Near-Integrability of Periodic Klein-Gordon Lattices

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Abstract: In this paper, we study the Klein-Gordon (KG) lattice with periodic boundary conditions. It is an N degrees of freedom Hamiltonian system with linear inter-site forces and nonlinear on-site potential, which here is taken to be of the ϕ^4 form. First, we prove that the system in consideration is non-integrable in Liouville sense. The proof is based on the Morales-Ramis-Simó theory. Next, we deal with the resonant Birkhoff normal form of the KG Hamiltonian, truncated to order four. Due to the choice of potential, the periodic KG lattice shares the same set of discrete symmetries as the periodic Fermi-Pasta-Ulam (FPU) chain. Then we show that the above normal form is integrable. To do this we use the results of B. Rink on FPU chains. If N is odd this integrable normal form turns out to be KAM nondegenerate Hamiltonian. This implies that almost all low-energetic motions of the periodic KG lattice are quasi-periodic. We also prove that the KG lattice with Dirichlet boundary conditions (that is, with fixed endpoints) admits an integrable, nondegenerate normal fourth order form. Then, the KAM theorem applies as above.

Keywords: Klein-Gordon lattice; integrability; resonant Birkhoff normal form; KAM theory

MSC: 37J40; 70H06; 70H08

1. Introduction

Let us introduce the Klein-Gordon (KG) lattice described by the Hamiltonian

$$H = \sum_{j \in \mathbb{Z}} \left[\frac{p_j^2}{2} + \frac{C}{2} (q_{j+1} - q_j)^2 + V(q_j) \right], \quad p_j = \dot{q}_j. \quad (1)$$

The constant $C > 0$ measures the interaction to nearest neighbor particles (with unit masses) and $V(x)$ is a nonlinear potential. This lattice appears, for instance, as a spatial discretization of the Klein-Gordon equation

$$u_{tt} = u_{xx} + V(u). \quad (2)$$

Both models subjected to different boundary conditions are used to describe a wide variety of physical phenomena: crystal dislocation, localized excitations in ionic crystals (see e.g., [1,2]). In particular, the model (1) with the Morse potential $V(x) = D(e^{-\alpha x} - 1)^2$ has been applied in the study of the thermal denaturation of DNA [3].

Our aim is to study the regular behavior of the trajectories in the Hamiltonian system with the Hamiltonian (1), in particular for its complete integrability. When $C = 0$ the Hamiltonian is separable, and hence, integrable. There are many periodic and quasi-periodic solutions in the dynamics of (1). It is natural to investigate whether this behavior persists for C small enough (see e.g., [4]). At this point it is worth mentioning that in the anticontinuous limit $C \rightarrow 0$, there exist self-localized periodic oscillations called also *discrete breathers* [5,6]). Here we assume that C is neither very small nor too large and put $C = 1$, which can be achieved by rescaling t .

We are interested mainly in the behavior at low energy, so we take quartic (ϕ^4) potential

$$V(x) = \frac{a}{2}x^2 + \frac{\beta}{4}x^4, \quad a > 0, \quad (3)$$

which is frequently used in the research on the subject (see [1] and the literature therein).

First, periodic boundary conditions for (1) are assumed. Then one gets a system with N degrees of freedom, described by the Hamiltonian

$$H = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} \left[\frac{p_j^2}{2} + \frac{1}{2}(q_{j+1} - q_j)^2 + \frac{a}{2}(q_j)^2 + \frac{\beta}{4}(q_j)^4 \right], \quad p_j = \dot{q}_j. \quad (4)$$

Our first result concerns the integrability of the Hamiltonian system governed by (4).

Theorem 1. *The periodic KG lattice with Hamiltonian (4) is integrable only when $\beta = 0$.*

In other words, the Hamiltonian system under consideration is integrable only when it is linear. The proof of the above theorem is based on the Morales-Ramis-Simó result, which gives necessary conditions for integrability with meromorphic integrals, or, equivalently, sufficient conditions for non-integrability.

Motivated by the works of Rink [7,8], who presented the periodic FPU chain as a perturbation of an integrable and KAM nondegenerated system, namely the truncated Birkhoff-Gustavson (or resonant Birkhoff) normal form of order 4 in the neighborhood of an equilibrium, we would like to verify whether this can be done for KG lattices. As in the case of periodic FPU chain the properties of the periodic KG lattice near the equilibrium strongly depend on the parity of the number of the particles N . Assume, in addition, that $a = 1$ (see an explanation for this choice in the next section). Denote by $\bar{H}$ the resonant Birkhoff normal form of the Hamiltonian (4) truncated to order four. Then we have the following.

Theorem 2. *The fourth order normal form $\bar{H} = H_2 + \bar{H}_4$ of the periodic KG lattice is:*

- (i) *completely integrable and KAM nondegenerate for N odd;*
- (ii) *completely integrable for N even.*

Remark 1. *The statement of Theorem 2 will be made more precise in Section 4. Please note that the cases with $N = 2$ and $N = 4$ particles are rather exceptions to the general situation. The corresponding first integrals are quadratic and the KAM conditions are trivially checked.*

Therefore, for the periodic KG lattices with N odd almost all low-energy solutions are quasi-periodic.

Next, Dirichlet boundary conditions for (1) are considered. Due to Rink [9] in the case of FPU chain such a system can be viewed as an invariant symplectic submanifold of a periodic FPU chain. This approach also works for KG lattice and the corresponding result is as follows.

Theorem 3. *The fourth order normal form $\bar{H} = H_2 + \bar{H}_4$ of KG lattice with fixed endpoints is completely integrable and KAM nondegenerate.*

This result and KAM theorem show the existence of large-measure set of low-energy quasi-periodic solutions of KG lattice with fixed endpoints.

Recall that the FPU chain with n particles (with unit masses) and with periodic boundary conditions can be described by a Hamiltonian system with the Hamiltonian

$$H = \sum_j \frac{1}{2} p_j^2 + W(q_{j+1} - q_j), \quad p_j = \dot{q}_j, \quad (q_0, p_0) = (q_n, p_n), \quad (5)$$

where W is a real potential of the kind $W(x) = \frac{1}{2!}x^2 + \frac{\alpha}{3!}x^3 + \frac{\beta}{4!}x^4$. When $\alpha \neq 0, \beta = 0$ the chain is called an α -chain. Accordingly, when $\alpha = 0, \beta \neq 0$ the chain is known as a β -chain. Nishida [10] and Sanders [11] are among the first who have calculated the normal form of the FPU chain with fixed endpoints and with periodic boundary conditions, respectively. By imposing some very strong non-resonant assumptions, they verify the KAM theory conditions, yet in general resonances do exist. We already mentioned the works of Rink who proved rigorously that the periodic FPU Hamiltonian is a perturbation of a nondegenerate Liouville integrable Hamiltonian, namely the normal form of order 4 [8]. Furthermore, he described the geometry of even FPU lattice in [12], and finally rigorously proved Nishida's conjecture stating that almost all low-energetic motions in FPU with fixed endpoints are quasi-periodic [9]. One should note that Rink's results are consequence of the special symmetry and resonance properties of the FPU chain and should not be expected for lower-order resonant Hamiltonian systems (see e.g., [13]). However, several problems remain open: some of them purely computational and some of them of a more philosophical nature. Henrici and Kappeler [14,15] managed to solve practically all these problems, and generalized the results of Rink, by applying special sets of canonical variables, initially designed for the Toda chain.

Concerning the integrability of nonlinear lattices we refer to the paper of Yoshimura and Umeno [16], where the following periodic lattice is studied

$$H = \sum_j^N \frac{1}{2} p_j^2 + W(q_{j+1} - q_j) + V(q_j). \quad (6)$$

The on-site potential $V(x)$ and the nearest neighbor interaction potential $W(x)$ are of the form

$$V(x) = \sum_{k=2}^{2m} \frac{\mu_k}{k} x^k, \quad W(x) = \sum_{k=2}^{2m} \frac{\kappa_k}{k} x^k, \quad m \geq 2,$$

with $\mu_k, \kappa_k \in \mathbb{R}$ and the coefficients μ_k with odd k are assumed to be zero, hence $V(-x) = V(x)$. Under certain conditions ($N = 4n, n \in \mathbb{N}$ among them) the authors applied Ziglin's theorem [17,18] and proved that this lattice is not completely integrable with analytic first integrals. Ziglin's approach is based on the study of the monodromy group of the variational equation (see Section 3 for the definitions and the corresponding results). The authors evaluated the elements of the monodromy group at low and high energies. When examining the dynamics of the lattices at a low energy level (which is the subject of the present paper), the lowest order terms in the above potentials become dominant. This justifies our choice of the potentials in the KG model. In particular, the variational equation in [16] is closely related to ours. However, this approach does not work here—the monodromy group is abelian and cannot serve as an obstacle to integrability. We show the non-integrability of Hamiltonian system (4) through meromorphic first integrals, without any restrictive conditions, using Morales-Ramis-Simó theory [19].

Our study on the normal forms of the KG lattice (in particular, Theorems 2 and 3) is related to the results of Rink on the normal form of the periodic FPU chain and uses his approach. Notice that there are some differences between the potentials defining the models describing FPU and KG lattices. Moreover, the periodic FPU chain with three particles is integrable while Theorem 1 shows that this is not the case for the periodic KG lattice. Our goal is to see whether these differences affect the integrability and eventually the dynamics of the system corresponding to the truncated normal form. In view of the wide applicability of the KG models, we think that this study is naturally motivated.

This paper is organized as follows. In Section 2 we recall some known facts about Liouville integrable systems in real domains, action-angle variables, KAM conditions, and normal forms. We also consider the resonances and discrete symmetries of the considered Hamiltonian system. In Section 3, the necessary notions and facts about the differential Galois theory and its relations with the integrability of Hamiltonian systems in complex domains, and in particular fundamental results of Ziglin, Morales, Ramis and Simò, are presented. Then the proof of Theorem 1 is carried out. In Section 4 we give the truncated to order four normal form for the periodic KG lattice and consider the integrability of this normal form for N odd and even, respectively. For any of the cases, a more detailed description of the commuting first integrals is given. This proves Theorem 2. Section 5 is devoted to the KG lattice with fixed endpoints. Making use of the result from the previous section, the truncated normal form of the corresponding Hamiltonian is derived and Theorem 3 is proved. We finish with some remarks and possible directions to extend our results.

2. Resonances and Symmetries

In this section, we briefly recall some notions and facts about integrability of Hamiltonian systems in real domain, action-angle variables, perturbation of integrable systems and normal forms. More complete exposition can be found in [20].

Let $H(q, p)$ be a real analytic Hamiltonian defined on a $2n$ -dimensional symplectic manifold. The corresponding Hamiltonian system is

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (7)$$

It is said that a Hamiltonian system is completely integrable if there exist n independent integrals $F_1 = H, F_2, \dots, F_n$ in involution, namely $\{F_i, F_j\} = 0$ for all i and j , where $\{, \}$ is the Poisson bracket. On a neighborhood U of the connected compact level sets of the integrals $M_c = \{F_j = c_j, j = 1, \dots, n\}$ by Liouville—Arnold theorem one can introduce a special set of symplectic coordinates, I_j, φ_j , called action—angle variables. Then, the integrals $F_1 = H, F_2, \dots, F_n$ are functions of action variables only and the flow of X_H is simple

$$\dot{I}_j = 0, \quad \dot{\varphi}_j = \frac{\partial H}{\partial I_j}, \quad j = 1, \dots, n. \quad (8)$$

Therefore, near M_c , the phase space is foliated with X_{F_i} invariant tori over which the flow of X_H is quasi—periodic with frequencies $(\omega_1(I), \dots, \omega_n(I)) = (\frac{\partial H}{\partial I_1}, \dots, \frac{\partial H}{\partial I_n})$. The map

$$(I_1, I_2, \dots, I_n) \rightarrow \left(\frac{\partial H}{\partial I_1}, \frac{\partial H}{\partial I_2}, \dots, \frac{\partial H}{\partial I_n} \right) \quad (9)$$

is called frequency map.

Consider a small perturbation of an integrable Hamiltonian $H_0(I)$. According to Poincaré the main problem of mechanics is to study the perturbation of quasi-periodic motions in the system given by the Hamiltonian

$$H = H_0(I) + \varepsilon H_1(I, \varphi), \quad \varepsilon \ll 1.$$

KAM theory gives conditions on the integrable Hamiltonian H_0 which ensures the survival of most of the invariant tori. The following condition, usually called Kolmogorov's condition, is that the frequency map should be a local diffeomorphism, or equivalently

$$\det \left(\frac{\partial^2 H_0}{\partial I_i \partial I_j} \right) \neq 0 \quad (10)$$

on an open and dense subset of U . We should note that the measure of the surviving tori decreases with the increase of both perturbation and the measure of the set where above Hessian is too close to zero.

In a neighborhood of the equilibrium $(q, p) = (0, 0)$ we have the following expansion of H

$$H = H_2 + H_3 + H_4 + \dots, \\ H_2 = \sum \omega_j (q_j^2 + p_j^2), \quad \omega_j > 0.$$

We assume that H_2 is a positively defined quadratic form. The frequency $\omega = (\omega_1, \dots, \omega_n)$ is said to be in resonance if there exists a vector $k = (k_1, \dots, k_n)$, $k_j \in \mathbb{Z}$, $j = 1, \dots, n$, such that $(\omega, k) = \sum k_j \omega_j = 0$, where $|k| = \sum |k_j|$ is the order of resonance.

With the help of a series of canonical transformations close to the identity, H simplifies. In the absence of resonances the simplified Hamiltonian is called Birkhoff normal form, otherwise—Birkhoff-Gustavson normal form or resonant Birkhoff normal form, which may contain combinations of angles arising from resonances. Often to detect the behavior in a small neighborhood of the equilibrium, instead of the Hamiltonian H one considers the normal form truncated to some order

$$\bar{H} = H_2 + \dots + \bar{H}_m.$$

It is known that the truncated to any order Birkhoff normal form is integrable [20]. The truncated Birkhoff-Gustavson normal form has at least two integrals— H_2 and $\bar{H}$. Therefore, the truncated resonant normal form of two degrees of freedom Hamiltonian is integrable.

To obtain estimates of the approximation by normalization in a neighborhood of an equilibrium point we scale $q \rightarrow \varepsilon \tilde{q}$, $p \rightarrow \varepsilon \tilde{p}$. Here ε is a small positive parameter and ε^2 is a measure for the energy relative to the equilibrium energy. Then, dividing by ε^2 and removing tildes we get

$$\bar{H} = H_2 + \varepsilon \bar{H}_3 + \dots + \varepsilon^{m-2} \bar{H}_m.$$

Provided that $\omega_j > 0$ it is proven in [21] that $\bar{H}$ is an integral for the original system with error $O(\varepsilon^{m-1})$ and H_2 is an integral for the original system with error $O(\varepsilon)$ for the whole time interval. If we have more independent integrals, then they are integrals for the original Hamiltonian system with error $O(\varepsilon^{m-2})$ on the time scale $1/\varepsilon$. The first integrals for the normal form $\bar{H}$ are approximate integrals for the original system, that is, if the normal form is integrable then the original system is **near integrable** in the above sense.

Returning to the Hamiltonian of the periodic KG lattice (4) we see that its quadratic part H_2 is not in diagonal form

$$H_2 = \frac{1}{2} p^T p + \frac{1}{2} q^T L_N q. \quad (11)$$

Here L_N is the following $N \times N$ matrix

$$L_N := \begin{pmatrix} 2+a & -1 & & & -1 \\ -1 & 2+a & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2+a & -1 \\ -1 & & & -1 & 2+a \end{pmatrix}. \quad (12)$$

The eigenvalues of L_N are of the form $\omega_k^2 = a + 4 \sin^2 \frac{k\pi}{N}$. There is a symplectic Fourier-transformation $q = MQ, p = MP$ which brings H_2 in diagonal form (see e.g., [7,8])

$$H_2 = \frac{1}{2} P^T P + \frac{1}{2} Q^T \Omega Q, \quad (13)$$

where $M^{-1} L_N M = \Omega := \text{diag}(\omega_1^2, \dots, \omega_N^2)$. The variables (Q, P) are known as **phonons**. Denote for short

$$c_{kj} := \cos\left(\frac{2kj\pi}{N}\right), \quad s_{kj} := \sin\left(\frac{2kj\pi}{N}\right).$$

Then the transformation $(q, p) \rightarrow (Q, P)$ in coordinate form is

$$\begin{aligned} q_j &= \sqrt{\frac{2}{N}} \sum_{1 \leq k < \frac{N}{2}} c_{kj} Q_k + s_{kj} Q_{N-k} + \frac{(-1)^j}{\sqrt{N}} Q_{\frac{N}{2}} + \frac{1}{\sqrt{N}} Q_N \\ p_j &= \sqrt{\frac{2}{N}} \sum_{1 \leq k < \frac{N}{2}} c_{kj} P_k + s_{kj} P_{N-k} + \frac{(-1)^j}{\sqrt{N}} P_{\frac{N}{2}} + \frac{1}{\sqrt{N}} P_N \end{aligned} \quad (14)$$

From these formulas one can easily get the explicit form of the matrix M . Later on, we will make use of the inverse transform: for $1 \leq j < \frac{N}{2}$ we have

$$\begin{aligned} Q_j &= \sqrt{\frac{2}{N}} \sum_{k=1}^N c_{kj} q_k, & P_j &= \sqrt{\frac{2}{N}} \sum_{k=1}^N c_{kj} p_k, \\ Q_{N-j} &= \sqrt{\frac{2}{N}} \sum_{k=1}^N s_{kj} q_k, & P_{N-j} &= \sqrt{\frac{2}{N}} \sum_{k=1}^N s_{kj} p_k, \\ Q_N &= \frac{1}{\sqrt{N}} \sum_{k=1}^N q_k, & P_N &= \frac{1}{\sqrt{N}} \sum_{k=1}^N p_k \end{aligned} \quad (15)$$

and if N is even

$$Q_{\frac{N}{2}} = \frac{1}{\sqrt{N}} \sum_{k=1}^N (-1)^k q_k, \quad P_{\frac{N}{2}} = \frac{1}{\sqrt{N}} \sum_{k=1}^N (-1)^k p_k.$$

Finally, by simple scaling $Q_k \rightarrow \frac{1}{\sqrt{\omega_k}} Q_k, P_k \rightarrow \sqrt{\omega_k} P_k$ we get the desired form of H_2

$$H_2 = \sum_{k=1}^N \frac{\omega_k}{2} (P_k^2 + Q_k^2). \quad (16)$$

Next, we need to study the possible relations between the frequencies ω_k . In general, $\omega_k = \omega_j$ holds if only if $j = N - k$. The resonances $\omega_k : \omega_{N-k} = (1 : 1)$ (known also as internal resonances [7]) are important for the construction of the normal form. The assumption on a does not prevent the appearance of more complicated resonances. It is clear that there are plenty of resonances when $a \in \mathbb{Q}$. Moreover, there are certain irrational values of $a > 0$ for which fourth order resonances exist in the low-dimensional periodic KG lattices (see [22]). Such values of a are difficult to control in the higher dimensions, which is why from now on we put $a = 1$

$$\omega_k = \sqrt{1 + 4 \sin^2 \frac{k\pi}{N}}. \quad (17)$$

Please note that another resonant relation $2 : 2 : 1$ appears when $N = 3s$. The other possible fourth order relations are

$$\begin{aligned} (1) \quad \omega_k &= 3\omega_{k'} & (2) \quad \omega_k &= \omega_{k'} + \omega_{k''} + \omega_{k'''} \\ (3) \quad \omega_k + \omega_{k'} &= 2\omega_{k''} & (4) \quad \omega_k + \omega_{k'} &= \omega_{k''} + \omega_{k'''} \end{aligned}$$

for some $1 \leq k, k', k'', k''' \leq N$. It is clear that there are no such k, k', k'', k''' to fulfill (1) and (2) because $1 \leq \omega_k \leq \sqrt{5}$. Since (3) is a particular case of the last relation, let us consider (4). Suppose the tuple (k, k', k'', k''') is a solution of the relation

$$\sqrt{1 + 4 \sin^2 \frac{k\pi}{N}} + \sqrt{1 + 4 \sin^2 \frac{k'\pi}{N}} = \sqrt{1 + 4 \sin^2 \frac{k''\pi}{N}} + \sqrt{1 + 4 \sin^2 \frac{k'''\pi}{N}},$$

then a solution is also the tuple $(N - k, N - k', N - k'', N - k''')$, as well as any combination of the elements of these tuples. Therefore, when searching for these tuples, which are solutions of the above relation, we can consider only the cases where $k + k' + k'' + k''' \equiv 0 \pmod{N}$.

Assertion. The only possible tuples (k, k', k'', k''') , which satisfies (4) are those that can be derived from $(k, k', N - k, N - k')$ by permutations.

Remark 2. So far we have no rigorous proof for that claim, but we believe so. Direct computations show that this assertion is true for low-dimensional cases $N = 2, \dots, 6$. Numerical simulations are confirmative.

Finally, to keep symmetry in the formulas we continue to write ω_N and $\omega_{N/2}$ instead of their particular values 1 and $\sqrt{5}$.

The Hamiltonian (4) of the periodic KG lattice possesses discrete symmetries. Two of them, important for the dynamics and exactly the same as in the periodic FPU chain, are the linear mappings $R, S : T^*\mathbb{R}^N \rightarrow T^*\mathbb{R}^N$ defined by (see [8,9])

$$R : (q_1, q_2, \dots, q_{N-1}, q_N; p_1, p_2, \dots, p_{N-1}, p_N) \mapsto (q_2, q_3, \dots, q_N, q_1; p_2, p_3, \dots, p_N, p_1) \quad (18)$$

and

$$\begin{aligned} S : & (q_1, q_2, \dots, q_{N-1}, q_N; p_1, p_2, \dots, p_{N-1}, p_N) \\ & \mapsto -(q_{N-1}, q_{N-2}, \dots, q_1, q_N; p_{N-1}, p_{N-2}, \dots, p_1, p_N). \end{aligned} \quad (19)$$

It is easily seen that R can serve as a generator of a group $\langle R \rangle$, isomorphic to the cyclic group of order N and S as a generator of $\langle S \rangle$, isomorphic to the cyclic group of order two. Please note that R and S are canonical transformations $R^*(dq \wedge dp) = S^*(dq \wedge dp) = dq \wedge dp$ and they leave the Hamiltonian invariant $R^*H = (H \circ R) = S^*H = H$. Moreover, they leave the Hamiltonian vector field X_H invariant, which implies that they commute with the flow of X_H . It is observed in [8] that the subgroup $\langle R, S \rangle := \{Id, R, \dots, R^{N-1}, SR, \dots, SR^{N-1}\}$ of the symmetry group of H , with the relations $R^N = S^2 = Id, SR = R^{-1}S$, is isomorphic to the N th dihedral group. For Hamiltonian systems with symmetries, we have

Theorem 4. (see [23,24]). Let $H = H_2 + H_3 + \dots$ be the expansion of H in a neighborhood of equilibrium and G be a group of linear symplectic symmetries of H . Then a normal form $\bar{H} = H_2 + \dots + \bar{H}_m$ for H can be constructed in such a way that $\bar{H}$ is also G -symmetric.

3. Proof of Theorem 1

In this section, before giving the proof of Theorem 1, we briefly summarize the Ziglin-Morales-Ruiz-Ramis theory. More detailed description about Differential Galois theory and its relations with the integrability of Hamiltonian systems, can be found in [19,25–27].

Given an analytic Hamiltonian H , defined on a complex $2n$ -dimensional manifold $\mathcal{N}$, defining the system

$$\dot{x} = X_H(x), \quad t \in \mathbb{C}, \quad x \in \mathcal{N}. \quad (20)$$

Again, we call such Hamiltonian system integrable in Liouville sense if there exist n independent first integrals $F_1 = H, F_2, \dots, F_n$ in involution.

Suppose the system (20) has a non-equilibrium solution $\Psi(t)$, $t \in \hat{I} \subset \mathbb{C}$. The image of $\hat{I}$ by Ψ is a Riemann surface Γ . We can write the equation in variation (VE) along this solution

$$\dot{\xi} = A(t)\xi, \quad \xi \in T_{\Gamma}\mathcal{N}, \quad (21)$$

where $A(t) := DX_H(\Psi(t))$. By the existence theorem there is a fundamental solution $\Xi(t)$ of (21), analytic in vicinity of a nonsingular point t_0 . The continuation of $\Xi(t)$ along a nontrivial loop on Γ defines a linear automorphism of the vector space of solutions, analytic in the neighborhood of t_0 , called monodromy transformation. Analytically this transformation can be presented in the following way. The linear automorphism Δ_γ , associated with a loop $\gamma \in \pi_1(\Gamma, t_0)$ corresponds to multiplication of $\Xi(t)$ from the right by a constant matrix M_γ , called monodromy matrix

$$\Delta_\gamma = \Xi(t)M_\gamma.$$

The set of all these matrices forms the monodromy group $\mathcal{M}$. Ziglin [17,18] introduced the monodromy group approach to study the variational equation as a fundamental tool for obtaining necessary conditions for integrability of complex-analytic Hamiltonian system. In particular, Ziglin's approach is adopted in the above-mentioned paper of Yoshimura and Umeno [16].

We may attach another object to the system (21)—a differential Galois group. A differential field K is a field with a derivation $\partial ='$, i.e., an additive mapping satisfying Leibnitz rule. A differential automorphism of K is an automorphism commuting with the derivation. Denote by K the coefficient field in (21). Let ξ_{ij} be the elements of the fundamental matrix $\Xi(t)$ and let $L_1(\xi_{ij})$ be the extension of K generated by K and ξ_{ij} —a differential field. This extension is called a Picard-Vessiot extension. Similarly to classical Galois Theory, we define the Galois group $G := \text{Gal}_K(L_1) = \text{Gal}(L_1/K)$ to be the group of all differential automorphisms of L_1 leaving the elements of K fixed. The Galois group is, in fact, an algebraic group. It has a unique connected component G^0 which contains the identity and which is a normal subgroup of finite index. The Galois group G can be represented as an algebraic linear subgroup of $\text{GL}(n, \mathbb{C})$ by

$$\sigma(\Xi(t)) = \Xi(t)R_\sigma,$$

where $\sigma \in G$ and $R_\sigma \in \text{GL}(n, \mathbb{C})$. Please note that by its definition the monodromy group is contained in the differential Galois group of the corresponding system. Then, the following result has established

Theorem 5. (Morales-Ramis [25]) *Suppose that a Hamiltonian system has n meromorphic first integrals in involution. Then the identity component G^0 of the Galois group $G = \text{Gal}(L_1/K)$ is abelian.*

Once it is proven that G^0 is not abelian, the respective Hamiltonian system is non-integrable in the Liouville sense. Please note that the fact that G^0 is abelian does not imply necessarily integrability of the Hamiltonian system. Thus, one needs other obstructions to the integrability. A method based on the higher variational equations has been introduced in [25] and the previous Theorem has been extended in [19]. Before formulating this result let us give an idea of higher variational equations. For the system (20) with a particular solution $\Psi(t)$ we put

$$x = \Psi(t) + \varepsilon \xi^{(1)} + \varepsilon^2 \xi^{(2)} + \dots + \varepsilon^k \xi^{(k)} + \dots, \quad (22)$$

where ε is a formal small parameter. Substituting the above expression into Equation (20) and comparing terms with the same order in ε we obtain the following chain of linear non-homogeneous equations

$$\dot{\xi}^{(k)} = A(t)\xi^{(k)} + f^{(k)}(\xi^{(1)}, \dots, \xi^{(k-1)}), \quad k = 1, 2, \dots, \quad (23)$$

where $f^{(1)} \equiv 0$. The Equation (23) is called k -th variational equation (VE_k) . Of course, $(VE_1) = (VE)$. With $\Xi(t)$, the solutions of $(VE_k), k > 1$ can be found by

$$\xi^{(k)} = \Xi(t)r_k(t), \quad (24)$$

where $r_k(t)$ is a solution of

$$\dot{r}_k = \Xi^{-1}(t)f^{(k)}. \quad (25)$$

Although (VE_k) are not actually homogeneous equations, they can be put in that frame, and therefore, one can define successive extensions $K \subset L_1 \subset L_2 \subset \dots \subset L_k$, where L_k is the extension obtained by adjoining the solutions of (VE_k) . Correspondingly one can define the Galois groups $Gal(L_1/K), \dots, Gal(L_k/K)$. The following result is proven in [19].

Theorem 6. (Morales-Ramis-Simó) *If the Hamiltonian system (20) is integrable in Liouville sense then the identity component of every Galois group $Gal(L_k/K)$ is abelian.*

Notice that we apply Theorem 6 in the situation when the identity component of the Galois group $Gal(L_1/K)$ is abelian. This means that the first variational equation is solvable. Once we have the solution of (VE_1) , then the solutions of (VE_k) can be found by the method of variations of constants as explained above. Hence, the Galois groups $Gal(L_k/K)$ are solvable. One possible way to show that some of them is not commutative is to find a logarithmic term in the corresponding solution. We will explain this argument in more details in the line of the proof.

Proof of Theorem 1. Suppose $\beta \neq 0$. The following proposition is immediate.

Proposition 1. *The Hamiltonian system corresponding to (4) admits a particular solution*

$$q_j(t) := q(t) = \operatorname{sn}(\sqrt{a + \beta/2}t, \kappa), \quad p_j(t) := p(t) = \frac{d}{dt}q(t), \quad j = 1, \dots, N \quad (26)$$

and Γ is its phase curve

$$\Gamma : p^2 = a + \frac{\beta}{2} - aq^2 - \frac{\beta}{2}q^4. \quad (27)$$

Here sn is the Jacobi elliptic function with the modulus $\kappa = \sqrt{\frac{-\beta/2}{a + \beta/2}}$.

Remark 3. *It is assumed that $\beta > 0$, which is not restrictive. In any case, the solution is expressed via Jacobi elliptic functions and one can proceed in the same way.*

It is straightforward that $T_1 = \frac{4K}{\sqrt{a + \beta/2}}$ and $T_2 = \frac{2iK'}{\sqrt{a + \beta/2}}$ are the periods of (26). Here K, K' are the complete elliptic integrals of the first kind. In the parallelogram of the periods, the solution (26) has two simple poles

$$t_1 = \frac{iK'}{\sqrt{a + \beta/2}}, \quad t_2 = \frac{2K + iK'}{\sqrt{a + \beta/2}}, \quad (28)$$

hence, Γ is a complex torus with two points removed.

Recall that in our case $\mathcal{N} = \mathbb{C}^{2N} = \{(q, p)\}$ and S is a symplectic involution (19), which preserves the Hamiltonian H (4) (and also the Hamiltonian vector field). Let $\mathcal{N}' = \{(q, p) \in \mathcal{N} | S(q, p) \neq (q, p)\}$, let $\hat{\mathcal{N}} := \mathcal{N}'/S$ be the quotient manifold, which is also symplectic. The Hamiltonian H (4) is naturally mapped into Hamiltonian function $\hat{H}$ (as it is invariant with respect to S), which in turn defines an induced Hamiltonian system on $\hat{\mathcal{N}}$. It can be shown that if the Hamiltonian system defined by H (4) has m meromorphic independent first integrals, then the induced Hamiltonian system on $\hat{\mathcal{N}}$ has m meromorphis independent first integrals (see Ziglin [17]). The involution S maps Γ onto itself, interchanging the places of the two poles $t_{1,2}$. Then $\hat{\Gamma} = \Gamma/S$ is a torus with one point removed. Hence,

the solution $\hat{q}(t)$ of the induced system is single-valued, double periodic with periods $\hat{T}_1 = \frac{2K}{\sqrt{a+\beta/2}}$ and $\hat{T}_2 = \frac{2iK'}{\sqrt{a+\beta/2}}$ with one simple pole t_1 in the parallelogram of the periods. To simplify the notations, we will keep writing $q(t)$.

Next, we need the expansion of the $q(t)$ in the neighborhood of the pole t_1 . To obtain it we proceed as follows. The series expansion of $\text{sn}(u, \kappa)$ reads

$$\text{sn}(u, \kappa) = u - \frac{1+\kappa^2}{3!}u^3 + \frac{\kappa^4+14\kappa^2+1}{5!}u^5 - \frac{(1+\kappa^2)(\kappa^4+134\kappa^2+1)}{7!}u^7 + \dots$$

Furthermore, from addition theorem for sn (see e.g., [28]) we have

$$\text{sn}(u + iK') = \frac{1}{\kappa} \text{ns}(u) = \frac{1}{\kappa} \frac{1}{\text{sn}(u)}.$$

Straightforward computations give

$$\text{sn}(u + iK') = \frac{1}{\kappa} \left[\frac{1}{u} + \frac{1+\kappa^2}{6}u + \frac{7\kappa^4-22\kappa^2+7}{360}u^3 + \frac{(1+\kappa^2)(31\kappa^4-46\kappa^2+31)}{15120}u^5 + \dots \right]$$

Finally, after shifting $u \rightarrow u - iK'$ and taking into account that $u = \sqrt{a+\beta/2}t$ and $\kappa = \sqrt{\frac{-\beta/2}{a+\beta/2}}$, we get the formula

$$q(t) = \frac{1}{\sqrt{-\beta/2}} \left[\frac{1}{t-t_1} + \frac{a}{6}(t-t_1) + \gamma_3(t-t_1)^3 + \gamma_5(t-t_1)^5 + \dots \right], \quad (29)$$

where

$$\gamma_3 = \frac{1}{360}(9\beta^2 + 18a\beta + 7a^2), \quad \gamma_5 = \frac{a}{42}(3\gamma_3 + \frac{a^2}{36}). \quad (30)$$

Denoting by $\tilde{\zeta}_j^{(1)} = dq_j, \eta_j^{(1)} = dp_j, j = 1, 2, \dots, N$, the variational equations (VE) (written as second order equations) are

$$\ddot{\tilde{\zeta}}_j^{(1)} + (2 + V''(q(t)))\tilde{\zeta}_j^{(1)} - \tilde{\zeta}_{j+1}^{(1)} - \tilde{\zeta}_{j-1}^{(1)} = 0, \quad j = 1, \dots, N. \quad (31)$$

Proposition 2. The identity component of the differential Galois group of (VE) (31) is abelian.

Proof. To see this, we first denote

$$\tilde{\zeta}^{(1)} := \begin{pmatrix} \tilde{\zeta}_1^{(1)} \\ \tilde{\zeta}_2^{(1)} \\ \vdots \\ \tilde{\zeta}_{N-1}^{(1)} \\ \tilde{\zeta}_N^{(1)} \end{pmatrix}, \quad K_N(q(t)) := \begin{pmatrix} 2 + V''(q(t)) & -1 & & & -1 \\ -1 & 2 + V''(q(t)) & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 + V''(q(t)) & -1 \\ -1 & & & -1 & 2 + V''(q(t)) \end{pmatrix}. \quad (32)$$

Then (VE) can be written as

$$\ddot{\tilde{\zeta}}^{(1)} + K_N(q(t)) \tilde{\zeta}^{(1)} = 0. \quad (33)$$

The structure of the matrix $K_N(q(t))$ is similar to that of L_N in (12). This suggests using the linear transformation

$$\tilde{\zeta}^{(1)} = M y^{(1)} \quad (34)$$

with already defined matrix M , which decouples the system (33)

$$\ddot{y}^{(1)} + D_N(q(t)) y^{(1)} = 0, \quad (35)$$

where $D_N(q(t)) := M^{-1}K_N(q(t))M = \text{diag}(\Lambda_1^2, \dots, \Lambda_N^2)$, with $\Lambda_j^2 := V''(q(t)) + 4\sin^2 \frac{\pi j}{N}$, $j = 1, \dots, N$. In coordinate form the above system can be written as

$$\ddot{y}_j^{(1)} + [a + 4\sin^2 \frac{\pi j}{N} + 3\beta \text{sn}^2(\sqrt{a + \beta/2}t, \kappa)]y_j^{(1)} = 0, \quad j = 1, \dots, N. \quad (36)$$

After changing the independent variable $\tau := \sqrt{a + \beta/2}t$, $' = d/d\tau$ we can see that each of these equations is a Lamé equation in Jacobi form.

$$\left(y_j^{(1)}\right)'' = \left[(\tilde{n} + 1)\tilde{n}\kappa^2 \text{sn}^2(\tau, \kappa) - \frac{a + 4\sin^2 \frac{\pi j}{N}}{a + \beta/2}\right]y_j^{(1)}, \quad (37)$$

with $\tilde{n} = 2$.

The monodromy group of any of the Lamé equations (36) (or (37)) is abelian. Indeed, let g_1, g_2 are the monodromies corresponding to the periods $\hat{T}_1, \hat{T}_2$. The commutator $g_* := [g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ corresponds to one winding around the regular singular point t_1 . The eigenvalues of g_* are just $\exp(2\pi i \rho_{1,2})$ [28], where $\rho_{1,2}$ are the roots of indicial equation $\rho(\rho - 1) - 6 = 0$. Since they are 3, -2 , then $g_* = Id$.

Let us consider now the Galois group of any of the Lamé equations (37). The coefficient field K is the field of elliptic functions with periods $\hat{T}_1, \hat{T}_2$. It is a classical result that there is a solution which is an elliptic function, i.e., it belongs to the field K . The other independent solution is given by a quadrature $\int f, f \in K$. The Picard-Vessiot extension is $L_1 = K(\int f)$ and it is known that the identity component of the Galois group $\text{Gal}(L_1/K)$ is isomorphic to $\begin{pmatrix} 1 & 0 \\ v_j & 1 \end{pmatrix}, v_j \in \mathbb{C}$. Therefore, the identity component G^0 of the Galois group of (VE) is represented by the matrix group

$$G^0 = \left\{ \begin{pmatrix} 1 & 0 & & & \\ v_1 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & v_N & 1 \end{pmatrix}, \quad v_j \in \mathbb{C}, j = 1, \dots, N \right\}$$

and it is clearly abelian. $\square$

To find another obstruction to integrability, let us consider the higher VE along the particular solution (26). We write

$$q_j = q(t) + \varepsilon \tilde{\xi}_j^{(1)} + \varepsilon^2 \tilde{\xi}_j^{(2)} + \varepsilon^3 \tilde{\xi}_j^{(3)} + \dots, \quad p_j = \dot{q}_j, \quad (38)$$

where ε is a formal parameter and substitute these expressions into the Hamiltonian system governed by (4). Comparing the terms with the same order in ε we consequently obtain the VE up to any order.

The first variational equation is, of course, (31) $(\text{VE}_1) = (\text{VE})$. For the second variational equation we have

$$\ddot{\tilde{\xi}}_j^{(2)} + (2 + V''(q(t)))\tilde{\xi}_j^{(2)} - \tilde{\xi}_{j+1}^{(2)} - \tilde{\xi}_{j-1}^{(2)} = -\frac{1}{2}V'''(q(t))(\tilde{\xi}_j^{(1)})^2, \quad j = 1, \dots, N. \quad (39)$$

Denote

$$\tilde{\xi}^{(2)} := (\tilde{\xi}_1^{(2)}, \dots, \tilde{\xi}_N^{(2)})^T, \quad f^{(2)} := -3\beta q(t) ((\tilde{\xi}_1^{(1)})^2, \dots, (\tilde{\xi}_N^{(1)})^2)^T. \quad (40)$$

Then the second variational equation (VE₂) can be written as

$$\ddot{\xi}^{(2)} + K_N(q(t)) \xi^{(2)} = f^{(2)}. \quad (41)$$

The third variational equation (VE₃) is

$$\ddot{\xi}_j^{(3)} + (2 + V''(q(t)))\xi_j^{(3)} - \xi_{j+1}^{(3)} - \xi_{j-1}^{(3)} = -\beta(\xi_j^{(1)})^3 - 6\beta q(t)\xi_j^{(1)}\xi_j^{(2)} \quad j = 1, \dots, N \quad (42)$$

and it can be written as

$$\ddot{\xi}^{(3)} + K_N(q(t)) \xi^{(3)} = f^{(3)}, \quad (43)$$

where

$$f^{(3)} = -\beta((\xi_1^{(1)})^3, \dots, (\xi_N^{(1)})^3)^T - 6\beta q(t)(\xi_1^{(1)}\xi_1^{(2)}, \dots, \xi_N^{(1)}\xi_N^{(2)}). \quad (44)$$

In this way we can obtain any (VE_k), $k > 3$, but we do not need them.

To study the local solutions of (VE₂) and (VE₃), again we make the linear change (with above introduced matrix M)

$$\xi^{(k)} = M y^{(k)}, \quad k = 2, 3.$$

In this way we obtain

$$\ddot{y}^{(k)} + D_N(q(t))y^{(k)} = \tilde{f}^{(k)}, \quad k = 2, 3, \quad (45)$$

where

$$\tilde{f}^{(k)} := M^{-1} f^{(k)},$$

that is, in these coordinates each of the variational equations (VE_k), $k = 2, 3$ decomposes into N second order differential equations. Therefore, to obtain non-integrability of the Hamiltonian system under consideration, it is enough to show that the identity component of the Galois group of one of them is non-commutative, which implies non-commutativity of the Galois group of (VE_k).

Remark 4. Now, we are ready to explain why we need a logarithmic term in the solution of some (VE_k) around a singular point. In [25] the following necessary and sufficient condition for abelian-ness of the Galois groups $G_k = \text{Gal}(L_k/K)$ of the variational equation of order k , (VE_k) of the Hamiltonian system, along the elliptic curves is given (in fact, every Galois group G_k is connected $G_k = (G_k)^0$):

The Galois group G_k is abelian, if and only if, the local monodromy g_* of the variational equation (VE_k) around the singular point of the coefficients is identity.

Suppose for some k ($k = 3$ in our case), we obtain a logarithm around the singular point (or equivalently, a residue different from zero in the Laurent expansions of some integrand). Then g_* will be represented by an upper (or lower) triangular matrix, i.e., $g_* \neq \text{Id}$, hence G_k is not abelian.

To keep the things simple, we take as a solution of (VE₁) (36)

$$y_1^{(1)} = y_2^{(1)} = \dots = y_{N-1}^{(1)} = 0 \quad (46)$$

and $y_N^{(1)} \neq 0$ satisfying

$$\ddot{y}_N^{(1)} + [a + 3\beta(\text{sn}(\sqrt{a + \beta/2}t, \kappa))^2]y_N^{(1)} = 0. \quad (47)$$

In what follows we need the expansions around t_1 of the fundamental system of the solutions with unit Wronskian for (47). Please note that all expansions below are convergent in a neighborhood of t_1 (see, for instance [28]). We get

$$\begin{aligned} y_{N,1}^{(1)} &= \frac{1}{(t-t_1)^2} - \frac{a}{6} - 3\gamma_3(t-t_1)^2 - 5\gamma_5(t-t_1)^4 + \dots \\ y_{N,2}^{(1)} &= \frac{1}{5}(t-t_1)^3 + \frac{a}{70}(t-t_1)^5 + \dots \end{aligned} \quad (48)$$

Then the fundamental matrix $\Xi_N(t)$ of (47) can be written as

$$\Xi_N(t) = \begin{pmatrix} y_{N,1}^{(1)} & y_{N,2}^{(1)} \\ \dot{y}_{N,1}^{(1)} & \dot{y}_{N,2}^{(1)} \end{pmatrix} \quad \Xi_N^{-1}(t) = \begin{pmatrix} \dot{y}_{N,2}^{(1)} & -y_{N,2}^{(1)} \\ -\dot{y}_{N,1}^{(1)} & y_{N,1}^{(1)} \end{pmatrix}. \quad (49)$$

Next, we examine the local solutions around t_1 of (VE₂). Taking into account (14), (15), (46) and (47) one gets

$$\begin{aligned} \ddot{y}_j^{(2)} + [a + 4\sin^2 \frac{\pi j}{N} + 3\beta(\operatorname{sn}(\sqrt{a+\beta/2}t, \kappa))^2] y_j^{(2)} &= 0, \quad j = 1, \dots, N-1 \\ \ddot{y}_N^{(2)} + [a + 3\beta(\operatorname{sn}(\sqrt{a+\beta/2}t, \kappa))^2] y_N^{(2)} &= -\frac{3\beta}{\sqrt{N}} q(t) (y_N^{(1)})^2. \end{aligned} \quad (50)$$

There are no logarithms in the expansions around $t = t_1$ of the local solutions of the above system. Moreover, we can take for the sake of simplicity

$$y_1^{(2)} = y_2^{(2)} = \dots = y_{N-1}^{(2)} = 0. \quad (51)$$

For the last equation in (50) one obtains by the method of variations of constants a fundamental system of solutions as follows

$$\begin{aligned} y_{N,1}^{(2)} &= y_{N,1}^{(1)} + y_p \\ y_{N,2}^{(2)} &= y_{N,2}^{(1)} + y_p \end{aligned} \quad (52)$$

where

$$y_p = -\frac{\beta}{2\sqrt{N}} \frac{1}{(t-t_1)^3} + d_1(t-t_1) + d_3(t-t_1)^3 + \dots$$

with

$$d_1 = -\frac{\beta}{4200\sqrt{N}}(139a^2 + 336a\beta + 168\beta^2), \quad d_3 = \frac{\beta}{31500\sqrt{N}}(227a^3 + 348a^2\beta + 174a\beta^2 - 18900).$$

Finally, we turn on studying the local solutions around t_1 of (VE₃). Due to (14), (15), (46) and (51) we can focus only on the equation for $y_N^{(3)}$, which reads

$$\ddot{y}_N^{(3)} + [a + 3\beta(\operatorname{sn}(\sqrt{a+\beta/2}t, \kappa))^2] y_N^{(3)} = -\frac{\beta}{N} (y_N^{(1)})^3 - \frac{6\beta}{\sqrt{N}} q(t) y_N^{(1)} y_N^{(2)}. \quad (53)$$

In this case, (25) becomes

$$\frac{d}{dt} \begin{pmatrix} r_{31} \\ r_{32} \end{pmatrix} = \Xi_N^{-1}(t) \begin{pmatrix} 0 \\ -\frac{\beta}{N} (y_N^{(1)})^3 - \frac{6\beta}{\sqrt{N}} q(t) y_N^{(1)} y_N^{(2)} \end{pmatrix} = \begin{pmatrix} y_{N,2}^{(1)} \left(\frac{\beta}{N} (y_N^{(1)})^3 + \frac{6\beta}{\sqrt{N}} q(t) y_N^{(1)} y_N^{(2)} \right) \\ -y_{N,1}^{(1)} \left(\frac{\beta}{N} (y_N^{(1)})^3 + \frac{6\beta}{\sqrt{N}} q(t) y_N^{(1)} y_N^{(2)} \right) \end{pmatrix}$$

We are looking for a component of above vector with a nonzero residue at $t = t_1$. This would imply the appearance of a logarithmic term. After some calculations making use of (29), (48) and (52),

the residue at $t = t_1$ of the first component $y_{N,2}^{(1)} \left(\frac{\beta}{N} (y_N^{(1)})^3 + \frac{6\beta}{\sqrt{N}} q(t) y_N^{(1)} y_N^{(2)} \right)$ with the specific representatives turns out to be

$$\text{Res}_{|t=t_1} y_{N,2}^{(1)} \left(\frac{\beta}{N} (y_N^{(1)})^3 + \frac{6\beta}{\sqrt{N}} q(t) y_N^{(1)} y_N^{(2)} \right) = \frac{3a\beta}{70N} (\beta + 2), \quad (54)$$

which is nonzero for $\beta > 0$. Thus, we have obtained a nonzero residue at $t = t_1$, which implies the appearance of a logarithmic term in the solutions of (VE₃). Then its Galois group is solvable, but not abelian. Hence, the non-integrability of the Hamiltonian system (4) follows from Theorem 6. $\square$

4. The Birkhoff Normal Form

In phonon coordinates, the Hamiltonian (4) is

$$H = \sum_{k=1}^N \frac{1}{2} \omega_k (P_k^2 + Q_k^2) + H_4(Q_1, \dots, Q_N). \quad (55)$$

Furthermore, we introduce the complex variables

$$z_k = Q_k + iP_k, \quad \bar{z}_k = Q_k - iP_k,$$

which are not symplectic, but are natural in the construction of the normal form. In these variables H_2 reads

$$H_2 = \sum_{1 \leq k < \frac{N}{2}} \omega_k (z_k \bar{z}_k + z_{N-k} \bar{z}_{N-k}) + \omega_{\frac{N}{2}} z_{\frac{N}{2}} \bar{z}_{\frac{N}{2}} + \omega_N z_N \bar{z}_N. \quad (56)$$

Next, we are looking for the monomials $z^\Theta \bar{z}^\theta$, Θ, θ being multi-indices, which commute with H_2 , i.e., $ad_{H_2}(z^\Theta \bar{z}^\theta) = 0$. These monomials are called resonant monomials and cannot be removed in the process of normalization. We then get

$$ad_{H_2}(z^\Theta \bar{z}^\theta) = \nu(\Theta, \theta) z^\Theta \bar{z}^\theta \quad (57)$$

with

$$\nu(\Theta, \theta) := \sum_{1 \leq k < \frac{N}{2}} i\omega_k (\Theta_k - \theta_k + \Theta_{N-k} - \theta_{N-k}) + i\omega_{\frac{N}{2}} (\Theta_{\frac{N}{2}} - \theta_{\frac{N}{2}}) + i\omega_N (\Theta_N - \theta_N). \quad (58)$$

Hence, the resonant monomials are ones with $\nu(\Theta, \theta) = 0$. Therefore, modulo the Remark 2 from the Section 2 we have that the set of multi-indices (Θ, θ) for which $|\Theta| + |\theta| = 4$ and $\nu(\Theta, \theta) = 0$ is contained in the set given by the relations

$$\Theta_k - \theta_k + \Theta_{N-k} - \theta_{N-k} = \Theta_{\frac{N}{2}} - \theta_{\frac{N}{2}} = \Theta_N - \theta_N = 0$$

This means that $\overline{H}_4$ is generated by

$$z_k \bar{z}_k, z_{N-k} \bar{z}_{N-k}, z_k \bar{z}_{N-k}, \bar{z}_k z_{N-k}, \quad 1 \leq k < \frac{N}{2} \quad \text{and} \quad z_{\frac{N}{2}} \bar{z}_{\frac{N}{2}}, z_N \bar{z}_N$$

However, we want a normal form which is invariant under R^* and S^* . To obtain such we define for $1 \leq j < \frac{N}{2}$

$$\begin{aligned} a_k &:= \frac{1}{2}(z_k \bar{\zeta}_k + z_{N-k} \bar{\zeta}_{N-k}) = \frac{1}{2}(Q_k^2 + P_k^2 + Q_{N-k}^2 + P_{N-k}^2), \\ b_k &:= \frac{i}{2}(z_k \bar{\zeta}_{N-k} - z_{N-k} \bar{\zeta}_k) = Q_k P_{N-k} - Q_{N-k} P_k, \\ c_k &:= \frac{1}{2}(z_k \bar{\zeta}_k - z_{N-k} \bar{\zeta}_{N-k}) = \frac{1}{2}(Q_k^2 + P_k^2 - Q_{N-k}^2 - P_{N-k}^2), \\ d_k &:= \frac{1}{2}(z_k \bar{\zeta}_{N-k} + z_{N-k} \bar{\zeta}_k) = Q_k Q_{N-k} + P_k P_{N-k}, \end{aligned} \quad (59)$$

$a_N := \frac{1}{2} z_N \bar{\zeta}_N = \frac{1}{2}(Q_N^2 + P_N^2)$ and if N is even $a_{\frac{N}{2}} := \frac{1}{2} z_{\frac{N}{2}} \bar{\zeta}_{\frac{N}{2}} = \frac{1}{2}(Q_{\frac{N}{2}}^2 + P_{\frac{N}{2}}^2)$. These quantities are known as Hopf variables and they satisfy the relations

$$a_k^2 = b_k^2 + c_k^2 + d_k^2, \quad 1 \leq k < \frac{N}{2} \quad (60)$$

and in these variables H_2 is

$$H_2 = \sum_{1 \leq k < \frac{N}{2}} \omega_k a_k + \omega_{\frac{N}{2}} a_{\frac{N}{2}} + \omega_N a_N. \quad (61)$$

The nontrivial Poisson brackets between these quantities are

$$\{b_k, c_k\} = 2d_k, \quad \{b_k, d_k\} = -2c_k, \quad \{c_k, d_k\} = 2b_k. \quad (62)$$

It is observed in [8] that a_k , $a_{\frac{N}{2}}$ and a_N are invariant under R^* and S^* and the products $a_k a_l, a_N a_k, b_k b_l, 1 \leq k, l < \frac{N}{2}$ and if N is even $a_{\frac{N}{2}} a_k, 1 \leq k \leq \frac{N}{2}$ and $d_k d_{\frac{N}{2}-k} - c_k c_{\frac{N}{2}-k}, 1 \leq k \leq \frac{N}{4}$, so $\bar{H}_4$ must be a linear combination of the above four order terms. Indeed, we have

Theorem 7. *The truncated up to order four normal form for the periodic KG lattice is*

$$\begin{aligned} \bar{H} &= H_2 + \bar{H}_4, \\ \bar{H}_4 &= \frac{\beta}{2N} \left\{ \frac{3}{2} \left(\frac{a_{\frac{N}{2}}^2}{\omega_{\frac{N}{2}}^2} + \frac{a_N^2}{\omega_N^2} \right) + 6 \frac{a_{\frac{N}{2}} a_N}{\omega_{\frac{N}{2}} \omega_N} + 6 \left(\frac{a_{\frac{N}{2}}}{\omega_{\frac{N}{2}}} + \frac{a_N}{\omega_N} \right) \sum_{1 \leq k < \frac{N}{2}} \frac{a_k}{\omega_k} + \frac{3}{4} \sum_{1 \leq k < \frac{N}{2}} \frac{3a_k^2 - b_k^2}{\omega_k^2} \right. \\ &\quad \left. + 6 \sum_{1 \leq k < l < \frac{N}{2}} \frac{a_k a_l}{\omega_k \omega_l} + 3 \sum_{1 \leq k < \frac{N}{4}} \frac{c_k c_{\frac{N}{2}-k} - d_k d_{\frac{N}{2}-k}}{\omega_k \omega_{\frac{N}{2}-k}} + \frac{3}{4} \frac{c_{\frac{N}{4}}^2 - d_{\frac{N}{4}}^2}{\omega_{\frac{N}{4}}^2} \right\}. \end{aligned} \quad (63)$$

In the above formula the terms with subscripts $\frac{N}{2}, \frac{N}{4}$ appear if $\frac{N}{2} \in \mathbb{N}, \frac{N}{4} \in \mathbb{N}$, respectively.

The calculation of the above normal form is long, tedious, but straightforward, which is why it is not presented here. Instead, we proceed with two important corollaries, which prove the assertions in Theorem 2.

Corollary 1. *When N is odd, the truncated normal form $\bar{H} = H_2 + \bar{H}_4$ of the periodic KG lattice is Liouville integrable with the quadratic integrals $a_j, b_j, 1 \leq j \leq \frac{N-1}{2}$ and a_N . Moreover, this normal form is KAM nondegenerate.*

Proof. H_2 is a linear combination of a_j and a_N . When N is odd, $\bar{H}_4$ becomes

$$\bar{H}_4 = \frac{\beta}{2N} \left\{ \frac{3}{4} \sum_{k=1}^{(N-1)/2} \frac{3a_k^2 - b_k^2}{\omega_k^2} + \frac{3}{2} \frac{a_N^2}{\omega_N^2} + 6 \frac{a_N}{\omega_N} \sum_{k=1}^{(N-1)/2} \frac{a_k}{\omega_k} + 6 \sum_{1 \leq k < l \leq \frac{N-1}{2}} \frac{a_k a_l}{\omega_k \omega_l} \right\} \quad (64)$$

and it is clear that $a_j, b_j, 1 \leq j \leq \frac{N-1}{2}$ and a_N commute with $\bar{H}_4$ and with each other.

To introduce action-angle variables we follow the scheme from [8], slightly adjusted to our case. We need to find the set of regular values of the energy momentum map

$$EM : (Q, P) \rightarrow (a_j, b_j, a_N).$$

Denote it by $U_r = \{(a_j, b_j, a_N) \in \mathbb{R}^N, a_j > 0, |b_j| < a_j, a_N > 0\}$. Then for all $(a_j, b_j, a_N) \in U_r$ the level sets of $EM^{-1}(a_j, b_j, a_N)$ are diffeomorphic to N -tori. Let $\arg : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ be the argument function $\arg(r \cos \Phi, r \sin \Phi) \rightarrow \Phi$. Define the following set of variables $(a_j, b_j, a_N, \phi_j, \psi_j, \varphi_N)$ a_j, b_j, a_N as above and

$$\begin{aligned} \phi_j &:= \frac{1}{2} \arg(-P_{N-j} - Q_j, P_j - Q_{N-j}) + \frac{1}{2} \arg(P_{N-j} - Q_j, P_j + Q_{N-j}), \\ \psi_j &:= \frac{1}{2} \arg(-P_{N-j} - Q_j, P_j - Q_{N-j}) - \frac{1}{2} \arg(P_{N-j} - Q_j, P_j + Q_{N-j}), \\ \varphi_N &:= \arctan P_N / Q_N. \end{aligned} \quad (65)$$

Using the formula $d \arg(x, y) = \frac{xdy - ydx}{x^2 + y^2}$, one can verify that $(a_j, b_j, a_N, \phi_j, \psi_j, \varphi_N)$ are indeed canonical coordinates $\sum dP_j \wedge dQ_j = \sum da_j \wedge d\phi_j + db_j \wedge d\psi_j + da_N \wedge d\varphi_N$. Since a_j, b_j and a_N are quadratic functions in the phase variables, they can be extended to global action variables.

Finally, to check the nondegeneracy condition, we compute the Hessian of $\bar{H}_4$ with respect to a_j, a_N, b_k . Denote $\lambda_j := 1/\omega_j$. Then

$$\frac{\partial^2 \bar{H}_4}{\partial a_j \partial a_k} = \frac{3\beta}{2N} \begin{pmatrix} \frac{3}{4} \lambda_1^2 & \lambda_1 \lambda_2 & & \lambda_1 \lambda_N \\ \lambda_2 \lambda_1 & \frac{3}{4} \lambda_2^2 & & \lambda_2 \lambda_N \\ \vdots & & \ddots & \vdots \\ \lambda_{N-1} \lambda_1 & \lambda_{N-1} \lambda_2 & & \frac{3}{4} \lambda_{N-1}^2 \\ \lambda_N \lambda_1 & \lambda_N \lambda_2 & & \lambda_{N-1} \lambda_N \\ & & & \frac{1}{2} \lambda_N^2 \end{pmatrix} \quad (66)$$

$$\frac{\partial^2 \bar{H}_4}{\partial b_j \partial b_k} = -\frac{3\beta}{4N} \begin{pmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_{\frac{N-1}{2}}^2 \end{pmatrix} \quad (67)$$

Clearly $\frac{\partial^2 \bar{H}_4}{\partial b_j \partial b_k}$ is nondegenerate. After some algebra, one can check that

$$\det \left(\frac{\partial^2 \bar{H}_4}{\partial a_j \partial a_k} \right) = \left(\frac{3\beta}{2N} \right)^N \frac{2N-1}{2^N} \prod_{j=1}^N \lambda_j^2,$$

i.e., it is also nondegenerate. $\square$

Thus, the periodic KG lattice (4) with an odd number of particles can, after normalization, be viewed as a perturbation of a nondegenerate integrable Hamiltonian system, namely its fourth order normal form. Therefore, by the KAM theorem almost all low-energy solutions of (4) are periodic or quasi-periodic and live on invariant tori.

Corollary 2. When N is even, the truncated normal form $\bar{H} = H_2 + \bar{H}_4$ of the periodic KG lattice is Liouville integrable with the quadratic integrals a_k , $1 \leq k \leq \frac{N}{2}$ and $a_N, b_k - b_{\frac{N}{2}-k}$, $1 \leq k < \frac{N}{4}$ and $c_{\frac{N}{4}}$ (if $\frac{N}{4} \in \mathbb{N}$) and the quartics

$$\mathcal{K}_k = 3 \frac{c_k c_{\frac{N}{2}-k} - d_k d_{\frac{N}{2}-k}}{\omega_k \omega_{\frac{N}{2}-k}} - \frac{3}{4} \left(\frac{b_k^2}{\omega_k^2} + \frac{b_{\frac{N}{2}-k}^2}{\omega_{\frac{N}{2}-k}^2} \right), \quad 1 \leq k < \frac{N}{4}. \quad (68)$$

Proof. This follows from simple calculations of all Poisson brackets using (62). Let us deal with the exceptional cases. First, in the case of $N = 2$ particles the frequencies are incommensurable, i.e., there are no resonances. From (63) we obtain

$$\bar{H} = \omega_1 a_1 + \omega_2 a_2 + \frac{\beta}{4} \left\{ \frac{3}{2} \left(\frac{a_1^2}{\omega_1^2} + \frac{a_2^2}{\omega_2^2} \right) + 6 \frac{a_1 a_2}{\omega_1 \omega_2} \right\},$$

where $a_k = \frac{1}{2}(Q_k^2 + P_k^2)$, $k = 1, 2$. The action variables a_k can be extended to global action-angle variables (a_k, φ_k) usually called symplectic polar coordinates. Then the KAM condition is immediate.

Furthermore, for the case with $N = 4$ particles, we get from (60) and (63)

$$\bar{H}_4 = \frac{\beta}{8} \left\{ \frac{3}{2} \left(\frac{a_2^2}{\omega_2^2} + \frac{a_4^2}{\omega_4^2} \right) + 6 \frac{a_2 a_4}{\omega_2 \omega_4} + 6 \frac{a_1}{\omega_1} \left(\frac{a_2}{\omega_2} + \frac{a_4}{\omega_4} \right) + \frac{3}{2\omega_1^2} (a_1^2 + c_1^2) \right\}.$$

Denote $I_k = \frac{1}{2}(Q_k^2 + P_k^2)$, $k = 1, \dots, 4$. Please note that $a_1 = I_1 + I_3$, $c_1 = I_1 - I_3$, $a_2 = I_2$, $a_4 = I_4$. Then $\bar{H}_4$ becomes

$$\bar{H}_4 = \frac{\beta}{8} \left\{ \frac{3}{2} \left(\frac{I_2^2}{\omega_2^2} + \frac{I_4^2}{\omega_4^2} \right) + 6 \frac{I_2 I_4}{\omega_2 \omega_4} + 6 \frac{I_1 + I_3}{\omega_1} \left(\frac{I_2}{\omega_2} + \frac{I_4}{\omega_4} \right) + \frac{3}{\omega_1^2} (I_1^2 + I_3^2) \right\}.$$

The actions I_k can be extended to global action-angle variables (I_k, φ_k) (symplectic polar coordinates) and the KAM condition is straightforward (compare with [22]). $\square$

Proof of Theorem 2. Part (i) is proved by Corollary 1, whereas part (ii) comes after Corollary 2. $\square$

5. KG Lattice with Fixed Endpoints

In this section, we consider the KG lattice with n ($n \geq 3$, not necessarily even) particles and with fixed endpoints

$$q_0 = q_{n+1} = p_0 = p_{n+1} = 0. \quad (69)$$

It was realized in [9] that such the FPU lattice with the fixed boundary conditions can be viewed as an invariant subsystem of the periodic FPU lattice with $N = 2n + 2$ particles. This invariant subsystem is obtained by the fixed point set of the compact group $\langle S \rangle = \{Id, S\}$. Since S is also a symmetry for the periodic KG Hamiltonian this constriction is applicable here and we will describe it briefly. Define the set

$$Fix\langle S \rangle := \{(q, p) \in T^*\mathbb{R}^N | S(q, p) = (q, p)\}. \quad (70)$$

By the definition of S (19) we get

$$q_j = -q_{2n+2-j}, \quad p_j = -p_{2n+2-j} \quad \forall j$$

from where it follows that $q_0 = q_{n+1} = q_0 = q_{n+1} = 0$. $Fix\langle S \rangle$ is a symplectic manifold with symplectic form obtained from $dq \wedge dp$ with restriction on $Fix\langle S \rangle$. One can take as coordinates on $Fix\langle S \rangle$ $(q_1, \dots, q_n, p_1, \dots, p_n)$. Then the lattice with fixed endpoints and n particles is described on $Fix\langle S \rangle$ by the Hamiltonian $H_{Fix\langle S \rangle}$.

Let us consider the periodic KG Hamiltonian with $N = 2n + 2$ particles. From the definition of the phonon particles (15), we can obtain that S acts on them in the following way

$$\begin{aligned} S : & (Q_1, \dots, -Q_{n+1}, Q_{n+2}, \dots, Q_{2n+1}, -Q_{2n+2}; P_1, \dots, P_{n+1}, P_{n+2}, \dots, P_{2n+1}, P_{2n+2}) \\ \mapsto & (-Q_1, \dots, -Q_{n+1}, Q_{n+2}, \dots, Q_{2n+1}, Q_{2n+2}; -P_1, \dots, -P_{n+1}, P_{n+2}, \dots, P_{2n+1}, -P_{2n+2}). \end{aligned} \quad (71)$$

Hence,

$$\text{Fix}\langle S \rangle := \{(Q, P) \in T^*\mathbb{R}^N | Q_k = P_k = 0, 1 \leq k \leq n+1, Q_{2n+2} = P_{2n+2} = 0\}, \quad (72)$$

which is a symplectic manifold isomorphic to $T^*\mathbb{R}^n$. We can take as coordinates on $\text{Fix}\langle S \rangle$ $(x_k, y_k) = (Q_{2n+2-k}, P_{2n+2-k})$, $k = 1, \dots, n$.

We have already constructed an S -invariant truncated normal form $\overline{H}$ for the periodic KG lattice. It was realized in [9] that to obtain the normal form of the Hamiltonian $H_{\text{Fix}\langle S \rangle}$, one needs to restrict the symmetric normal form $\overline{H}$ to $\text{Fix}\langle S \rangle$, that is,

$$\overline{H}_{\text{Fix}\langle S \rangle} = \overline{H}|_{\text{Fix}\langle S \rangle}.$$

Clearly, $b_k = d_k = 0$, $1 \leq k \leq n$, $a_N = a_{\frac{N}{2}} = 0$ and $c_k = -a_k$, $1 \leq k \leq n$. Introduce

$$I_k := a_k = \frac{1}{2}(x_k^2 + y_k^2), \quad 1 \leq k \leq n. \quad (73)$$

Then we have

Theorem 8. *The KG lattice with n particles and fixed boundary conditions has the fourth order normal form $\overline{H} = H_2 + \overline{H}_4$, where $H_2 = \sum_{k=1}^n \omega_k I_k$, $\omega_k := \sqrt{1 + 4 \sin^2 \frac{k\pi}{2n+2}}$ and*

$$\overline{H}_4 = \frac{\beta}{2(2n+2)} \left\{ \frac{9}{4} \sum_{1 \leq k \leq n} \frac{I_k^2}{\omega_k^2} + 6 \sum_{1 \leq k < l \leq n} \frac{I_k I_l}{\omega_k \omega_l} + 3 \sum_{1 \leq k < \frac{n+1}{2}} \frac{I_k I_{n+1-k}}{\omega_k \omega_{n+1-k}} + \frac{3}{4} \frac{I_{\frac{n+1}{2}}^2}{\omega_{\frac{n+1}{2}}^2} \right\}. \quad (74)$$

In the above formula the term with subscripts $\frac{n+1}{2}$ appears if n is odd. This normal form is completely integrable with the quadratic first integrals I_k , $k = 1, \dots, n$ and it is KAM nondegenerate.

Proof. It follows from Theorem 7 and the explanations above that the quantities I_k are Poisson commuting, so the complete integrability is clear. The variables I_k can be extended to global action-angle variables (I_k, φ_k) —symplectic polar coordinates. It remains to verify the nondegeneracy of the normal form $\overline{H}$. Denote the Hessian of $\overline{H}$ with $\mathcal{H}$ and let $\lambda_j = 1/\omega_j$ as before. We have

$$\mathcal{H} = \frac{\beta}{2(2n+2)} \Delta_n \frac{3}{2} F_n \Delta_n, \quad (75)$$

where $\Delta_n = \text{diag}(\lambda_1, \dots, \lambda_n)$ and F_n is an $n \times n$ matrix which for n even, respectively odd takes the form

$$\underbrace{\begin{pmatrix} 3 & 4 & \dots & & \dots & 4 & 6 \\ 4 & 3 & 4 & \dots & \dots & 4 & 6 & 4 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ & & & 3 & 6 & & & \\ & & & 6 & 3 & & & \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 4 & 6 & 4 & \dots & \dots & 4 & 3 & 4 \\ 6 & 4 & \dots & & \dots & 4 & 3 \end{pmatrix}}_{n \text{ even}}, \quad \underbrace{\begin{pmatrix} 3 & 4 & \dots & & \dots & 4 & 6 \\ 4 & 3 & 4 & \dots & \dots & 4 & 6 & 4 \\ \vdots & & \ddots & & & \ddots & & \vdots \\ & & & 3 & 4 & 6 & & \\ 4 & 4 & & 4 & 4 & 4 & & 4 & 4 \\ & & & 6 & 4 & 3 & & \\ \vdots & & \ddots & & & \ddots & & \vdots \\ 4 & 6 & 4 & \dots & \dots & 4 & 3 & 4 \\ 6 & 4 & \dots & & \dots & 4 & 3 \end{pmatrix}}_{n \text{ odd}}.$$

Then

$$\det(\mathcal{H}) = \left(\frac{\beta}{2(2n+2)} \right)^n \prod_{j=1}^n \lambda_j^2 \left(\frac{3}{2} \right)^n \det(F_n).$$

After some linear algebra F_n is shown to be nonsingular, from where the nondegeneracy of $\bar{H}$ follows. $\square$

As a consequence of the above result, we conclude that almost all low-energy solutions of the KG lattice with fixed endpoints $H_{\text{Fix}\langle S \rangle}$ are quasi-periodic and live on invariant tori.

6. Discussion

In the present paper, we deal with the integrability of the KG lattices. First, we study the periodic KG lattice with N particles (4) and quartic potential

$$V(x) = \frac{a}{2}x^2 + \frac{\beta}{4}x^4.$$

We have shown that this Hamiltonian system is non-integrable by means of meromorphic integrals unless $\beta = 0$. For this we use Differential Galois theory and Morales-Ramis-Simó approach. Hence, Theorem 1 can be considered as an improvement of the Yoshimura and Umeno's result [16].

The considered system enjoys the same important discrete symmetries R, S (18), (19) as in the periodic FPU chain. Following [8] we construct an R, S -symmetric resonant fourth order normal form $\bar{H}$. This normal form happens to be Liouville integrable. It is similar to the normal form of the periodic FPU β -chain, but it is natural to be expected. Hence, the periodic KG lattice can be considered as a perturbation of its integrable Birkhoff normal form.

If N is odd the integrals of the normal form are quadratic. The global action-angle variables can be introduced, and it turns out that this normal form is KAM nondegenerated. This proves the existence of large-measure set of quasi-periodic solutions in the dynamics of the periodic KG lattice at low-energy level.

The resonant normal form with N even admits certain set of quartic integrals in addition to the quadratic ones. Probably, it would be interesting to explore the geometry of the system, defined by this normal form. One can assume that the things are similar to the even periodic FPU chain [12] up to small modifications due to the extra degree of freedom. The close analogy with the FPU chain implies that in this case global action variables do not exist. This makes the verification of the Kolmogorov's condition difficult.

Next, we consider the KG lattice with fixed endpoints. Such a system can be considered as an invariant symplectic submanifold of a larger periodic KG lattice. For this the discrete symmetry S is used. Then the normal form in this case is easy to get from the previous result, and hence, it is integrable. Furthermore, KAM theorem applies which implies that almost all low-energetic solutions are quasi-periodic.

Finally, we notice that the results of this paper do not provide an answer to one of the most important problems: what happens in the dynamics of the KG lattice when the number of particles becomes larger and larger.

Let us emphasize again the importance of the discrete symmetries, and in particular, the symmetry S in the carrying out of the above analysis. This leads to the following question: what happens when we drop the symmetry S ? We can ask the same thing in a different way: Can the results of this paper be extended for the KG lattice with the potential

$$V(x) = \frac{a}{2}x^2 + \alpha x^3 + \frac{\beta}{4}x^4, \quad \alpha \neq 0,$$

which is more relevant in studying the dynamics of low-energetic solutions in the DNA model?

It is clear that the non-integrability result of Theorem 1 can easily be extended in a similar line as in Section 3. The formal computation of the normal form would be more difficult, because we must transform away the third order terms. However, straightforward calculations for the low-dimensional periodic KG lattices with $a = 1$ and $N = 2, \dots, 6$ show that resonant third order terms do not appear in the corresponding normal forms (see [22]). Hence, these normal forms remain integrable for the latter potential. In view of the applications to the DNA models, it is clearly of some interest to calculate these normal forms in the general case.

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