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A Common Fixed-Point Theorem for Iterative Contraction of Seghal Type

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Abstract: In this paper, we consider a common fixed-point theorem with a contractive iterative at a point in the setting of complete dislocated b-metric space that was initiated by Seghal. We shall consider an example and application in fractional differential equations to support the given results.

Keywords: dislocated metric space; Seghal type contraction; common fixed point

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1. Introduction and Preliminaries

It is quite natural to consider the distance of a thing to itself to be 0, which seems also very reasonable. For instance, let us consider the set of all infinite sequences endowed with a metric d such that $d(x,y) = \frac{1}{2^s}$ for $x = (x_i)_{i\in\mathbb{N}}$, and $y = (y_i)_{i\in\mathbb{N}}$ where $s := |i \in \mathbb{N} : x_i = y_i|$. It is evident that s is infinity in the case of $x = y$ and hence $d(x,x) = 0$. On the other hand, in computer science, infinite sequences are not useful because of time restriction. On the contrary, finite sequences are more useful and reasonable in programming. Using the finite sequences to infinite sequence by keeping the definition of the metric stable, we shall get a very interesting scenario. More precisely, for a finite sequence, for example for $x = (x_1, \cdots, x_7)$, the self-distance of x to itself is not 0. Indeed, here $s = 7$ and self-distance is $\frac{1}{2^7}$.

On account of such motivation, the notion of dislocated metric was proposed by Hitzler [1] by claiming that self-distance may not be 0.

Definition 1. *Suppose that $\mathcal{X}$ is not empty. A dislocated metric is a function $\delta : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ such that for all $\varsigma, \kappa, \epsilon \in \mathcal{X}$:*

(δ1) $\delta(\varsigma,\kappa) = 0 \Rightarrow \varsigma = \kappa$,
(δ2) $\delta(\varsigma,\kappa) = \delta(\kappa,\varsigma)$,
(δ3) $\delta(\varsigma,\kappa) \le \delta(\varsigma,\epsilon) + \delta(\epsilon,\kappa)$.

The pair of the letters $(\mathcal{X},\delta)$ represent a dislocated metric space, in short DMS.

Another extension of metric is a $b-$metric which has been introduced by Czerwik [2], see also e.g., [3,4].

Definition 2. *Suppose that $\mathcal{X}$ is not empty and $s \ge 1$ is given. A b-metric is a function $d : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ such that for all $\varsigma, \kappa, \epsilon \in \mathcal{X}$:*

$(b1)$ $\varsigma = \kappa \Rightarrow d(\varsigma, \kappa) = 0$,
$(b2)$ $d(\varsigma, \kappa) = 0 \Rightarrow \varsigma = \kappa$,
$(b3)$ $d(\varsigma, \kappa) = d(\kappa, \varsigma)$,
$(b4)$ $d(\varsigma, \kappa) \leq s[d(\varsigma, \epsilon) + d(\epsilon, \kappa)]$.

The pair of letters $(\mathcal{X}, d)$ is called a b-metric space, in short b-MS. Notice that in some paper, this spaces was called quasi-metric space, see e.g., [5,6].

In what follows, we shall consider the unification of the above-mentioned notions:

Definition 3. *Suppose that $\mathcal{X}$ is not empty and $s \geq 1$ is given. A dislocated b-metric is a function $\delta_d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that for all $\varsigma, \kappa, \epsilon \in \mathcal{X}$:*

$(\delta b1)$ $\delta_d(\varsigma, \kappa) = 0 \Rightarrow \varsigma = \kappa$,
$(\delta b2)$ $\delta_d(\varsigma, \kappa) = \delta_d(\kappa, \varsigma)$,
$(\delta b3)$ $\delta_d(\varsigma, \kappa) \leq s[\delta_d(\varsigma, \epsilon) + \delta_d(\epsilon, \kappa)]$.

The pair $(\mathcal{X}, \delta_d, s)$ is said to be a dislocated b-metric space, in short b-DMS.

Example 1. *Let $\mathcal{X} = \mathbb{R}_0^+$ and $\delta_d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ defined by $\delta_d(\varsigma, \kappa) = |\varsigma - \kappa|^2 + \max\{\varsigma, \kappa\}$. Then, $\mathcal{X}$ with δ_d is a dislocated b-metric space with $s = 2$.*

It is obvious that b-metric spaces are b-DMS, but conversely this is not true.

Example 2. *Let $\mathcal{X} = \mathbb{R}_0^+$ and $\delta_d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ defined by $\delta_d(\varsigma, \kappa) = (\varsigma + \kappa)^2$. The pair $(\mathcal{X}, \delta_d, s)$ is a dislocated b-metric space with $s = 2$ but is not a b-metric space.*

For more examples see e.g., [7–12].

The topology of dislocated b-metric space $(\mathcal{X}, \delta_d, s)$ was generated by the family of open balls

$$B(\varsigma, r) = \{y \in \mathcal{X} : |\delta_d(\varsigma, \kappa) - \delta_d(\varsigma, \varsigma)| < r\}, \text{ for all } \varsigma \in \mathcal{X} \text{ and } r > 0.$$

On a b-DMS $(\mathcal{X}, \delta_d, s)$, a sequence $\{\varsigma_n\}$ in $\mathcal{X}$ is called *convergent to a point* $\varsigma \in \mathcal{X}$ if the limit

$$\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma) = \delta_d(\varsigma, \varsigma) \tag{1}$$

exists and is finite. In addition, if the following limit

$$\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma_m)$$

exists and is finite we say that the sequence $\{\varsigma_n\}$ is Cauchy. Moreover, if $\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma_m) = 0$, then we say that $\{\varsigma_n\}$ is a 0-Cauchy sequence.

Definition 4. *The b-DMS $(\mathcal{X}, \delta_d, s)$ is complete if for each Cauchy sequence $\{\varsigma_n\}$ in $\mathcal{X}$, there is some $\varsigma \in \mathcal{X}$ such that*

$$L = \lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma) = \delta_d(\varsigma, \varsigma) = \lim_{n,m\to\infty} \delta_d(\varsigma_n, \varsigma_m). \tag{2}$$

Moreover, a b-DMS $(\mathcal{X}, \delta_d, s)$ is said to be 0-complete if for each 0-Cauchy sequence $\{\varsigma_n\}$ converges to a point $\varsigma \in \mathcal{X}$ so that $L = 0$ in (2).

Let $(\mathcal{X}, \delta_d, s)$ be a b-DMS. A mapping $f : \mathcal{X} \to \mathcal{X}$ is continuous if $\{f\varsigma_n\}$ converges to $f\varsigma$ for any sequence $\{\varsigma_n\}$ in $\mathcal{X}$ converges to $\varsigma \in \mathcal{X}$.

Proposition 1. [7] *Let $(\mathcal{X}, \delta_d, s)$ be a b-DMS and $\{\varsigma_n\}$ be a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma) = 0$. Then,*

(i) ς *is unique;*
(ii) $\frac{1}{s}\delta_d(\varsigma, \kappa) \leq \lim_{n\to\infty} \delta_d(\varsigma_n, \kappa) \leq s\delta_d(\varsigma, \kappa)$, *for all* $\kappa \in \mathcal{X}$.

Proposition 2. [7] *Let $(\mathcal{X}, \delta_d, s)$ be a b-DMS. For any $\varsigma, \kappa \in \mathcal{X}$,*

(i) *if* $\delta_d(\varsigma, \kappa) = 0$ *then* $\delta_d(\varsigma, \varsigma) = \delta_d(\kappa, \kappa) = 0$;
(ii) *if* $\varsigma \neq \kappa$ *then* $\delta_d(\varsigma, \kappa) > 0$;
(iii) *if* $\{\varsigma_n\}$ *is a sequence in* $\mathcal{X}$ *such that* $\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma_{n+1}) = 0$, *then*

$$\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma_n) = \lim_{n\to\infty} \delta_d(\varsigma_{n+1}, \varsigma_{n+1}) = 0.$$

We need the following definitions from [6,13] in our main results.

Definition 5. *A comparison function is a function $\varphi : [0, \infty) \to [0, \infty)$ for which the following statements are true:*

(1^*) φ *is increasing;*
(2^*) $\lim_{n\to\infty} \varphi^n(v) = 0$, *for* $v \in [0, \infty)$.

We denote by Φ the class of the comparison functions $\varphi : [0, \infty) \to [0, \infty)$.

Proposition 3. *If φ is a comparison function then:*

(i) *each* φ^k *is a comparison function, for all* $k \in \mathbb{N}$;
(ii) φ *is continuous at 0;*
(iii) $\varphi(v) < u$ *for all* $v > 0$.

Definition 6. *A function $\varphi : [0, \infty) \to [0, \infty)$ is called a c-comparison function if:*

$(c1)$ φ *is monotone increasing;*
$(c2)$ $\sum_{n=0}^{\infty} \varphi^n(v) < \infty$, *for all* $v \in (0, \infty)$.

We denote by Φ_c the family of c-comparison functions.

Remark 1. *If φ is a c-comparison function, then $\varphi(v) < v$ for all $v > 0$.*

Remark 2. *Any c-comparison function is a comparison function.*

Definition 7. [6] *A function $\varphi : [0, \infty) \to [0, \infty)$ is called a b-comparison function if:*

$(b1)$ φ *is monotone increasing;*
$(b2)$ $\sum_{n=0}^{\infty} s^n \varphi^n(v) < \infty$, *for all* $u \in (0, \infty)$ *and* $s \geq 1$ *a real number.*

We denote by Φ_b the family of b-comparison functions.

Remark 3. *Any b-comparison function is a comparison function.*

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ such that

(ψ_1) ψ is lower semicontinuous,
(ψ_2) $\psi(v) = 0$ if and only if $v = 0$.

In what follows, we shall mention one of the interesting extensions of the Banach contraction principle [14] that was given by Seghal [15]:

Theorem 1. *([15]) Let $(\mathcal{M}, d)$ be a complete metric space, T a continuous self-mapping of $\mathcal{M}$ that satisfies the condition that there exists a real number q, $0 < q < 1$ such that for each $v \in \mathcal{M}$ there exists a positive integer $m(v)$ such that for each $w \in \mathcal{M}$,*

$$d(T^{m(v)}v, T^{m(v)}w) \leq qd(v, w). \tag{3}$$

Then T has a unique fixed point in $\mathcal{M}$.

In this paper, we shall investigate the fixed point of a certain mapping with a contractive iterate at a point in the setting of dislocated b-metric space. Such fixed-point results were introduced by Seghal [15] and continued by many others; see e.g., [16,17]. Furthermore, we shall consider an application to support the obtained result.

2. Main Results

In this section, we prove some new fixed-point results in the setting of b

Theorem 2. *Let U, V be two self-mappings on a complete b-MS $(\mathcal{X}, \delta, s)$. Suppose that for any $\varsigma, \kappa \in \mathcal{X}$ there exist positive integers $p(\varsigma), q(\kappa)$, and that there exist $\psi \in \Psi$ and an upper semicontinuous $\varphi \in \Phi_b$ such that*

$$\begin{aligned} \delta(U^{p(\varsigma)}\varsigma, V^{q(\kappa)}) \quad &\leq \varphi\left(\max\left\{\delta(\varsigma,\kappa), \delta(\varsigma, U^{p(\varsigma)}\varsigma), \delta(\kappa, V^{q(\kappa)}\kappa), \tfrac{\delta(\kappa, U^{p(\varsigma)}\varsigma)+\delta(\varsigma, V^{q(\kappa)}\kappa)}{2s}\right\}\right) \\ &+\psi\left(\min\left\{\delta(\varsigma, U^{p(\varsigma)}\varsigma), \delta(\kappa, \mathcal{V}^{q(\kappa)}\kappa), \delta(\kappa, U^{p(\varsigma)}\varsigma), \delta(\varsigma, V^{q(\kappa)}\kappa)\right\}\right) \end{aligned} \tag{4}$$

Then the pair of the functions U, V has exactly one fixed point ς^*.

Proof. Consider the initial value $\varsigma_0 \in \mathcal{X}$ and define a sequence $\{\varsigma_n\}$ as follows:

$$\varsigma_1 = V^{q(\varsigma_0)}\varsigma_0,\ \varsigma_2 = U^{p(\varsigma_1)}\varsigma_1, \dots\ \varsigma_{2i+1} = V^{q(\varsigma_{2i})}\varsigma_{2i},\ \varsigma_{2i+2} = U^{p(\varsigma_{2i+1})}\varsigma_{2i+1}, \dots \tag{5}$$

or, if we denote $p_{i-1} = p(\varsigma_{2i-1})$ and $q_i = q(\varsigma_{2i})$, for any $i \in \mathbb{N}$, we can write $\varsigma_{2i} = U^{p_{i-1}}\varsigma_{2i-1}$ and $\varsigma_{2i+1} = V^{q_i}\varsigma_{2i}$. In the initial inequality (3), we let $\varsigma = \varsigma_{2i-1}$, $\kappa = \varsigma_{2i}$ we have

$$\begin{aligned} \delta(\varsigma_{2i}, \varsigma_{2i+1}) \quad &= \delta(U^{p_{i-1}}\varsigma_{2i-1}, V^{q_i}\varsigma_{2i}) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} \delta(\varsigma_{2i-1}, \varsigma_{2i}), \delta(\varsigma_{2i-1}, U^{p_{i-1}}\varsigma_{2i-1}), \delta(\varsigma_{2i}, V^{q_i}\varsigma_{2i}), \\ \frac{\delta(\varsigma_{2i}\mathcal{U}^{p_{i-1}}\varsigma_{2i-1})+\delta(\varsigma_{2i-1}, V^{q_i}\varsigma_{2i})}{2s} \end{array}\right)\right\} \\ &\quad +\psi\left(\min\left\{\begin{array}{c} \delta(\varsigma_{2i-1}, U^{p_{i-1}}\varsigma_{2i-1}), \delta(\varsigma_{2i}, V^{q_i}\varsigma_{2i}), \delta(\varsigma_{2i}, U^{p_{i-1}}\varsigma_{2i-1}), \\ \delta(\varsigma_{2i-1}, V^{q_i}\varsigma_{2i}) \end{array}\right\}\right) \\ &= \varphi\left(\max\left\{\delta(\varsigma_{2i-1}, \varsigma_{2i}), \delta(\varsigma_{2i-1}, \varsigma_{2i}), \delta(\varsigma_{2i}, \varsigma_{2i+1}), \tfrac{\delta(\varsigma_{2i}, \varsigma_{2i})+\delta(\varsigma_{2i-1}, \varsigma_{2i+1})}{2s}\right\}\right) \\ &+\psi\left(\min\left\{\delta(\varsigma_{2i-1}, \varsigma_{2i}), \delta(\varsigma_{2i}, \varsigma_{2i+1}), \delta(\varsigma_{2i}, \varsigma_{2i}), \delta(\varsigma_{2i-1}, \varsigma_{2i+1})\right\}\right). \end{aligned} \tag{6}$$

By using $(b3)$,

$$\begin{aligned} \frac{\delta(\varsigma_{2i-1}, \varsigma_{2i+1})}{2s} \quad &\leq \frac{s\cdot[\delta(\varsigma_{2i-1}, \varsigma_{2i})+\delta(\varsigma_{2i}, \varsigma_{2i+1})]}{2s} = \frac{\delta(\varsigma_{2i-1}, \varsigma_{2i})+\delta(\varsigma_{2i}, \varsigma_{2i+1})}{2} \\ &\leq \max\left\{\delta(\varsigma_{2i-1}, \varsigma_{2i}), \delta(\varsigma_{2i}, \varsigma_{2i+1})\right\} \end{aligned}$$

and (6) becomes

$$\begin{aligned} \delta(\varsigma_{2i}, \varsigma_{2i+1}) \quad &\leq \varphi\left(\max\left\{\delta(\varsigma_{2i-1}, \varsigma_{2i}), \delta(\varsigma_{2i}, \varsigma_{2i+1})\right\}\right) + \psi(0) \\ &< \max\left\{\delta(\varsigma_{2i-1}, \varsigma_{2i}), \delta(\varsigma_{2i}, \varsigma_{2i+1})\right\}. \end{aligned} \tag{7}$$

If for some $i \in \mathbb{N}$, $\max\{\delta(\varsigma_{2i-1},\varsigma_{2i}),\delta(\varsigma_{2i},\varsigma_{2i+1})\} = \delta(\varsigma_{2i},\varsigma_{2i+1})$, then (7) turns into $\delta(\varsigma_{2i},\varsigma_{2i+1}) < \delta(\varsigma_{2i},\varsigma_{2i+1})$ which is a contradiction. Hence

$$\max\{\delta(\varsigma_{2i-1},\varsigma_{2i}),\delta(\varsigma_{2i},\varsigma_{2i+1})\} = \delta(\varsigma_{2i-1},\varsigma_{2i})$$

and so

$$\delta(\varsigma_{2i},\varsigma_{2i+1}) \le \varphi(\delta(\varsigma_{2i-1},\varsigma_{2i})) \tag{8}$$

By continuing this process, since φ is monotone increasing, we find that

$$\delta(\varsigma_{2i},\varsigma_{2i+1}) \le \varphi^{2i}(\delta(\varsigma_0,\varsigma_1)). \tag{9}$$

Similarly, if $\varsigma = \varsigma_{2i+1}$ and $\kappa = \varsigma_{2i}$, then, by the inequality (4) we get

$$\begin{aligned}
\delta(\varsigma_{2i+2},\varsigma_{2i+1}) &= \delta(U^{p_{i+1}}\varsigma_{2i+1}, V^{q_i}\varsigma_{2i}) \\
&\le \varphi\left(\max\left\{\begin{array}{c}\delta(\varsigma_{2i+1},\varsigma_{2i}),\delta(\varsigma_{2i+1},U^{p_{i+1}}\varsigma_{2i+1}),\delta(\varsigma_{2i},V^{q_i}\varsigma_{2i}),\\ \frac{\delta(\varsigma_{2i},U^{p_{i+1}}\varsigma_{2i+1})+\delta(\varsigma_{2i+1},V^{q_i}\varsigma_{2i})}{2s}\end{array}\right\}\right) \\
&\quad +\psi\left(\min\{\delta(\varsigma_{2i+1},U^{p_{i+1}}\varsigma_{2i+1}),\delta(\varsigma_{2i},\mathcal{V}^{q_i}\varsigma_{2i}),\delta(\varsigma_{2i},U^{p_{i+1}}\varsigma_{2i+1}),\delta(\varsigma_{2i+1},\mathcal{V}^{q_i}\varsigma_{2i})\}\right) \\
&= \varphi\left(\max\left\{\delta(\varsigma_{2i+1},\varsigma_{2i}),\delta(\varsigma_{2i+1},\varsigma_{2i+2}),\delta(\varsigma_{2i},\varsigma_{2i+1}),\frac{\delta(\varsigma_{2i},\varsigma_{2i+2})+\delta(\varsigma_{2i+1},\varsigma_{2i+1})}{2s}\right\}\right) \\
&\quad +\psi\left(\min\{\delta(\varsigma_{2i+1},\varsigma_{2i+2}),\delta(\varsigma_{2i},\varsigma_{2i+1}),\delta(\varsigma_{2i},\varsigma_{2i+2}),\delta(\varsigma_{2i+1},\varsigma_{2i+1})\}\right) \\
&\le \varphi\left(\max\{\delta(\varsigma_{2i+1},\varsigma_{2i}),\delta(\varsigma_{2i+1},\varsigma_{2i+2}))\}+\psi(0)\right. \\
&< \max\{\delta(\varsigma_{2i+1},\varsigma_{2i}),\delta(\varsigma_{2i+1},\varsigma_{2i+2})).
\end{aligned} \tag{10}$$

As above, if there is $i \in \mathbb{N}$ such that $\max\{\delta(\varsigma_{2i+1},\varsigma_{2i}),\delta(\varsigma_{2i+1},\varsigma_{2i+2})\} = \delta(\varsigma_{2i+1},\varsigma_{2i+2})$ then from (10) we get $\delta(\varsigma_{2i+1},\varsigma_{2i+2}) < \delta(\varsigma_{2i+1},\varsigma_{2i+2})$ which is a contradiction.

Therefore, $\max\{\delta(\varsigma_{2i+1},\varsigma_{2i}),\delta(\varsigma_{2i+1},\varsigma_{2i+2})\} = \delta(\varsigma_{2i+1},\varsigma_{2i})$ and

$$\delta(\varsigma_{2i+1},\varsigma_{2i+2}) \le \varphi(\delta(\varsigma_{2i},\varsigma_{2i+1})) \le \ldots \le \varphi^{2i}(\delta(\varsigma_1,\varsigma_2)) \tag{11}$$

Let $D(x_0) = \max\{\delta(\varsigma_0,\varsigma_1),\delta(\varsigma_1,\varsigma_2)\}$. Combining (9), (11) and taking into account the property of function φ we conclude that for all $m \in \mathbb{N}$

$$\delta(\varsigma_m,\varsigma_{m+1}) \le \varphi^m(D(\varsigma_0)) \tag{12}$$

and we have

$$\lim_{m\to\infty}\delta(\varsigma_m,\varsigma_{m+1}) = 0. \tag{13}$$

Using triangle inequality, for $j \in \mathbb{N}$, we have

$$\begin{aligned}
\delta(\varsigma_m,\varsigma_{m+j}) &\le s\cdot\left[\delta(\varsigma_m,\varsigma_{m+1})+\delta(\varsigma_{m+1},\varsigma_{m+j})\right] \\
&\le s\cdot\delta(\varsigma_m,\varsigma_{m+1})+s^2\delta(\varsigma_{m+1},\varsigma_{m+2})+\ldots+s^j\cdot\delta(\varsigma_{m+j-1},\varsigma_{m+j}) \\
&\le s\cdot\varphi^m D(\varsigma_0)+s^2\cdot\varphi^{m+1}D(\varsigma_0)+\ldots+s^j\cdot\varphi^{m+j-1}D(\varsigma_0) \\
&= \sum_{l=m}^{m+j-1} s^{l-m+1}\cdot\varphi^l(D(\varsigma_0)) \\
&\le \sum_{l=m}^{\infty} s^l\cdot\varphi^l(D(\varsigma_0)) \to 0
\end{aligned} \tag{14}$$

as $n\to\infty$, and therefore $\{\varsigma_n\}$ is a Cauchy sequence. By completeness of $(\mathcal{X},\delta,s)$, there is some point $\varsigma^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty}\delta(\varsigma_n,\varsigma^*) = 0. \tag{15}$$

We claim that ς^* is a common fixed point of $U^{p(\varsigma^*)}$, respectively $V^{q(\varsigma^*)}$. Indeed, taking $\varsigma = \varsigma_{2i-1}$ and $\kappa = \varsigma^*$ in (4), we have

$$\begin{aligned}
&\delta(U^{p_{i-1}}\varsigma_{2i-1}, V^{q(\varsigma^*)}\varsigma^*) \leq \\
&\leq \varphi\left(\max\left\{\delta(\varsigma_{2i-1},\varsigma^*), \delta(\varsigma_{2i-1}, U^{p_{i-1}}\varsigma_{2i-1}), \delta(\varsigma^*, V^{q(\varsigma^*)}\varsigma^*), \frac{\delta(\varsigma_{2i-1}, V^{q(\varsigma^*)}\varsigma^*)+\delta(\varsigma^*, U^{p_{i-1}}\varsigma_{2i-1})}{2s}\right\}\right) \\
&\quad +\psi\left(\min\left\{\delta(\varsigma_{2i-1}, U^{p_{i-1}}\varsigma_{2i-1}), \delta(\varsigma^*, V^{q(\varsigma^*)}\varsigma^*), \delta(\varsigma_{2i-1}, V^{q(\varsigma^*)}\varsigma^*), \delta(\varsigma^*, U^{p_{i-1}}\varsigma_{2i-1})\right\}\right) \\
&= \varphi\left(\max\left\{\delta(\varsigma_{2i-1},\varsigma^*), \delta(\varsigma_{2i-1},\varsigma_{2i}), \delta(\varsigma^*, \mathcal{V}^{q(\varsigma^*)}\varsigma^*), \frac{\delta(\varsigma_{2i-1}, \mathcal{V}^{q(\varsigma^*)}\varsigma^*)+\delta(\varsigma^*,\varsigma_{2i})}{2s}\right\}\right) \\
&\quad +\psi\left(\min\left\{\delta(\varsigma_{2i-1},\varsigma_{2i}), \delta(\varsigma^*, \mathcal{V}^{q(\varsigma^*)}\varsigma^*), \delta(\varsigma_{2i-1}, \mathcal{V}^{q(\varsigma^*)}\varsigma^*), \delta(\varsigma^*,\varsigma_{2i})\right\}\right).
\end{aligned} \tag{16}$$

Let $i \to \infty$ in the above inequality, and taking (15) into account, we find that

$$\delta(\varsigma^*, \mathcal{V}^{q(\varsigma^*)}\varsigma^*) \leq \lim_{n\to\infty} \delta(\varsigma_{2i}, \mathcal{V}^{q(\varsigma^*)}\varsigma^*) \leq \varphi\left(\delta(\varsigma^*, \mathcal{V}^{q(\varsigma^*)}\varsigma^*)\right) < \delta(\varsigma^*, \mathcal{V}^{q(\varsigma^*)}\varsigma^*), \tag{17}$$

which implies that $\delta(\varsigma^*, \mathcal{V}^{q(\varsigma^*)}\varsigma^*) = 0$. Hence, $V^{q(\varsigma^*)}\varsigma^* = \varsigma^*$. Supposing that $U^{p(\varsigma^*)}\varsigma^* \neq \varsigma^*$, from (4) and (17), we have

$$\begin{aligned}
0 < \delta(U^{p(\varsigma^*)}\varsigma^*, \varsigma^*) &= \delta(U^{p(\varsigma^*)}\varsigma^*, V^{q(\varsigma^*)}\varsigma^*) \leq \varphi(\max\left\{\delta(\varsigma^*, \mathcal{U}^{p(\varsigma^*)}\varsigma^*), \delta(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*)/2s\right\}) \\
&< \delta(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*)
\end{aligned} \tag{18}$$

which is a contradiction, and hence, $U^{p(\varsigma^*)}\varsigma^* = \varsigma^*$.

Be $\kappa^* \in \mathcal{X}$ another point such that $U^{p(\kappa^*)}\kappa^* = \kappa^* = V^{q(\kappa^*)}\kappa^*$ and $\varsigma^* \neq \kappa^*$. Since U, V satisfy (4), we have

$$\begin{aligned}
0 &< \delta(\varsigma^*, \kappa^*) = \delta(U^{p(\varsigma^*)}\varsigma^*, V^{q(\kappa^*)}\kappa^*)) \\
&\leq \varphi\left(\max\left\{\begin{array}{c} \delta(\varsigma^*,\kappa^*), \delta(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*), \delta(\kappa^*, V^{q(\kappa^*)}\kappa^*), \\ \left[\delta(\kappa^*, U^{p(\varsigma^*)}\varsigma^*) + \delta(\varsigma^*, V^{q(\kappa^*)}\kappa^*)\right]/2s \end{array}\right\}\right) \\
&\quad +\psi\left(\min\left\{\delta(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*), \delta(\kappa^*, \mathcal{V}^{q(\kappa^*)}\kappa^*), \delta(\kappa^*, U^{p(\varsigma^*))}\varsigma^*), \delta(\varsigma^*, V^{q(\kappa^*)}\kappa^*)\right\}\right) \\
&= \varphi(\delta(\varsigma^*,\kappa^*)) < \delta(\varsigma^*,\kappa^*),
\end{aligned} \tag{19}$$

but, the above inequality is possible only if $\delta(\varsigma^*, \kappa^*) = 0$ that is $\varsigma^* = \kappa^*$. Very easy, due to the uniqueness of the fixed point we can conclude that ς^* is a common fixed point for U and V. Indeed,

$$U\varsigma^* = \mathcal{U}(U^{p(\varsigma^*)}\varsigma^*) = U^{p(\varsigma^*)}(U\varsigma^*) \tag{20}$$

shows that $U\varsigma^*$ is also fixed point of $U^{p(\varsigma^*)}$. However, $U^{p(\varsigma^*)}$ has exactly one fixed point ς^*, so $U\varsigma^* = \varsigma^*$. Similarly, $V\varsigma^* = \varsigma^*$. □

If we take $c \in [0, \frac{1}{s})$, $k \geq 1$, $\varphi(x) = cx$ and $\psi(x) = kx$ for all $x > 0$ then, we get the following result.

Corollary 1. *Let $U, \mathcal{V}$ be two self-mappings on a complete b-MS $(\mathcal{X}, \delta, s)$. Suppose that there exist $0 \leq c < \frac{1}{s}$ and $k \geq 1$ such that for all $\varsigma, \kappa \in \mathcal{X}$ there exist positive integers $p(\varsigma), q(\kappa)$ such that*

$$\begin{aligned}
\delta(U^{p(\varsigma)}\varsigma, V^{q(\kappa)}\kappa) &\leq c \cdot \max\left\{\delta(\varsigma,\kappa), \delta(\varsigma, U^{p(\varsigma)}\varsigma), \delta(\kappa, V^{q(\kappa)}\kappa), \frac{\delta(\kappa, U^{p(\varsigma)}\varsigma)+\delta(\varsigma, V^{q(\kappa)}\kappa)}{2s}\right) \\
&\quad +k \cdot \min\left\{\delta(\varsigma, U^{p(\varsigma)}\varsigma), \delta(y, V^{q(\kappa)}\kappa), \delta(\kappa, U^{p(\varsigma)}\varsigma), d(\varsigma, V^{q(\kappa)}\kappa)\right\},
\end{aligned} \tag{21}$$

then the pair of the mappings U, V possesses a common fixed point ς^*.

Corollary 2. *Let U be a self-mapping on a complete b-MS $(\mathcal{X},\delta,s)$. Suppose that for any $\varsigma,\kappa\in\mathcal{X}$ there exist positive integer $p(\varsigma)$ and there exist $\psi\in\Psi$ and upper semicontinuous $\varphi\in\Phi_b$ such that*

$$\begin{aligned}\delta(U^{p(\varsigma)}\varsigma,U^{p(\varsigma)}\kappa) &\le \varphi\left(\max\left\{\delta(\varsigma,\kappa),\delta(\varsigma,U^{p(\varsigma)}\varsigma),\delta(\kappa,U^{p(\varsigma)}\kappa),\tfrac{\delta(\kappa,U^{p(\varsigma)}\varsigma)+\delta(\varsigma,U^{p(\varsigma)}\kappa)}{2s}\right)\right\}\\ &+\psi\left(\min\left\{\delta(\varsigma,U^{p(\varsigma)}\varsigma),\delta(\kappa,U^{p(\varsigma)}\kappa),\delta(\varsigma,U^{p(\varsigma)}\varsigma),\delta(\varsigma,U^{p(\varsigma)}\kappa)\right\}\right),\end{aligned} \tag{22}$$

then the map U has a unique fixed point ς^*.

Now we take the same idea in the context of b-DMS.

Theorem 3. *Let $(\mathcal{X},\delta_d,s)$ be a 0-complete b-DMS and $U,V:(\mathcal{X},\delta_d,s)\to(\mathcal{X},\delta_d,s)$ be two functions. Let the function $\varphi\in\Phi_b$. Suppose that for all $\varsigma,\kappa\in\mathcal{X}$ we can find the positive integers $p(\varsigma),q(\kappa)$ such that*

$$\delta_d(U^{p(\varsigma)}\varsigma,V^{q(\kappa)}\kappa) \le \varphi\left(\max\left\{\delta_d(\varsigma,\kappa),\delta_d(\varsigma,U^{p(\varsigma)}\varsigma),\delta_d(\kappa,V^{q(\kappa)}\kappa),\tfrac{\delta_d(\kappa,U^{p(\varsigma)}\varsigma)+\delta_d(\varsigma,V^{q(\kappa)}\kappa)}{4s}\right)\right\} \tag{23}$$

Then the pair of the functions U,V has exactly one fixed point ς^*.

Proof. Consider a point $\varsigma_0\in\mathcal{X}$ and as in above theorem we shall define the sequence $\{\varsigma_n\}$ in $\mathcal{X}$ as follows:

$$\varsigma_1=V^{q(\varsigma_0)}\varsigma_0,\ x_2=U^{p(\varsigma_1)}\varsigma_1,\ldots\ \varsigma_{2i+1}=V^{q(\varsigma_{2i})}\varsigma_{2i},\ \varsigma_{2i+2}=U^{p(\varsigma_{2i+1})}\varsigma_{2i+1},\ldots \tag{24}$$

Denoting $p_{i-1}=p(\varsigma_{2i-1})$ and $q_i=q(\varsigma_{2i})$, for any $i\in\mathbb{N}$, we can write $\varsigma_{2i}=U^{p_{i-1}}\varsigma_{2i-1}$ and $\varsigma_{2i+1}=V^{q_i}\varsigma_{2i}$. As we have seen in Theorem 2, the first purpose is to show that the sequence $\{\varsigma_n\}$ is Cauchy. For this, let us get in (23) $\varsigma=\varsigma_{2i-1}$ and $\kappa=\varsigma_{2i}$. We have,

$$\begin{aligned}\delta_d(\varsigma_{2i},\varsigma_{2i+1}) &= \delta_d(U^{p_{i-1}}\varsigma_{2i-1},\mathcal{V}^{q_i}\varsigma_{2i})\\ &\le \varphi\left(\max\left\{\begin{array}{c}\delta_d(\varsigma_{2i-1},\varsigma_{2i}),\delta_d(\varsigma_{2i-1},U^{p_{i-1}}\varsigma_{2i-1}),\delta_d(\varsigma_{2i},V^{q_i}\varsigma_{2i}),\\ \frac{\delta_d(\varsigma_{2i},U^{p_{i-1}}\varsigma_{2i-1})+\delta_d(\varsigma_{2i-1},\mathcal{V}^{q_i}\varsigma_{2i})}{4s}\end{array}\right\}\right)\\ &= \varphi\left(\max\left\{\begin{array}{c}\delta_d(\varsigma_{2i-1},\varsigma_{2i}),\delta_d(\varsigma_{2i-1},\varsigma_{2i}),\delta_d(\varsigma_{2i},\varsigma_{2i+1}),\\ \frac{\delta_d(\varsigma_{2i},\varsigma_{2i})+\delta_d(\varsigma_{2i-1},\varsigma_{2i+1})}{4s}\end{array}\right\}\right)\\ &< \max\left\{\begin{array}{c}\delta_d(\varsigma_{2i-1},\varsigma_{2i}),\delta_d(\varsigma_{2i},\varsigma_{2i+1}),\\ \frac{s[\delta_d(\varsigma_{2i},\varsigma_{2i-1})+\delta_d(\varsigma_{2i-1},\varsigma_{2i})]+s[\delta_d(\varsigma_{2i-1},\varsigma_{2i})+\delta_d(\varsigma_{2i},\varsigma_{2i+1})]}{4s}\end{array}\right\}\\ &= \max\left\{\delta_d(\varsigma_{2i-1},\varsigma_{2i}),\delta_d(\varsigma_{2i},\varsigma_{2i+1}),\tfrac{3s\delta_d(\varsigma_{2i},\varsigma_{2i-1})+s\delta_d(\varsigma_{2i},\varsigma_{2i+1})]}{4s}\right\}\end{aligned} \tag{25}$$

and then two situations can be considerate. If $\delta_d(\varsigma_{2i-1},\varsigma_{2i})\le\delta_d(\varsigma_{2i},\varsigma_{2i+1})$, then the inequality (25) becomes

$$\delta_d(\varsigma_{2i},\varsigma_{2i+1})<\delta_d(\varsigma_{2i},\varsigma_{2i+1})$$

which is a contradiction. However, this tells us that $\delta_d(\varsigma_{2i-1},\varsigma_{2i})>\delta_d(\varsigma_{2i},\varsigma_{2i+1})$ for all $i\in\mathbb{N}$. Thus, regarding at (25)

$$\delta_d(\varsigma_{2i},\varsigma_{2i+1})\le\varphi(\delta_d(\varsigma_{2i-1},\varsigma_{2i}))$$

Since $\varphi\in\Phi_b$, we know that φ is monotone increasing so, we obtain

$$\delta_d(\varsigma_{2i},\varsigma_{2i+1})\le\varphi(\delta_d(\varsigma_{2i-1},\varsigma_{2i}))\le\varphi^2(\delta_d(\varsigma_{2i-2},\varsigma_{2i-1}))\le\ldots\le\varphi^{2i}(\delta_d(\varsigma_0,\varsigma_1)). \tag{26}$$

Similarly, we can observe that if we replace ς and κ in (23) by ς_{2i+1} respectively ς_{2i} we have

$$\begin{aligned}
\delta_d(\varsigma_{2i+2}, \varsigma_{2i+1}) &= \delta_d(\mathcal{U}^{p_{i+1}}\varsigma_{2i+1}, \mathcal{V}^{q_i}\varsigma_{2i}) \\
&\leq \varphi\left(\max\left\{\begin{array}{c} \delta_d(\varsigma_{2i+1}, \varsigma_{2i}), \delta_d(\varsigma_{2i+1}, \mathcal{U}^{p_{i+1}}\varsigma_{2i+1}), \delta_d(\varsigma_{2i}, \mathcal{V}^{q_i}\varsigma_{2i}), \\ \frac{\delta_d(\varsigma_{2i}, \mathcal{U}^{p_{i+1}}\varsigma_{2i+1}) + \delta_d(\varsigma_{2i+1}, \mathcal{V}^{q_i}\varsigma_{2i})}{4s} \end{array}\right\}\right) \\
&= \varphi\left(\max\left\{\begin{array}{c} \delta_d(\varsigma_{2i+1}, \varsigma_{2i}), \delta_d(\varsigma_{2i+1}, \varsigma_{2i+2}), \delta_d(\varsigma_{2i}, \varsigma_{2i+1}), \\ \frac{\delta_d(\varsigma_{2i}, \varsigma_{2i+2}) + \delta_d(\varsigma_{2i+1}, \varsigma_{2i+1})}{4s} \end{array}\right\}\right) \\
&< \max\left\{\begin{array}{l} \delta_d(\varsigma_{2i+1}, \varsigma_{2i}), \delta_d(\varsigma_{2i+1}, \varsigma_{2i+2}), \delta_d(\varsigma_{2i}, \varsigma_{2i+1}), \\ \frac{s[\delta_d(\varsigma_{2i}, \varsigma_{2i+1}) + \delta_d(\varsigma_{2i+1}, \varsigma_{2i+2})] + s[\delta_d(\varsigma_{2i+1}, \varsigma_{2i}) + \delta_d(\varsigma_{2i}, \varsigma_{2i+1})]}{4s} \end{array}\right\} \\
&= \max\left\{\begin{array}{l} \delta_d(\varsigma_{2i+1}, \varsigma_{2i}), \delta_d(\varsigma_{2i+1}, \varsigma_{2i+2}), \\ \frac{3s\delta_d(\varsigma_{2i}, \varsigma_{2i+1}) + s\delta_d(\varsigma_{2i+1}, \varsigma_{2i+2})}{4s} \end{array}\right\}.
\end{aligned} \tag{27}$$

Again, if there is $N \in \mathbb{N}$ such that $\delta_d(\varsigma_{2i+1}, \varsigma_{2i}) \leq \delta_d(\varsigma_{2i+1}, \varsigma_{2i+2})$ for any $i > N$, then

$$\begin{aligned}
\delta_d(\varsigma_{2i+2}, \varsigma_{2i+1}) &\leq \varphi(\delta_d(\varsigma_{2i+1}, \varsigma_{2i+2})) \\
&< \delta_d(\varsigma_{2i+1}, \varsigma_{2i+2}).
\end{aligned}$$

From this contradiction we get that $\delta_d(\varsigma_{2i+1}, \varsigma_{2i}) > \delta_d(\varsigma_{2i+1}, \varsigma_{2i+2})$ and with the same reasoning as above, we can conclude that

$$\delta_d(\varsigma_{2i+1}, \varsigma_{2i+2}) \leq \varphi(\delta_d(\varsigma_{2i}, \varsigma_{2i+1})) \leq \varphi^2(\delta_d(\varsigma_{2i-1}, \varsigma_{2i})) \leq \ldots \leq \varphi^{2i}(\delta_d(\varsigma_1, \varsigma_2)). \tag{28}$$

Certainly, combining (26) and (28) we find that

$$\delta_d(\varsigma_n, \varsigma_{n+1}) \leq \varphi^n(D(\varsigma_0)), \tag{29}$$

for any $n \in \mathbb{N}$, where $D(\varsigma_0) = \max\{\delta_d(\varsigma_0, \varsigma_1), \delta_d(\varsigma_1, \varsigma_2)\}$. On one hand the inequality (29) shows us, taking into account (2^*) from Definition 5 that

$$\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma_{n+1}) = 0. \tag{30}$$

On the other hand, as in (14), we have

$$\begin{aligned}
\delta_d(\varsigma_n, \varsigma_{n+r}) &\leq s \cdot [\delta_d(\varsigma_n, \varsigma_{n+1}) + \delta_d(\varsigma_{n+1}, \varsigma_{n+r})] \\
&\leq s \cdot \delta_d(\varsigma_n, \varsigma_{n+1}) + s^2 \cdot \delta_d(\varsigma_{n+1}, \varsigma_{n+2}) + \ldots + s^r \cdot \delta_d(\varsigma_{n+r-1}, \varsigma_{n+r}) \\
&\leq s \cdot \varphi^n(D(\varsigma_0)) + s^2 \cdot \varphi^{n+1}(D(\varsigma_0)) + \ldots + s^r \cdot \varphi^{n+r-1}(D(\varsigma_0)) \\
&= \sum_{j=n}^{n+r-1} b^{j-n+1} \cdot \varphi^j(D(\varsigma_0)) \\
&\leq \sum_{j=n}^{\infty} b^j \cdot \varphi^j(D(\varsigma_0)) \to 0
\end{aligned} \tag{31}$$

as $n \to \infty$. Hence the sequence $\{\varsigma_n\}$ is 0-Cauchy. Since $(\mathcal{X}, \delta_d, s)$ is a 0-complete space, every 0-Cauchy sequence is convergent. Then there is some point $\varsigma^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma_{n+r}) = \lim_{n\to\infty} \delta_d(\varsigma_n, \varsigma^*) = d(\varsigma^*, \varsigma^*) = 0. \tag{32}$$

We prove that the limit of sequence $\{\varsigma_n\}$ is a fixed point for U and V. For this, we are considering in inequality (23), $\varsigma = \varsigma^*$ and $\kappa = \varsigma_{2n}$

$$\begin{aligned} \delta_d(U^{p(\varsigma^*)}\varsigma^*, \varsigma_{2n+1}) &= \delta_d(U^{p(\varsigma^*)}\varsigma^*, \mathcal{V}^{q_n}\varsigma_{2n}) \\ &\leq \varphi\left(\max\left\{\begin{array}{c} \delta_d(\varsigma^*, \varsigma_{2n}), \delta_d(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*), \delta_d(\varsigma_{2n}, V^{q_n}\varsigma_{2n})), \\ \frac{\delta_d(\varsigma_{2n}, U^{p(\varsigma^*)}\varsigma^*) + \delta_d(\varsigma^*, \mathcal{V}^{q_n}\varsigma_{2n})}{4s} \end{array}\right\}\right) \\ &= \varphi\left(\max\left\{\begin{array}{c} \delta_d(\varsigma^*, \varsigma_{2n}), \delta_d(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*), \delta_d(\varsigma_{2n}, \varsigma_{2n+1}) \\ \frac{\delta_d(\varsigma_{2n}, U^{p(\varsigma^*)}\varsigma^*) + \delta_d(\varsigma^*, \varsigma_{2n+1})}{4s} \end{array}\right\}\right) \\ &< \varphi\left(\max\left\{\begin{array}{c} \delta_d(\varsigma^*, \varsigma_{2n}), \delta_d(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*), \delta_d(\varsigma_{2n}, \varsigma_{2n+1}), \\ \frac{s\left[\delta_d(\varsigma_{2n}, \varsigma^*) + \delta_d(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*)\right] + \delta_d(\varsigma^*, \varsigma_{2n+1})}{4s} \end{array}\right\}\right). \end{aligned} \tag{33}$$

Letting $n \to \infty$ in the both sides of the above inequality and considering (30), (32) we get that

$$\lim_{n\to\infty} \sup \delta_d(U^{p(\varsigma^*)}\varsigma^*, \varsigma_{2n+1}) \leq \varphi(\delta_d(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*)) < \delta_d(\varsigma^*, U^{p(\varsigma^*)}\varsigma^*),$$

a contradiction. Thus, $\delta_d(U^{p(\varsigma^*)}\varsigma^*, \varsigma^*) = 0$ and from $(\delta b1)$ in Definition 3 we get $U^{p(\varsigma)}\varsigma = \varsigma^*$. Analogously, if we substitute ς by ς_{2n-1} and κ by ς^* we will find that $V^{q(\varsigma^*)}\varsigma^* = \varsigma^*$. In concluding this proof we wish to show that the common fixed point is unique. Supposing by contradiction that there is $\kappa^* \in \mathcal{X}$ a point such that $U^{p(\kappa^*)}\kappa^* = \kappa^* = V^{q(\kappa^*)}\kappa^*$ and $\kappa^* \neq \varsigma^*$. Replacing in (23) we have:

$$\begin{aligned} \delta_d(\varsigma^*, \kappa^*) &= \delta_d(U^{p(\varsigma^*)}\varsigma^*, V^{q(\kappa^*)}\kappa^*) \\ &\leq \varphi(\max\{\delta_d(\varsigma^*, \kappa^*), \delta_d(\varsigma^*, \mathcal{U}^{p(\varsigma^*)}\varsigma^*), \delta_d(\kappa^*, V^{q(\kappa^*)}\kappa^*), \tfrac{\delta_d(\kappa^*, U^{p(\varsigma^*)}\varsigma^*) + \delta_d(\varsigma^*, V^{q(\kappa^*)}\kappa^*)}{4s}\}) \end{aligned}$$

thus

$$\delta_d(\varsigma^*, \kappa^*) < \delta_d(\varsigma^*, \kappa^*),$$

which is a contradiction. Therefore $\delta_d(\varsigma^*, \kappa^*) = 0$, which implies $\varsigma^* = \kappa^*$. □

Example 3. *Let* $\mathcal{X} = \left\{X = \begin{pmatrix} 4\varsigma & \varsigma \\ \kappa & -2\kappa \end{pmatrix} : \varsigma, \kappa \in \mathbb{R}\right\}$ *and consider the* 2*-dislocated metric* $\delta_d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ *defined by* $\delta_d(X, Y) = (|trX| + |trY|)^2$. *Define two maps* $U, V : \mathcal{X} \to \mathcal{X}$ *by*

$$U(X) = AX \text{ respectively } V(X) = XB,$$

where

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}.$$

Let $X, Y \in \mathcal{X}$, $X = \begin{pmatrix} 4\varsigma_1 & \varsigma_1 \\ \kappa_1 & -2\kappa_1 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 4\varsigma_2 & \varsigma_2 \\ \kappa_2 & -2\kappa_2 \end{pmatrix}$ *where* $\varsigma_1, \varsigma_2, \kappa_1, \kappa_2 \in \mathcal{X}$.

By elementary calculation, we get $U(X) = \begin{pmatrix} 2\kappa_1 & -4\kappa_1 \\ \kappa_1 & -2\kappa_1 \end{pmatrix}$ *and* $V^2(Y) = YB^2 = \begin{pmatrix} 0 & -9\kappa_2 \\ -9\kappa_2 & 0 \end{pmatrix}$. *Since* $\delta_d(U(X), V^2(Y))) = (|tr(U(X))| + |tr(V^2(Y))|)^2 = 0$ *we conclude that for* $p = 1$ *and* $q = 2$ *all the presumptions of Theorem 3 are satisfied. Accordingly, the maps U and V have a unique fixed point. In other words, there is a unique matrix* $X \in \mathcal{X}$ *such that* $AX = X = XB$, *namely* $X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

3. Application

Let $0 < \gamma$ be a real number and $\varsigma : [1,\infty) \to \mathbb{R}$ be a function. Throughout this part, we consider that $[\gamma]$ represents the integer part of real number γ and by $\log(\cdot)$ we denote $\log_e(\cdot)$.

The Hadamard derivative of fractional order γ for ς is defined by

$$D^{\gamma}\varsigma(\theta) = \frac{1}{\Gamma(n-\gamma)}\left(\theta\frac{d}{d\theta}\right)^{n}\int_{1}^{\theta}\left(\log\frac{\theta}{s}\right)^{n-\gamma-1}\frac{\varsigma(s)}{s}ds,\ n-1<\gamma<n. \tag{34}$$

The Hadamard fractional integral of order γ for ς is given by

$$I^{\gamma}\varsigma(\theta) = \frac{1}{\Gamma(\gamma)}\int_{1}^{\theta}\left(\log\frac{\theta}{s}\right)^{\gamma-1}\frac{\varsigma(s)}{s}ds,\ \gamma>0, \tag{35}$$

provided the integral exists.

Starting from [18], where the problems involving Hadamard-type fractional derivatives are studied, we discuss here the existence of a solution for the following system of fractional functional differential equations with initial values:

$$\begin{cases} D^{\gamma}\varsigma(\theta) & = \xi(\theta,\varsigma_{\theta}), \ \text{ for each } \theta\in[0,t], 0<\gamma<1 \\ D^{\gamma}\kappa(\theta) & = \eta(\theta,\kappa_{\theta}), \\ \varsigma(\theta) & = \kappa(\theta) = f(\theta),\ \theta\in[1-y,1] \end{cases} \tag{36}$$

where the functions $\xi,\eta : [1,t]\times C([-y,0],\mathbb{R}) \to \mathbb{R}$ are given, $f\in C([1-y,1],\mathbb{R})$ is such that $f(1)=0$ and for any ς,κ defined on $[1-y,t]$ the functions $\varsigma_{\theta},\kappa_{\theta}$ are elements of $C([-y,0],\mathbb{R})$ such that

$$\varsigma_{\theta}(\tau) = \varsigma(\theta+\tau),\ \kappa_{\theta}(\tau) = \kappa(\theta+\tau)$$

for any $\theta\in[0,t]$. Let $\mathcal{X} = C([1-y,t],\mathbb{R})$ be the set of real continuous functions and consider the distance $d : \mathcal{X}\times\mathcal{X}\to[0,\infty)$ defined as

$$d(\varsigma,\kappa) = \sup_{\theta\in[1-y,t]}|\varsigma(\theta)-\kappa(\theta)|,\ \forall\varsigma,\kappa\in\mathcal{X}.$$

For $r\geq 1$ we take the b-distance $\delta : \mathcal{X}\times\mathcal{X}\to[0,\infty]$ given by

$$\delta(\varsigma,\kappa) = (d(\varsigma,\kappa))^{r} = \sup_{\theta\in[1-y,t]}|\varsigma(\theta)-\kappa(\theta)|^{r},\ \forall\varsigma,\kappa\in\mathcal{X}.$$

Certainly, $(\mathcal{X},\delta,s)$is a complete b-metric space, where $s=2^{r-1}$.

Theorem 4. *Let $\lambda>0$ such that $\lambda\frac{(\log t)^{\gamma}}{\Gamma(\gamma+1)} < 2^{\frac{1}{r}-1}$. Assume that*

$$|\xi(\theta,\varsigma)-\eta(\theta,\kappa)| \leq c\sup_{\theta\in[1,t]}\left|\sqrt{|\varsigma|}-\sqrt{|\kappa|}\right|$$
$$|\xi(\theta,\varsigma)|+|\eta(\theta,\kappa)| \leq c\sup_{\theta\in[1,t]}\left|\sqrt{|\varsigma|}+\sqrt{|\kappa|}\right|$$

for $\theta\in[1,t]$ and every $\varsigma,\kappa\in\mathcal{X}$. Then the system (36) possesses a unique solution on the interval $[1-y,t]$.

Proof. Define $U, V : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned} U\varsigma(\theta) &= \begin{cases} f(\theta), & \text{if } \theta \in [1-y,1] \\ \frac{1}{\Gamma(\gamma)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\gamma-1}\frac{\xi(s,\varsigma_s)}{s}ds, & \text{if } \theta \in [1,t] \end{cases} \\ V\kappa(\theta) &= \begin{cases} f(\theta), & \text{if } \theta \in [1-y,1] \\ \frac{1}{\Gamma(\gamma)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\gamma-1}\frac{\eta(s,\kappa_s)}{s}ds, & \text{if } \theta \in [1,t] \end{cases} \end{aligned} \tag{37}$$

(We should mention that the system (36) has a unique solution if and only if the operators U and V have exactly one common fixed point.)

Now we have for $\theta \in [1,t]$:

$$\begin{aligned} |U\varsigma(\theta) - V\kappa(\theta)| &\le \frac{1}{\Gamma(\gamma)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\gamma-1}\frac{|\xi(s,\varsigma_s)-\eta(s,\kappa_s)|}{s}ds \\ &\le \lambda\cdot\frac{1}{\Gamma(\gamma)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\gamma-1}\sup_{\theta\in[1,t]}\left|\sqrt{|\varsigma_s|}-\sqrt{|\kappa_s|}\right|\frac{ds}{s} \\ &\le \lambda\cdot\frac{1}{\Gamma(\gamma)}\sup_{\theta\in[1-y,t]}\left|\sqrt{|\varsigma_s|}-\sqrt{|\kappa_s|}\right|\int_1^\theta\left(\log\frac{\theta}{s}\right)^{\gamma-1}\frac{ds}{s} \\ &= \lambda\frac{(\log\theta)^\gamma}{\Gamma(\gamma+1)}\sup_{\theta\in[1-y,t]}\left|\sqrt{|\varsigma_s|}-\sqrt{|\kappa_s|}\right|. \end{aligned}$$

At the same time,

$$\begin{aligned} |U\varsigma(\theta)| + |V\kappa(\theta)| &\le \frac{1}{\Gamma(\gamma)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\gamma-1}\frac{|\xi(s,\varsigma_s)|+|\eta(s,\kappa_s)|}{s}ds \\ &\le \lambda\cdot\frac{1}{\Gamma(\gamma)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\gamma-1}\sup_{\theta\in[1,t]}\left|\sqrt{|\varsigma_s|}+\sqrt{|\kappa_s|}\right|\frac{ds}{s} \\ &\le \lambda\cdot\frac{1}{\Gamma(\gamma)}\sup_{\theta\in[1-y,t]}\left|\sqrt{|\varsigma_s|}+\sqrt{|\kappa_s|}\right|\int_1^\theta\left(\log\frac{\theta}{s}\right)^{\gamma-1}\frac{ds}{s} \\ &= \lambda\frac{(\log\theta)^\gamma}{\Gamma(\gamma+1)}\sup_{\theta\in[1-y,t]}\left|\sqrt{|\varsigma_s|}+\sqrt{|\kappa_s|}\right|. \end{aligned}$$

Now, we have

$$\begin{aligned} \delta(U^2\varsigma, V^2\kappa) &= (\sup_{\theta\in[1-y,t]}\left|U^2\varsigma(\theta)-V^2\kappa(\theta)\right|)^r \\ &= (\sup_{\theta\in[1-y,t]}|U\varsigma(\theta)-V\kappa(\theta)| \times \sup_{\theta\in[1-y,t]}|U\varsigma(\theta)+V\kappa(\theta)|)^r \\ &\le (\sup_{\theta\in[1-y,t]}|U\varsigma(\theta)-V\kappa(\theta)| \times \sup_{\theta\in[1-y,t]}|U\varsigma(\theta)|+|V\kappa(\theta)|)^r \\ &\le (\lambda\frac{(\log\theta)^\gamma}{\Gamma(\gamma+1)}\sup_{\theta\in[1-y,t]}\left|\sqrt{|\varsigma_s|}-\sqrt{|\kappa_s|}\right| \times \sup_{\theta\in[1-y,t]}\left|\sqrt{|\varsigma_s|}+\sqrt{|\kappa_s|}\right|)^r \\ &= (\lambda\frac{(\log\theta)^\gamma}{\Gamma(\gamma+1)}\sup_{\theta\in[1-y,t]}||\varsigma_s|-|\kappa_s||)^r \\ &\le (\lambda\frac{(\log\theta)^\gamma}{\Gamma(\gamma+1)}\sup_{\theta\in[1-y,t]}|\varsigma_s-\kappa_s|)^r \\ &= (\lambda\frac{(\log\theta)^\gamma}{\Gamma(\gamma+1)})^r\delta(\varsigma,\kappa), \end{aligned}$$

for all $\varsigma, \kappa \in \mathcal{X}$. We conclude that for any $\varsigma, \kappa \in \mathcal{X}$ taking $p(\varsigma) = q(\kappa) = 2$ and $c = \left(\lambda\frac{(\log t)^\gamma}{\Gamma(\gamma+1)}\right)^r$ all presumptions of Corollary 1 are verified and the maps U and V have exactly one common fixed point on $\mathcal{X}$, so the system (36) has a unique common solution in $[1,t]$. □

4. Conclusions

In this paper, we have two main goals. The first one is to get the most general form of Seghal [15]-type fixed-point results, that is, investigating a fixed point of certain operators with a contractive iterate at a point in the setting of b-dislocated metric space. The second main goal of the paper is to underline the importance of the obtained fixed-point results by providing an application. As its origin, one of the pioneers of the fixed-point theorem, the Banach contraction principle, was derived from a proposed solution of a differential equation. Under this motivation, we investigate the solution of Hadamard-type fractional functional differential equations.

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