\{d_\theta(q, P) | q \in Q\}, \sup\{d_\theta(p, Q) | p \in P\}\right\}$$

and

$$\mathcal{H}_\theta^+(P, Q) = \frac{1}{2}\left[\sup\{d_\theta(q, P) | q \in Q\} + \sup\{d_\theta(p, Q) | p \in P\}\right],$$

where

$$d_\theta(p, Q) = \inf\{d_\theta(p, q) | q \in Q\}.$$

We call $\mathcal{H}_\theta$ a θ-Pompeiu–Hausdorff distance (see [20] and references therein). We also denote $\sup\{d_\theta(p, Q) | p \in P\}$ by $\rho_\theta(P, Q)$.

Theorem 4. *$(\mathcal{CB}(X), \mathcal{H}_\theta)$ is a θ-metric space if (X, d_θ) is θ-metric space.*

Proof. Clearly, due to non-negativity and symmetry of d_θ, $\mathcal{H}_\theta$ is also non-negative and symmetric. Next, we show $\mathcal{H}_\theta(P, Q) = 0$ if and only if $P = Q$. We only require to show that $\mathcal{H}_\theta(P, Q) = 0 \implies P = Q$; the converse will be true due to property (i) of Definition 2. For this, suppose that $\mathcal{H}_\theta(P, Q) = 0$ for any $P, Q \in \mathcal{CB}(X)$. This implies that $\sup\{d_\theta(q, P) | q \in Q\} = 0$, which gives us $d_\theta(q, P) = 0$ for $q \in Q$. This yields $q \in \overline{P}$. Thus, $Q \subset \overline{P} = P$. Similarly, $\sup\{d_\theta(p, Q) | p \in P\} = 0$ implies $p \in \overline{Q}$, which yields $P \subset \overline{Q} = Q$. Therefore, $P = Q$.

Now, it remains to prove that $\mathcal{H}_\theta(P, R) \leq \theta(\mathcal{H}_\theta(P, Q), \mathcal{H}_\theta(Q, R))$ for any $P, Q, R \in \mathcal{CB}(X)$. Suppose $P, Q, R \in \mathcal{CB}(X)$. Let $u \in P$ be arbitrary; there exists $v \in Q$ and $\epsilon > 0$ such that

$$d_\theta(u, v) \leq d_\theta(u, Q) + \frac{\epsilon}{2}.$$

In addition, there exists $w \in R$ such that

$$d_\theta(v, w) \leq d_\theta(v, R) + \frac{\epsilon}{2}.$$

Now,

$$\begin{aligned} d_\theta(u, R) &\leq d_\theta(u, w) \\ &\leq \theta(d_\theta(u, v), d_\theta(v, w)) \\ &< \theta\big(d_\theta(u, Q) + \frac{\epsilon}{2}, d_\theta(v, R) + \frac{\epsilon}{2}\big) \\ &< \theta\big(\mathcal{H}_\theta(P, Q) + \frac{\epsilon}{2}, \mathcal{H}_\theta(Q, R) + \frac{\epsilon}{2}\big). \end{aligned}$$

Since u is arbitrary in P, we have

$$\sup\{d_\theta(a, R) | a \in P\} \leq \theta\big(\mathcal{H}_\theta(P, Q) + \frac{\epsilon}{2}, \mathcal{H}_\theta(Q, R) + \frac{\epsilon}{2}\big).$$

Since ϵ is arbitrary, the above inequality yields

$$\sup\{d_\theta(a, R) | a \in P\} \leq \theta\big(\mathcal{H}_\theta(P, Q), \mathcal{H}_\theta(Q, R)\big). \tag{7}$$

Using the similar argument, we obtain

$$\sup\{d_\theta(c,P)|c\in R\}\leq\theta\big(\mathcal{H}_\theta(P,Q),\mathcal{H}_\theta(Q,R)\big). \tag{8}$$

Thus, from Inequalities (7) and (8), we get

$$\mathcal{H}_\theta(P,R)\leq\theta\big(\mathcal{H}_\theta(P,Q),\mathcal{H}_\theta(Q,R)\big).$$

□

Theorem 5. *$(\mathcal{CB}(X),\mathcal{H}_\theta^+)$ is θ-metric space if (X,d_θ) is θ-metric space.*

Proof. We only prove $\mathcal{H}_\theta^+(P,R)\leq\theta\big(\mathcal{H}_\theta^+(P,Q),\mathcal{H}_\theta^+(Q,R)\big)$ for any $P,Q,R\in\mathcal{CB}(X)$. Other things follow in the same way as in the proof of Theorem 4. Suppose $P,Q,R\in\mathcal{CB}(X)$. Letting $u\in P$, there exists $v\in Q$ and $\epsilon>0$ such that

$$d_\theta(u,v)\leq d_\theta(u,Q)+\frac{\epsilon}{2}.$$

In addition, there exists $w\in R$ such that

$$d_\theta(v,w)\leq d_\theta(v,R)+\frac{\epsilon}{2}.$$

Furthermore, for $\epsilon>0$, there exist $a\in P$ and $b\in Q$ such that

$$\frac{1}{2}\big[d_\theta(a,Q)+d_\theta(b,P)\big]+\frac{\epsilon}{2}\geq d_\theta(a,Q),$$

$$\frac{1}{2}\big[d_\theta(b,P)+d_\theta(a,Q)\big]+\frac{\epsilon}{2}\geq d_\theta(b,P).$$

Now,

$$\begin{aligned}
d_\theta(a,R)&\leq d_\theta(a,c)\\
&\leq\theta(d_\theta(a,b),d_\theta(b,c))\\
&<\theta\big(d_\theta(a,Q)+\frac{\epsilon}{2},d_\theta(b,R)+\frac{\epsilon}{2}\big)\\
&<\theta\big(\frac{1}{2}\big[d_\theta(a,Q)+d_\theta(b,P)\big]+\epsilon,\frac{1}{2}\big[d_\theta(b,R)+d_\theta(c,Q)\big]+\epsilon\big)\\
&\leq\theta\big(\mathcal{H}_\theta^+(P,Q)+\epsilon,\mathcal{H}_\theta^+(Q,R)+\epsilon\big).
\end{aligned}$$

Taking supremum in the above inequality, we get

$$\sup\{d_\theta(a,R)|a\in P\}\leq\theta\big(\mathcal{H}_\theta^+(P,Q)+\epsilon,\mathcal{H}_\theta^+(Q,R)+\epsilon\big).$$

Since ϵ is arbitrary, this yields

$$\sup\{d_\theta(a,R)|a\in P\}\leq\theta\big(\mathcal{H}_\theta^+(P,Q),\mathcal{H}_\theta^+(Q,R)\big). \tag{9}$$

Using a similar argument, we get

$$\sup\{d_\theta(c,P)|c\in R\}\leq\theta\big(\mathcal{H}_\theta^+(P,Q),\mathcal{H}_\theta^+(Q,R)\big). \tag{10}$$

Adding Inequalities (9) and (10), we get

$$\frac{1}{2}\big[\sup\{d_\theta(a,R)|a\in P\}+\sup\{d_\theta(c,P)|c\in R\}\big]\leq\theta\big(\mathcal{H}_\theta^+(P,Q),\mathcal{H}_\theta^+(Q,R)\big),$$

that is,

$$\mathcal{H}_{\theta}^{+}(P,R) \leq \theta\big(\mathcal{H}_{\theta}^{+}(P,Q), \mathcal{H}_{\theta}^{+}(Q,R)\big).$$

□

Remark 3. *$\mathcal{H}_\theta$ and $\mathcal{H}_\theta^+$ defined above are equivalent metrics on $\mathcal{CB}(X)$, since*

$$\frac{1}{2}\mathcal{H}_\theta(P,Q) \leq \mathcal{H}_\theta^+(P,Q) \leq \mathcal{H}_\theta(P,Q).$$

It is worth mentioning here that the equivalence of the two θ-metric does not mean that the results proved with one are equivalent to others. This is shown by means of some examples in [9] in the case of metric spaces.

5. Fixed Point Results for Set-Valued Mappings

This section presents some fixed point results in θ-metric spaces for multivalued mappings. Firstly, we obtain a fixed point theorem using θ–Pompeiu–Hausdorff metric. Secondly, we prove some fixed point theorems of Pathak and Sahzad type [9] for the multivalued case using the $\mathcal{H}_\theta^+$ metric. The results presented here generalize various results of the metric fixed point theory.

5.1. Lim–Nadler Type Fixed Point Theorems

Let Φ_θ be a collection of mappings $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with

(i) for each $\alpha > 0$, there exists $\beta > 0$ such that $\alpha < u < \beta$ implies $\varphi(u) \leq \alpha$,
(ii) $\varphi(\theta(u,v)) \leq \theta(\varphi(u), \varphi(v))$ for all $u, v \in Im(\theta)$,
(iii) $\varphi(u) = 0$ if and only if $u = 0$.

The following result is required to prove the fixed point theorem.

Lemma 3. *Let $\varphi \in \Phi_\theta$ such that for some $u > 0$, $\varphi(u) \leq u$. Then,*

(i) $\varphi(l) < l$ *for every* $l > 0$,
(ii) *for every sequence* $\{l_n\}$ *such that* $l_n \to l$ *as* $n \to \infty$, $l_n \geq l > 0$, *we have*

$$\limsup_{n\to\infty} \varphi(l_n) < l.$$

Proof. (i) Suppose there exists $c > 0$ such that $\varphi(c) = c$. Let $c \in Im(\theta)$ then for every $u \in Im(\theta)$ such that $u < c$, we can find $v \in Im(\theta)$ by (iii) of Definition 1 such that $\theta(u,v) = c$.

Therefore,

$$c = \varphi(c) = \varphi(\theta(u,v)) \leq \theta(\varphi(u), \varphi(v)) \leq \theta(u,v) = c.$$

This implies that $\varphi(u) = u$ for every $u \leq c$. Since $\varphi \in \Phi_\theta$, for $\alpha = \frac{c}{2}$, there exists $\beta > \frac{c}{2}$ such that $\varphi(l) < \frac{c}{2}$ for every $l \in (\frac{c}{2}, \beta)$. If $l \in (\frac{c}{2}, \beta) \cap (\frac{c}{2}, c)$, we have $\varphi(l) = l > \frac{c}{2}$, which is a contradiction. Thus, $\varphi(l) < l$ for every $l > 0$.

(ii) Letting $l_n \geq l$, then we can find $l^* > 0$ such that $\theta(l, l^*) = l_n$ by using (iii) of Definition 1. Now, we have

$$\varphi(l_n) = \varphi\big(\theta(l,l^*)\big) \leq \theta\big(\varphi(l), \varphi(l^*)\big) < \theta(l,l^*) = l_n.$$

Therefore, $\limsup_{n\to\infty} \varphi(l_n) < l$. □

Example 4. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\varphi(t) = \ln(1+t)$. Clearly, $\varphi(t) < t$. Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by $\theta(s,t) = s + t + st$. We now verify that $\varphi \in \Phi_\theta$.*

(i) *for $\alpha > 0$, we consider $\beta = \exp(\alpha) - 1$; then, $\beta > \alpha$. For $t \in (\alpha, \beta)$, we have $\varphi(t) = \ln(1+t) < \ln\exp(\alpha) = \alpha$.*

(ii) *For every $s, t \in Im(\theta)$, we have*

$$\begin{aligned}\theta(\varphi(s), \varphi(t)) &= \varphi(s) + \varphi(t) + \varphi(s)\varphi(t)\\ &= \ln(1+s) + \ln(1+t) + \ln(1+s)\ln(1+t)\\ &> \ln(1+s) + \ln(1+t)\\ &= \ln(1+s+t+st) = \varphi(s+t+st) = \varphi(\theta(s,t)).\end{aligned}$$

(iii) $\varphi(t) = 0 \iff \ln(1+t) = 0 \iff 1+t = 1 \iff t = 0.$

Thus, $\varphi \in \Phi_\theta$.

Theorem 6. *Let (X, d_θ) be a complete θ-metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued mapping such that there exists $\varphi \in \Phi_\theta$ satisfying*

$$\mathcal{H}_\theta(Ta, Tb) \le \varphi(d_\theta(a, b))$$

for all $a, b \in X$. Then, T has a fixed point.

Proof. Let us take arbitrary a_0 in X and Fix $a_1 \in Ta_0$. We choose $a_2 \in Ta_1$ such that

$$d_\theta(a_1, a_2) \le \mathcal{H}_\theta(Ta_0, Ta_1) \le \varphi(d_\theta(a_0, a_1)) < d_\theta(a_0, a_1).$$

In general, if a_n is chosen such that $a_n \notin Ta_n$, then we can choose $a_{n+1} \in Ta_n$ such that

$$d_\theta(a_n, a_{n+1}) \le \mathcal{H}_\theta(Ta_{n-1}, Ta_n) \le \varphi(d_\theta(a_{n-1}, a_n)) < d_\theta(a_{n-1}, a_n). \tag{11}$$

Then, $\{d_n = d_\theta(a_n, a_{n+1})\}$ is a strictly decreasing sequence. Thus, there exists some $d \ge 0$ such that $d_n \to d$. Suppose $d > 0$.

Then, from Inequality (11), we have

$$d_n \le \varphi(d_n) \le d_n.$$

Tending n to ∞, we get

$$d \le \lim_{n\to\infty} \varphi(d_n) \le d.$$

That is, $\lim_{n\to\infty} \varphi(d_n) = d$, which contradicts (ii) of Lemma 3. Thus, $d = 0$. Hence, $\mathcal{H}_\theta(Ta_n, Ta_{n+1}) \to 0$, $d_n \to 0$ as $n \to \infty$.

Supposing that $\{a_n\}$ is not Cauchy, then there exist two subsequences of $\{a_n\}$ say $\{a_{n(k)}\}$, $\{a_{m(k)}\}$ and $\epsilon > 0$ such that

$$d_\theta(a_{n(k)}, a_{m(k)}) \ge \epsilon$$

for all k, where $m(k) > n(k) \ge k$. Then, clearly

$$d_\theta(a_{n(k)-1}, a_{m(k)}) < \epsilon.$$

Thus, we have

$$\begin{aligned}\epsilon &\le d_\theta(a_{n(k)}, a_{m(k)})\\ &\le \theta\big(d_\theta(a_{n(k)}, a_{n(k)-1}), d_\theta(a_{n(k)-1}, a_{m(k)})\big)\\ &\le \theta\big(d_\theta(a_{n(k)}, a_{n(k)-1}), \epsilon\big).\end{aligned}$$

Tending k to ∞, we get

$$\lim_{k\to\infty} d_\theta(a_{n(k)}, a_{m(k)}) = \epsilon.$$

In addition, we have

$$
\begin{aligned}
\epsilon &\leq d_\theta(a_{n(k)+1}, a_{m(k)+1}) \\
&\leq \mathcal{H}_\theta(Ta_{n(k)}, Ta_{m(k)}) \\
&\leq \varphi(d_\theta(a_{n(k)}, a_{m(k)})) \\
&< d_\theta(a_{n(k)}, a_{m(k)}).
\end{aligned}
$$

Taking $k \to \infty$, we get

$$
\lim_{k\to\infty} \varphi(d_\theta(a_{n(k)}, a_{m(k)})) = \epsilon. \tag{12}
$$

However, due to Lemma 3(ii), we have

$$
\limsup_{k\to\infty} \varphi(d_\theta(a_{n(k)}, a_{m(k)})) < \epsilon,
$$

which contradicts Equation (12). Thus, $\{a_n\}$ is a Cauchy sequence and completeness of θ-metric space X gives rise to existence of $v \in X$ such that $a_n \to v$ as $n \to \infty$. Now,

$$
\begin{aligned}
d_\theta(v, Tv) &\leq \theta\big(d_\theta(v, a_{n+1}), \rho_\theta(Ta_n, Tv)\big) \\
&< \theta\big(d_\theta(v, a_{n+1}), \mathcal{H}_\theta(Ta_n, Tv)\big) \\
&< \theta\big(d_\theta(v, a_{n+1}), \varphi(d_\theta(a_n, v))\big) \\
&< \theta\big(d_\theta(v, a_{n+1}), d_\theta(a_n, v)\big).
\end{aligned}
$$

Letting $n \to \infty$, we get $d_\theta(v, Tv) = 0$, which implies $v \in Tv$. □

Example 5. *Let $X = [0, 10]$ be a θ-metric space with $d_\theta : X \times X \to \mathbb{R}^+$ defined as $d_\theta(x, y) = |x - y|$ and $\theta(s, t) = s + t + st$. Clearly, (X, d_θ) is complete. Let $T : X \to \mathcal{CB}(X)$ be given by*

$$
T(a) = \begin{cases} \{0\}, & \text{if } a = 0, \\ [0, \ln(1 + a)], & \text{if } a > 0. \end{cases}
$$

Let us define a mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(v) = \ln(1 + v)$. Then, clearly $\varphi \in \Phi_\theta$ and $\varphi(v) < v$, for every $v > 0$ (as shown in Example 4).

We now show that $\mathcal{H}_\theta(Ta, Tb) \leq \varphi(d_\theta(a, b))$ holds for all $a, b \in X$.

For this, let $b \geq a \geq 0$, then $Ta \subset Tb$, so we have

$$
\mathcal{H}_\theta(Ta, Tb) = \ln(1 + b) - \ln(1 + a) = \ln\left(\frac{1 + b}{1 + a}\right).
$$

Since $a \leq b$, we have $\frac{1+b}{1+a} \leq 1 + b - a$, which implies $\ln\left(\frac{1+b}{1+a}\right) \leq \ln(1 + |b - a|)$. Thus, we get $\mathcal{H}_\theta(Ta, Tb) \leq \varphi(d_\theta(a, b))$.

All of the requirements of Theorem 6 are fulfilled. Hence, T has a fixed point $a = 0$.

Remark 4. *In 1969, Meir and Keeler [4] obtained an interesting generalization of BCP on a complete metric space. In 2001, Lim [6] characterized the Meir–Keeler contraction by introducing an L-function φ which satisfies the condition (i) of class Φ_θ in metric context. Thus, our Theorem 6 characterizes Lim type fixed point results. Consequently, Theorem 6 generalizes various fixed point results of Lim–Nadler type in metric spaces in the context of both space and contractive conditions.*

5.2. Pathak and Sahzad Type Fixed Point Theorem

We require the following concepts to prove our results in this section.

Definition 7. *Let (X, d_θ) be a θ-metric space. A mapping $T : X \to \mathcal{CB}(X)$ is an $\mathcal{H}_\theta^+$-contraction if*

(i) there exists $L \in (0,1)$ such that $\mathcal{H}_\theta^+(Ta, Tb) \leq L d_\theta(a,b)$, for every $a, b \in X$,
(ii) for every $a \in X$, $b \in Ta$ and $k > 0$, there exists $c \in Tb$ such that

$$d_\theta(b,c) \leq \mathcal{H}_\theta^+(Ta, Tb) + k.$$

Definition 8. *A mapping $T : X \to \mathcal{CB}(X)$ is called generalized $\mathcal{H}_\theta^+$-contraction if $d_\theta(a,b)$ in (i) of Definition 7 is replaced by $m_\theta(a,b) = \max\{d_\theta(a,b), d_\theta(a,Ta), d_\theta(b,Tb)\}$.*

Theorem 7. *Every generalized $\mathcal{H}_\theta^+$-contraction on a complete θ-metric space has a fixed point.*

Proof. Let (X, d_θ) be a complete θ-metric space. We may choose $k > 0$ satisfying $0 < L + k = \beta < 1$. Let us take arbitrary a_0 in X and fix $a_1 \in Ta_0$. From (ii) of Definition 7, it follows that we can choose $a_2 \in Ta_1$ such that

$$\begin{aligned} d_\theta(a_1, a_2) &\leq \mathcal{H}_\theta^+(Ta_0, Ta_1) + k m_\theta(a_0, a_1) \\ &\leq (L + k) m_\theta(a_0, a_1) = \beta \ m_\theta(a_0, a_1). \end{aligned} \tag{13}$$

Similarly, there exists $a_3 \in Ta_2$ such that

$$d_\theta(a_2, a_3) \leq \beta \ m_\theta(a_1, a_2).$$

In general, if a_n be chosen, then we can choose $a_{n+1} \in Ta_n$ such that

$$\begin{aligned} d_\theta(a_{n+1}, a_{n+2}) &\leq \beta \ m_\theta(a_n, a_{n+1}) \\ &\leq \beta \max\{d_\theta(a_n, a_{n+1}), d_\theta(a_n, a_{n+1}), d_\theta(a_{n+1}, a_{n+2})\} \\ &\leq \beta \max\{d_\theta(a_n, a_{n+1}), d_\theta(a_{n+1}, a_{n+2})\}. \end{aligned} \tag{14}$$

If we take $\max\{d_\theta(a_n, a_{n+1}), d_\theta(x_{n+1}, x_{n+2})\} = d_\theta(a_{n+1}, a_{n+2})$, then, from Inequality (14), we get $d_\theta(a_{n+1}, a_{n+2}) \leq \beta \ d_\theta(a_{n+1}, a_{n+2})$, which is a contradiction. Thus, we have

$$d_\theta(a_{n+1}, a_{n+2}) \leq \beta \ d_\theta(a_n, a_{n+1}).$$

Inductively,

$$d_\theta(a_{n+1}, a_{n+2}) \leq \beta^n \ d_\theta(a_0, a_1).$$

Furthermore, we show that sequence $\{a_n\}$ is Cauchy sequence. Since we have

$$0 \leq d_\theta(a_n, a_{n+p}) \leq \theta\big(d_\theta(a_n, a_{n+1}), d_\theta(a_{n+1}, a_{n+p})\big),$$

tending n to ∞, we get

$$\begin{aligned}0 \leq \lim_{n\to\infty} d_\theta(a_n, a_{n+p}) &\leq \lim_{n\to\infty} \theta\big(\beta^n d_\theta(a_0, a_1), d_\theta(a_{n+1}, a_{n+p})\big)\\ &\leq \theta\big(0, \lim_{n\to\infty} d_\theta(a_{n+1}, a_{n+p})\big)\\ &\leq \lim_{n\to\infty} d_\theta(a_{n+1}, a_{n+p})\\ &\leq \lim_{n\to\infty} \theta\big(d_\theta(a_{n+1}, a_{n+2}), d_\theta(a_{n+2}, a_{n+p})\big)\\ &\leq \lim_{n\to\infty} \theta\big(\beta^n d_\theta(a_0, a_{n+1}), d_\theta(a_{n+2}, a_{n+p})\big)\\ &\leq \theta\big(0, \lim_{n\to\infty} d_\theta(a_{n+2}, a_{n+p})\big)\\ &\leq \cdots\\ &\leq \theta\big(0, \lim_{n\to\infty} d_\theta(a_{n+p-1}, a_{n+p})\big)\\ &\leq \theta\big(0, 0\big) = 0.\end{aligned}$$

Thus, we get $\lim_{n,m\to\infty} d_\theta(a_n, a_m) = \lim_{n\to\infty} d_\theta(a_n, a_{n+p}) = 0$. Therefore, $\{a_n\}$ is a Cauchy sequence and completeness of (X, d_θ) gives rise to existence of c in X such that $\lim_{n\to\infty} d_\theta(a_n, c) = 0$.

Now, since

$$\frac{1}{2}\big[\rho_\theta(Ta_n, Tc) + \rho_\theta(Tc, Ta_n)\big] = \mathcal{H}_\theta^+(Ta_n, Tc) \leq Lm_\theta(a_n, c),$$

where $\rho_\theta(Ta_n, Tc) = \sup\{d_\theta(a_{n+1}, Tc) | a_{n+1} \in Ta_n\}$. Thus, we have

$$\liminf_{n\to\infty} \frac{1}{2}\big[\rho_\theta(Ta_n, Tc) + \rho_\theta(Tc, Ta_n)\big] \leq Ld_\theta(c, Tc).$$

Now,

$$\begin{aligned}d_\theta(c, Tc) &= \frac{1}{2}[d_\theta(c, Tc) + d_\theta(c, Tc)]\\ &\leq \frac{1}{2}[\theta(d_\theta(c, Ta_n), \rho_\theta(Ta_n, Tc)) + \theta(d_\theta(c, Ta_n), \rho_\theta(Ta_n, Tc))].\end{aligned}$$

Taking limit as $n \to \infty$, we get

$$d_\theta(c, Tc) \leq Ld_\theta(c, Tc),$$

a contradiction. Thus, $d_\theta(c, Tc) = 0$, and hence $u \in \overline{Tc} = Tc$. □

Theorem 8. *Every $\mathcal{H}_\theta^+$-contraction on a complete θ-metric space has a fixed point.*

Proof. Proof follows from the proof of Theorem 7. □

Example 6. *Let $X = \{x, y, z\}$ and $d_\theta : X \times X \to \mathbb{R}^+$ be defined by*

$$d_\theta(x, y) = \frac{1}{5}, \quad d_\theta(x, z) = 1, \quad d_\theta(y, z) = \frac{3}{4},$$

$$d_\theta(a, a) = 0, \text{ and } d_\theta(a, b) = d_\theta(b, a) \text{ for every } a, b \in X.$$

Then, for $\theta(s, t) = s + t + st$, (X, d_θ) is a complete θ-metric space but not a metric space.

Let $T : X \to \mathcal{CB}(X)$ be such that

$$T(a) = \begin{cases} \{x\}, & \text{if } a = x, \\ \{x, y\}, & \text{if } a = y, \\ \{x, z\}, & \text{if } a = z. \end{cases}$$

First, we verify that $\mathcal{H}_\theta^+(Ta, Tb) \leq Lm_\theta(a, b) = \max\{d_\theta(a, b), d_\theta(a, Ta), d_\theta(b, Tb)\}$ holds for some $L \in (0, 1)$. Consider the following three cases:

- *If $a = x, b = y$, then $m_\theta(a, b) = \frac{1}{5}$ and $\mathcal{H}_\theta^+(Ta, Tb) = \frac{1}{10}$. Thus, $\mathcal{H}_\theta^+(Ta, Tb) \leq Lm_\theta(a, b)$ is satisfied for $L \geq \frac{1}{2}$,*
- *If $a = x, b = z$, then $m_\theta(a, b) = 1$ and $\mathcal{H}_\theta^+(Ta, Tb) = \frac{1}{2}$. Thus, $\mathcal{H}_\theta^+(Ta, Tb) \leq Lm_\theta(a, b)$ is satisfied for $L \geq \frac{1}{2}$*
- *If $a = y, b = z$, then $m_\theta(a, b) = \frac{3}{4}$ and $\mathcal{H}_\theta^+(Ta, Tb) = \frac{19}{40}$. Thus, $\mathcal{H}_\theta^+(Ta, Tb) \leq Lm_\theta(a, b)$ is satisfied for $L \geq \frac{19}{30}$.*

Now, we verify that, for every $a \in X$, $b \in Ta$ and $k > 0$, there exists $c \in Tb$ such that $d_\theta(b, c) \leq \mathcal{H}_\theta^+(Ta, Tb) + k$.

- *If $a = x$, $b = x \in T(a) = \{x\}$, $k > 0$, there exists $c = x \in T(b) = a$ such that $0 = d_\theta(b, c) \leq \mathcal{H}_\theta^+(Ta, Tb) + k$.*
- *If $a = y, b \in Ta = T(y) = \{x, y\}$,*
  (i)let $b = x, k > 0$, there exists $c = x \in Tb = x$ such that $0 = d_\theta(b, c) < \mathcal{H}_\theta^+(Ta, Tb) + k$,
  (ii) let $b = y, k > 0$, there exists $c \in Tb = \{x, y\}$ say $c = x$ such that $0 = d_\theta(b, c) < \mathcal{H}_\theta^+(Ta, Tb) + k$.
- *If $a = z, b \in Ta = T(z) = \{x, z\}$,*
  (i)let $b = x, k > 0$, there exists $c = x \in Tb = \{x\}$ such that $0 = d_\theta(b, c) < \mathcal{H}_\theta^+(Ta, Tb) + k$,
  (ii) let $b = z, k > 0$, there exists $c \in Tb = \{x, z\}$ say $c = z$ such that $0 = d_\theta(b, c) < \mathcal{H}_\theta^+(Ta, Tb) + k$.

Thus, T is a generalized $\mathcal{H}_\theta^+$-contraction for $L \in [\frac{19}{30}, 1)$. Therefore, all the requirements of Theorem 7 are fulfilled. Hence, T has at least one fixed point. Evidently, T has fixed points x, y and z here.

6. Conclusions

Khojasteh et al. [16] introduced θ-metric and generalized the notion of metric by replacing triangle inequality with a weaker form. They proved a Caristi type fixed point theorem by assigning partial order on the domain of operator and made use of Zorn's lemma. In this manuscript, we proved that the Caristi type fixed point Theorem 1 in an alternative, comparatively new and simple way.

The study of fixed points of multivalued mappings is of immense interest. For that, we investigated two θ-metrics (namely $\mathcal{H}_\theta$ and $\mathcal{H}_\theta^+$) on $\mathcal{CB}(X)$ that are equivalent. In Theorem 6, we used $\mathcal{H}_\theta$ metric and established a Lim–Nadler type fixed point theorem in the setting of θ-metric, whereas Theorems 7 and 8 are Pathak–Sahzad type fixed point results proved with the aid of $\mathcal{H}_\theta^+$ metric. Clearly, these results generalize that of Nadler [8], Lim [6], Pathak and Sahzad [9], etc. New illustrative examples are provided for better understanding of the results.

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