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Locating Chromatic Number of Powers of Paths and Cycles

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Abstract: Let c be a proper k-coloring of a graph G. Let $\pi = \{R_1, R_2, ..., R_k\}$ be the partition of $V(G)$ induced by c, where R_i is the partition class receiving color i. The color code $c_\pi(v)$ of a vertex v of G is the ordered k-tuple $(d(v, R_1), d(v, R_2), ..., d(v, R_k))$, where $d(v, R_i)$ is the minimum distance from v to each other vertex $u \in R_i$ for $1 \leq i \leq k$. If all vertices of G have distinct color codes, then c is called a locating k-coloring of G. The locating-chromatic number of G, denoted by $\chi_L(G)$, is the smallest k such that G admits a locating coloring with k colors. In this paper, we give a characterization of the locating chromatic number of powers of paths. In addition, we find sharp upper and lower bounds for the locating chromatic number of powers of cycles.

Keywords: locating chromatic number; powers of paths; powers of cycles

MSC: 05C76; 05C38

1. Introduction

All graphs considered in this paper are simple connected graphs. The m-th power graph, G^m, of a graph G is the graph whose vertex set is $V(G)$ and in which two distinct vertices are adjacent if and only if their distance in G is at most m. Let c be a proper k-coloring of a graph G and $\pi = \{R_1, R_2, ..., R_k\}$ be an ordered partition of $V(G)$ of the resulting color classes. For any vertex v of G, the color code of v with respect to π, $c_\pi(v)$, is defined as the ordered k-tuple $(d(v, R_1), d(v, R_2), ..., d(v, R_k))$, where $d(v, R_i)$ is the minimum distance from v to each other vertex $u \in R_i$ for $1 \leq i \leq k$. If distinct vertices of G have distinct color codes, then we call c a locating coloring of G. The locating chromatic number of G, $\chi_L(G)$, is the minimum number of colors needed in a locating coloring of G. The locating-chromatic number of a graph is a combined concept between the coloring and partition dimension of a graph. There are many applications of graph coloring and labeling in various fields, for instance, this notion relates to different applications in computer science and communication network and it plays an important role in solving scheduling problems, storage problem of chemical substances and placement problem of particular different objects—see, for example, [1,2]. The concept of locating chromatic number of a graph was introduced and studied by Chartrand et al. [3] in 2002. They established some bounds for the locating chromatic number of a connected graph. They also proved that, for a connected graph G with $n \geq 3$ vertices, $\chi_L(G) = n$ if and only if G is a complete multi-partite graph. Hence, the locating chromatic number of the complete graph K_n is n. In addition, for paths and cycles of order $n \geq 3$, they proved that $\chi_L(P_n) = 3$, $\chi_L(C_n) = 3$ when n is odd, and $\chi_L(C_n) = 4$ when n is even. The locating chromatic numbers of trees, and the amalgamation of stars, the graphs with dominant vertices are studied in [4–6], respectively.

The distance graph $G(D)$ with distance set $D = \{d_1, d_2, ...\} \subseteq \mathbb{N}$ is a graph with vertex set $\{x_i : i \in \mathbb{Z}\}$, and edge set $\{x_i x_j : |i - j| \in D\}$. The circulant graph can be defined as follows. Let n, r

be two positive integers and let $S = \{k_1, k_2, ..., k_r\}$ with $\{k_1 < k_2 < \cdots < k_r \leq \frac{n}{2}\}$. Then, the vertex set of the circulant graph $G(n;S)$ is $\{x_1, x_2, ..., x_n\}$ and the set of edges is $\{x_i x_j : i - j \equiv \pm k_l \bmod n,$ for some $k_l \in S\}$. The problem of coloring of this class of graphs has attracted considerable attention—see, for example, [7,8]. Circulant graphs have been extensively studied and have an immense number of applications to multicomputer networks and distributed computation—see, for example, [9,10]. The distance graph $G(D)$ with finite distance set $D = \{1, 2, ..., m\}$ is isomorphic to the m-th power graph of a path and the circulant graph $G(n;S)$ with $S = \{1, 2, ..., m\}$ is isomorphic to the m-th power graph of a cycle. In this paper, we investigate the locating chromatic number of powers of paths and powers of cycles. For further work, one might consider the locating chromatic number of circulant graphs $G(n;S)$ for any finite set S.

2. Locating Chromatic Number of Powers of Paths

Let P_n denote the path of order n with vertex set $V(P_n) = \{x_1, x_2, ..., x_n\}$ and edge set $E(P_n) = \{x_i x_{i+1} : i = 1, 2, ..., n\}$. Then, the m-th power graph of P_n, P_n^m, is the graph with the the same vertex set of P_n and the edge set $\{x_i x_j : 1 \leq |i - j| \leq m\}$.

In this section, we determine the locating chromatic number of the m-th power of the path P_n, P_n^m, where $m \leq n - 1$.

To clarify the proof of the next theorem, we give the following example.

Example 1. *Let P_9 be the path of length 9 with vertex set $V(P_9) = \{x_1, x_2, ..., x_9\}$ and edge set $E(P_9) = \{x_i x_{i+1} : i = 1, 2, ..., 8\}$. Then, the induced subgraph of P_9^3 by the vertices x_1, x_2, x_3 and x_4 form a clique. Thus, $\chi(P_9^3) \geq 4$. Now, define the function $k : V(P_9^3) \longrightarrow \{1, 2, 3, 4\}$ as follows:*

$$k(x_i) = \begin{cases} 1, & \text{if } i = 1, 5, 9; \\ 2, & \text{if } i = 2, 6; \\ 3, & \text{if } i = 3, 7; \\ 4, & \text{if } i = 4, 8. \end{cases}$$

Clearly, k is a coloring of P_9^3, and hence $\chi(P_9^3) = 4$. Since $\chi(P_9^3) \leq \chi_L(P_9^3)$, we have $\chi_L(P_9^3) \geq 4$. If $\chi_L(P_9^3) = 4$, then x_1 and x_5 share the same color in P_9^3 since they are both adjacent to the vertices x_2, x_3, and x_4 that have different colors. Therefore, x_1 and x_5 have the same coding color, a contradiction. Thus, $\chi_L(P_9^3) \geq 5$. Now, define the coloring function $c : V(P_9^3) \longrightarrow \{1, 2, 3, 4, 5\}$ by

$$c(x_i) = \begin{cases} 1, & \text{if } i = 1; \\ 2, & \text{if } i = 2, 6; \\ 3, & \text{if } i = 3, 7; \\ 4, & \text{if } i = 4, 8; \\ 5, & \text{if } i = 5, 9. \end{cases}$$

Then, $\pi = \{R_1 = \{x_1\}, R_2 = \{x_2, x_6\}, R_3 = \{x_3, x_7\}, R_4 = \{x_4, x_8\}, R_5 = \{x_5, x_9\}\}$ is the partition of $V(P_9^3)$ with respect to c. Since the color code of any vertex x_i with respect to the partition π is $c_\pi(x_i) = (d(x_i, R_1), d(x_i, R_2), ..., d(x_i, R_5))$, we get, $c_\pi(x_1) = (0, 1, 1, 1, 2)$, $c_\pi(x_2) = (1, 0, 1, 1, 1)$, $c_\pi(x_3) = (1, 1, 0, 1, 1)$, $c_\pi(x_4) = (1, 1, 1, 0, 1)$, $c_\pi(x_5) = (2, 1, 1, 1, 0)$, $c_\pi(x_6) = (2, 0, 1, 1, 1)$, $c_\pi(x_7) = (2, 1, 0, 1, 1)$, $c_\pi(x_8) = (3, 1, 1, 0, 1)$, and $c_\pi(x_9) = (3, 1, 1, 1, 0)$. Thus, $\chi_L(P_9^3) = 5$.

Theorem 1. *Let P_n be the path of order n and P_n^m be the m-th power of P_n. Then,*

$$\chi_L(P_n^m) = \begin{cases} n, & \text{if } m = n - 1; \\ m + 2, & \text{if } m < n - 1. \end{cases}$$

Proof. Clearly, when $n = m+1$, then P_n^m is a complete graph of order n, and thus $\chi(P_n^m) = n$. But $\chi(P_n^m) \leq \chi_L(P_n^m) \leq n$, so $\chi_L(P_n^m) = n$. Now, let $n \geq m+2$ and $x_1, x_2, ..., x_n$ be the vertices of P_n such that $x_i x_{i+1} \in E(P_n)$ for all $i = 1, 2, ..., n-1$. Then, the vertices $x_1, x_2, ..., x_{m+1}$ induce a clique in the graph P_n^m and thus each of these vertices should have a different color. Now, if $\chi_L(P_n^m) = m+1$, then there exists a coloring function $c : V(P_n^m) \longrightarrow \{1, 2, ..., m+1\}$ such that $c_\pi(x_i) \neq c_\pi(x_j)$ when $i \neq j$. Since x_1 and x_{m+2} are both adjacent to the vertices $x_2, ..., x_m, x_{m+1}$, they must have the same color and hence they share the same color code, a contradiction. Thus, $\chi_L(P_n^m) \geq m+2$, whenever $n \geq m+2$. Now, define the coloring function $c : V(P_n^m) \longrightarrow \{1, 2, ..., m+2\}$ such that

$$c(x_i) = \begin{cases} 1, & \text{if } i = 1; \\ j, & \text{if } i \equiv j \bmod (m+1); \text{ where } j \in \{2, 3, ..., m+1, m+2\} \text{ and } i \neq 1. \end{cases}$$

Then, $\pi = \{R_1, R_2, \cdots, R_{m+2}\}$ is a partition of $V(P_n^m)$, where R_i is the set of vertices receiving color i. Note that, for $k \neq 1$, the induced subgraph with vertex set $\{x_{k+1}, x_{k+2}, ..., x_{k+m+1}\}$ is a clique colored by the $m+1$ distinct colors $2, 3, ..., m+2$. Henceforth,

$$d(x_i, R_j) = \begin{cases} 0, & \text{if } c(x_i) = c(x_j); \\ 1, & \text{if } c(x_i) \neq c(x_j) \text{ and } i, j \neq 1 \text{ or } i = 1 \text{ and } 2 \leq j \leq m+1. \end{cases}$$

Moreover,

$$d(x_i, R_1) = d(x_i, x_1) = \begin{cases} 0, & \text{if } i = 1; \\ k+1, & \text{if } 2 + km \leq i \leq 1 + (k+1)m. \end{cases}$$

Since the induced subgraph with vertex set $\{x_i : 2 + km \leq i \leq 1 + (k+1)m\}$ form a clique, we have $d(x_i, R_1) \neq d(x_j, R_1)$ when $c(x_i) = c(x_j)$. Therefore, $c_\pi(x_i) \neq c_\pi(x_j)$ when $i \neq j$. Thus, $\chi_L(P_n^m) = m+2$ whenever $n \geq m+2$. □

3. Locating Chromatic Number of Powers of Cycles

Let C_n be the cycle of order n with the vertex set $\{x_1, x_2, ..., x_n\}$ and edge set $\{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\}$. For positive integers n and m, we denote by C_n^m the graph with the same vertex set of C_n and edge set $\{x_i x_j : i - j \equiv \pm k (\bmod n), 1 \leq k \leq m\}$. The graph C_n^m is the m-th power of the n-cycle C_n. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, the open neighborhood of v, denoted by $N(v)$, is defined by $N(v) = \{u \in V(G) \mid uv \in E(G)\}$.

In this section, we give an upper and a lower bound for the locating chromatic number of the m-th power of the cycle C_n, and we prove that these bounds are sharp. It should be mentioned that the power of cycle graph is highly symmetric and so we can start coloring from any vertex and this is simplify the coloring process through our work.

We start with the following lemma that helps us in our study.

Lemma 1 ([3]). *Let c be a locating-coloring in a connected graph G. If u and v are distinct vertices of G such that $d(u, w) = d(v, w)$ for all $w \in v(G) \setminus \{u, v\}$, then $c(u) \neq c(v)$. In particular, if u and v are non-adjacent vertices of G such that $N(u) = N(v)$, then $c(u) \neq c(v)$.*

Theorem 2. *Let C_n be a cycle of order n. Then, $\chi_L(C_n^m) = n$ for all $n \leq 2m+2$.*

Proof. Since C_n^m is a complete graph for any $n \leq 2m+1$, we have $\chi_L(C_n^m) = n$. If $n = 2m+2$, then $V(C_n^m) = \{x_1, x_2, ..., x_n\}$ and $E(C_n^m) = \{x_i x_j : i - j \equiv \pm k (\bmod n), 1 \leq k \leq m\}$. Clearly, $x_1 x_i \in E(C_n^m), i \neq m+2$ and $d(x_i, x_1) = d(x_i, x_{m+2})$ for all $i \neq 1, m+2$. Using Lemma 1, we get $c(x_1) \neq c(x_i)$ for all $i \geq 2$. Similarly, $x_2, x_3, ..., x_{2m+2}$ have different colors, so $\chi_L(C_n^m) = n$. □

Now, we give an upper bound for $\chi_L(C_n^m)$.

Theorem 3. *Let C_n be a cycle of order $n \geq 2m+3$. Then, $\chi_L(C_n^m) \geq m+3$.*

Proof. Clearly, $\chi_L(C_n^m) > m+1$. Now, assume that $\chi_L(C_n^m) = m+2$. Then, there exists $c : V(C_n) \longrightarrow \{1, 2, ..., m+2\}$ such that $c_\pi(x_i) = c_\pi(x_j)$ if and only if $x_i = x_j$. Let $\pi = \{R_1, R_2, \cdots, R_{m+2}\}$ be the partition of $V(C_n)$, where $c(x_i) = j$ for all $x_i \in R_j$ and let $|R_1| \leq |R_2| \leq ... \leq |R_{m+2}|$. Let $V(C_n^m) = \{x_1, x_2, ..., x_n\}$ and $E(C_n^m) = \{x_i x_j : i - j \equiv \pm k(\bmod n), 1 \leq k \leq m\}$. Then, $|R_1| \neq 1$, otherwise $R_1 = \{x_a\}$ and hence there exist $u \in \{x_{a-m}, x_{a-m+1}, ..., x_{a-2}, x_{a-1}\}$ and $v \in \{x_{a+1}, x_{a+2}\}$ that have the same color. Since $\{x_{a-m-1}, x_{a-m}, ..., x_{a-2}, x_{a-1}\}$ and $\{x_{a+1}, x_{a+2}, ..., x_{a+m}\}$ are subsets of $N_{C_n^m}(x_a)$ and each one of them induce a complete subgraph of C_n^m, we have $c_\pi(u) = c_\pi(v)$, a contradiction. Thus, we have two cases:

Case 1: $2 \leq |R_1| < |R_{m+2}|$.

Then, there exist $x_s, x_t \in R_1$ where $s < t$ such that $x_i \notin R_1$, for all $s < i < t$ and the number of vertices between x_s and x_t is greater than $m+1$. Thus, there exists $u \in \{x_{s+1}, x_{s+2}, ..., x_{s+m+2}\}$ such that $c(u) = c(x_{t-1})$. Thus, $c_\pi(u) = c_\pi(x_{t-1})$, a contradiction.

Case 2: $2 \leq |R_1| = |R_2| = ... = |R_{m+2}|$.

Assume that $x_s, x_t \in R_i$ such that $s < t$, $x_j \notin R_i$ for all $s < j < t$ and the number of vertices between x_s and x_t greater than $m+1$, then as in Case 1 we have a contradiction. Now, let the number of vertices between x_s and x_t in R_i is $m+1$ for all i. Then, c is not a locating coloring. □

In the following lemma, we will show that $m+3$ is a sharp upper bound for $\chi_L(C_n^m)$.

Lemma 2. *Suppose that C_n is a cycle of order $n \geq 2m+3$ and $n = q(m+1)$ or $q(m+1)+1$ where q is a positive integer. Then, $\chi_L(C_n^m) = m+3$.*

Proof. Let $V(C_n^m) = \{x_1, x_2, ..., x_n\}$ and $E(C_n^m) = \{x_i x_j : i - j \equiv \pm k(\bmod n), 1 \leq k \leq m\}$. Define

$$R_1 = \{x_1\}, \quad R_2 = \{x_2, x_{2+(m+1)}, x_{2+2(m+1)}, ..., x_{q(m+1)-(m-1)}\},$$

$$R_3 = \{x_3, x_{3+(m+1)}, x_{3+2(m+1)}, ..., x_{q(m+1)-(m-2)}\}, ...,$$

$$R_m = \{x_m, x_{m+(m+1)}, x_{m+2(m+1)}, ..., x_{q(m+1)-1}\}, \quad R_{m+1} = \{x_{m+1}\},$$

$$R_{m+2} = \{x_{1+(m+1)}, x_{1+2(m+1)}, ..., x_{q(m+1)-m}\} \text{ when } n = q(m+1),$$

$$R_{m+2} = \{x_{1+(m+1)}, x_{1+2(m+1)}, ..., x_{q(m+1)+1}\} \text{ when } n = q(m+1)+1,$$

$$R_{m+3} = \{x_{2(m+1)}, x_{3(m+1)}, ..., x_{q(m+1)}\}.$$

Then, $d(u, x_1) \neq d(v, x_1)$ whenever $\{u, v\} \subseteq R_i \cap \{x_2, x_3, ..., x_{\lfloor \frac{n}{2} \rfloor}\}$, or $\{u, v\} \subseteq R_i \cap \{x_{\lceil \frac{n}{2} \rceil}, x_{\lceil \frac{n}{2} \rceil+1}, ..., x_n\}$. In addition, $d(u, x_1) \neq d(v, x_1)$ or $d(u, x_{m+1}) \neq d(v, x_{m+1})$ for any $u \in R_i \cap \{x_2, x_3, ..., x_{\lfloor \frac{n}{2} \rfloor}\}$, $v \in R_i \cap \{x_{\lceil \frac{n}{2} \rceil}, x_{\lceil \frac{n}{2} \rceil+1}, x_{\lceil \frac{n}{2} \rceil+2}, ..., x_n\}$. Now, set $\pi = \{R_i : i = 1, ..., m+3\}$ and define $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+3\}$ by $c(x_i) = j$ for any $x_i \in R_j$. Then, for any $u, v \in V(C_n^m)$, $c_\pi(u) \neq c_\pi(v)$. □

Now, we give exact values of the locating chromatic number of certain powers of cycles (for $m = 2$ and $m = 3$, when $n \equiv 0, 1$ or $2 \bmod 4$).

Lemma 3.

(i) *If $n \geq 7$, then $\chi_L(C_n^2) = 5$.*

(ii) *If $n \geq 9$, then $\chi_L(C_n^3) = 6$ when $n \in \{4q, 4q+1, 4q+2\}$, and $6 \leq \chi_L(C_n^3) \leq 7$ when $n = 4q$.*

Proof.

(i) In view of Theorem 3 and Lemma 2, it is enough to show that $\chi_L(C_{3q+2}^2) \leq 5$. Assume that $n = 3q+2$, then $\pi = \{R_1 = \{x_1\}, R_2 = \{x_5, x_8, ..., x_{3q+2}\}, R_3 = \{x_3\}, R_4 = \{x_4, x_7, ..., x_{3q+1}\},$

$R_5 = \{x_2, x_6, x_9, ..., x_{3q}\}\}$ is a partition of $V(C_n^2)$. Now, define $c : V(C_n^2) \longrightarrow \{1,2,3,4,5\}$ by $c(x_i) = j$ for any $x_i \in R_j$. Then, it is easy to show that $c_\pi(u) \neq c_\pi(v)$ for any $u, v \in V(C_n^2)$.

(ii) Let $\pi_1 = \{R_1 = \{x_1\}, R_2 = \{x_6, x_{10}, ..., x_{4q+2}\}, R_3 = \{x_3, x_7, ..., x_{4q-1}\}, R_4 = \{x_4\}, R_5 = \{x_5, x_9, ..., x_{4q+1}\}, R_6 = \{x_2, x_8, x_{12} ..., x_{4q}\}\}$, and $\pi_2 = \pi_1 \cup R_7$, $R_7 = \{x_{4q+3}\}$. Then, π_k is a partition of $V(C^3_{4q+1+k})$ for $k = 1,2$. Now, let $c : V(C^3_{4q+1+k}) \longrightarrow \{1,2,...,5+k\}$ defined by $c(x_i) = j$ for any $x_i \in R_j$. Clearly, for any $u, v \in V(C^3_{4q+1+k})$, $c_{\pi_k}(u) \neq c_{\pi_k}(v)$ for $k = 1,2$.

□

In the following lemma, upper and lower bounds for some, $\chi_L(C_n^m)$, of a certain n are given.

Lemma 4.

(i) *Let* $m = 2t \geq 4$ *and* $n = q(m+1) + t, q \geq 2$. *Then,* $m + 3 \leq \chi_L(C_n^m) \leq m + t + 1$.

(ii) *Let* $m = 2t - 1 \geq 5$ *and* $n = q(m+1) + (t-1), q \geq 2$. *Then,* $m + 3 \leq \chi_L(C_n^m) \leq m + t$.

Proof.

(i) Assume that $m = 2t$ and $n = q(m+1) + t, q \geq 2$. Notice that the length of the path $x_{q(m+1)-t} - x_{q(m+1)-(t-1)} - \cdots - x_{q(m+1)+t} - x_1 - ... - x_t$ is $m + t$ and the length of the path $x_{q(m+1)} - x_{q(m+1)+1} - ... - x_{q(m+1)+t} - x_1 - ... - x_{t+1}$ is $m + 1$, while the length of the path $x_{q(m+1)-(t-i+1)} - x_{q(m+1)-(t-i)} - ... - x_{q(m+1)+t} - x_1 - ... - x_{t+i}$ is $m + t + 1$ for $2 \leq i \leq t$ Thus, $d(x_t, x_{q(m+1)-t}) = d(x_{t+1}, x_{q(m+1)}) = d(x_{t+i}, x_{q(m+1)-(t-i+1)}) = 2$. Now, let $R_1 = \{x_1\}, R_2 = \{x_2\}, ..., R_{t-1} = \{x_{t-1}\}, R_t = \{x_t, x_{(t+1)+(m+1)}, x_{(t+1)+2(m+1)}, ..., x_{q(m+1)-t}\}$, $R_{t+1} = \{x_{t+1}, x_{2(m+1)}, x_{3(m+1)}, ..., x_{q(m+1)}\}$, $R_{t+i} = \{x_{t+i}, x_{(t+i)+(m+1)}, x_{(t+i)+2(m+1)}, ..., x_{q(m+1)-(t-i+1)}\}$, $2 \leq i \leq t$, $R_{m+1} = \{x_{m+1}\}$, $R_{m+i} = \{x_{m+i}, x_{(m+i)+(m+1)}, x_{(m+i)+2(m+1)}, ..., x_{q(m+1)+(i-1)}\}$, $2 \leq i \leq t+1$. Then, $\pi = \{R_i : i = 1,2,...,m+t+1\}$ is a partition of $V(C_n^m)$ and $c : V(C_n^m) \longrightarrow \{1,2,...,m+t+1\}$ defined by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m. By using Theorem 3, we obtain $m + 3 \leq \chi_L(C_n^m) \leq m + t + 1$.

(ii) Assume that $m = 2t - 1$ and $n = q(m+1) + (t-1), q \geq 2$. Then, $d(x_{t-1}, x_{q(m+1)-t}) = d(x_t, x_{q(m+1)-(t-1)}) = d(x_{t+1}, x_{q(m+1)}) = d(x_{t+i}, x_{q(m+1)-(t-i)}) = 2$ for all $2 \leq i \leq t-1$. Set $R_1 = \{x_1\}, R_2 = \{x_2\}, ..., R_{t-2} = \{x_{t-2}\}, R_{t-1} = \{x_{t-1}, x_{t+(m+1)}, x_{t+2(m+1)}, ..., x_{q(m+1)-t}\}$, $R_t = \{x_t, x_{(t+1)+(m+1)}, x_{(t+1)+2(m+1)}, ..., x_{q(m+1)-(t-1)}\}$, $R_{t+1} = \{x_{t+1}, x_{2(m+1)}, x_{3(m+1)}, ..., x_{q(m+1)}\}$, $R_{t+i} = \{x_{t+i}, x_{(t+i)+(m+1)}, x_{(t+i)+2(m+1)}, ..., x_{q(m+1)-(t-i)}\}$, $2 \leq i \leq t-1$, $R_{m+1} = \{x_{m+1}\}$, $R_{m+i} = \{x_{m+i}, x_{(m+i)+(m+1)}, x_{(m+i)+2(m+1)}, ..., x_{q(m+1)+(i-1)}\}$, $2 \leq i \leq t$. Then, $\pi = \{R_i : i = 1, ..., m+t\}$ is a partition of $V(C_n^m)$ and $c : V(C_n^m) \longrightarrow \{1,2,...,m+t\}$ defined by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

□

In the following two lemmas, we give an upper bound for $\chi_L(C_n^m)$ whenever $m \geq 4$.

Lemma 5. *Let* $m = 2t \geq 4$ *and* $n \geq 2m + 3$. *Then,*

$$\chi_L(C_n^m) \leq \begin{cases} m+4, \text{ if } n \equiv 2 \bmod (m+1); \\ m+5, \text{ if } n \equiv 3 \bmod (m+1); \\ \vdots \\ m+t+1, \text{ if } n \equiv t-1, t \bmod (m+1) \\ m+t+2, \text{ if } n \equiv t+i \bmod (m+1) \text{ for } 1 \leq i \leq t. \end{cases}$$

Proof.

(1) For $n = q(m+1) + 2$, let $R_1 = \{x_1\}, R_2 = \{x_2, x_{2+(m+1)}, ..., x_{q(m+1)-(m-1)}\}$, $R_3 = \{x_3, x_{3+(m+1)}, ..., x_{q(m+1)-(m-2)}\}, ...,$ $R_m = \{x_m, x_{m+(m+1)}, ..., x_{q(m+1)-1}\}$, $R_{m+1} = \{x_{m+1}\}$, $R_{m+2} = \{x_{1+(m+1)}, x_{1+2(m+1)}, ..., x_{q(m+1)+1}\}$, $R_{m+3} = \{x_{2(m+1)}, x_{3(m+1)}, ..., x_{q(m+1)}\}$, $R_{m+4} =$

$\{x_{q(m+1)+2}\}$. Then, $\pi = \{R_i : i = 1, 2..., m+4\}$ is a partition of $V(C_n^m)$ and $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+t\}$ defined by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

(2) For $n = q(m+1)+i$, where $3 \leq i < t$ let $R_1, R_2, ..., R_{m+4}$ similar to the case $n = q(m+1)+2$. Set $R_{m+j} = \{x_{(m+1)q+(j-2)}\}, 5 \leq j \leq i+2$. Then, $\pi = \{R_i : i = 1, 2..., m+i+2\}$ is a partition of $V(C_n^m)$ and $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+i+2\}$ given by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

(3) By part (i) of Lemma 4, $\chi_L(C_n^m) \leq m+t+1$ when $n = q(m+1)+t, q \geq 2$.

(4) For $n = q(m+1)+(t+1)$, take $R_1, R_2, ..., R_{m+t+1}$ similar to the case $n = q(m+1)+(t-1)$ except R_{t+1} and R_{m+3}. Let $R_{t+1} = \{x_{t+1+(m+1)}, x_{t+1+2(m+1)}, ..., x_{t+1+q(m+1)}\}$, $R_{m+3} = \{x_{t+1}, x_{2(m+1)}, x_{3(m+1)}, ..., x_{q(m+1)}\}$, and $R_{m+t+2} = \{x_{t+q(m+1)}\}$. This implies that $\pi = \{R_i : i = 1, 2..., m+i+2\}$ is a partition of $V(C_n^m)$ and $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+t\}$ defined by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

(5) For $n = q(m+1)+(t+i)$ where $2 \leq i \leq t$, take $R_1, R_2, ..., R_{m+t+2}$ similar to the case $n = q(m+1)+(t+1)$ except R_{t+j} and R_{m+j+2}, where $2 \leq j \leq i$. Set $R_{t+j} = \{x_{(t+j)+(m+1)}, x_{(t+j)+2(m+1)}, ..., x_{(t+j)+q(m+1)}\}$, and $R_{m+j+2} = \{x_{q(m+1)+j}, x_{t+j}\}$. Note that $d(x_{q(m+1)+j}, x_{t+j}) = 2, d(x_{t+j}, x_{m+1}) = 1$ and $d(x_{q(m+1)+j}, x_{m+1}) = 2$. Then, it is easy to show that the function $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+t+2\}$ defined by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

□

Lemma 6. *Let $m = 2t-1 \geq 5$ and $n \geq 2m+3$. Then,*

$$\chi_L(C_n^m) \leq \begin{cases} m+4, \textit{ if } n \equiv 2 \textit{ mod } (m+1); \\ m+5, \textit{ if } n \equiv 3 \textit{ mod } (m+1); \\ \vdots \\ m+t, \textit{ if } n \equiv (t-2) \textit{ or } (t-1) \textit{ mod } (m+1); \\ m+t+1, \textit{ if } n \equiv (t+i) \textit{ mod } (m+1) \textit{ for } 0 \leq i \leq m-1; \\ m+t+2, \textit{ if } n \equiv m \textit{ mod } (m+1). \end{cases}$$

Proof.

(1) If $n = q(m+1)+2$, define $R_1 = \{x_1\}, R_2 = \{x_2, x_{2+(m+1)}, x_{2+2(m+1)}, ..., x_{q(m+1)-(m-1)}\}$, $R_3 = \{x_3, x_{3+(m+1)}, ..., x_{q(m+1)-(m-2)}\}, ..., R_m = \{x_m, x_{m+(m+1)}, ..., x_{q(m+1)-1}\}, R_{m+1} = \{x_{m+1}\}$, $R_{m+2} = \{x_{1+(m+1)}, x_{1+2(m+1)}, ..., x_{1+q((m+1)}\}$, $R_{m+3} = \{x_{2(m+1)}, x_{3(m+1)}, ..., x_{q(m+1)}\}$, $R_{m+4} = \{x_{q(m+1)+2}\}$. Clearly, $\pi = \{R_i : i = 1, 2, ..., m+4\}$ is a partition of $V(C_n^{m+1})$ and the function $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+4\}$ given by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

(2) If $n = q(m+1)+i$, where $3 \leq i \leq t-2$, take $R_1, R_2, ..., R_{m+4}$ similar to the case $n = q(m+1)+2$. Set $R_{m+j} = \{x_{(j-2)+q(m+1)}\}, 5 \leq j \leq i+2$. Then, $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+i+2\}$ defined by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

(3) From part (ii) of Lemma 4, we conclude that $\chi_L(C_n^m) \leq m+t$ when $n = q(m+1)+(t-1), q \geq 2$.

(4) If $n = q(m+1)+t$, take $R_1, R_2, ..., R_{m+t}$ similar to the case $n = q(m+1)+(t-2)$ except R_t and R_{m+3}. Set $R_t = \{x_{t+(m+1)}, x_{t+2(m+1)}, ..., x_{t+q(m+1)}\}$ and $R_{m+3} = \{x_t, x_{2(m+1)}, ..., x_{q(m+1)}\}$. Then, $\pi = \{R_i : i = 1, 2, ..., m+t\} \cup R_{m+t+1}$, where $R_{m+t+1} = \{x_{(t-1)+q(m+1)}\}$ is a partition of $V(C_n^m)$. Notice that $d(t, q(m+1)) = 2, d(t, m+1) = 1$ and $d(q(m+1), m+1) = 2$. Thus, the function $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+t+1\}$ given by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

(5) If $n = q(m+1)+(t+i)$, where $1 \leq i \leq t-2$, define $R_1, R_2, ..., R_{m+t+1}$ similar to the case $n = q(m+1)+t$ except R_{t+j} and R_{m+3+j}, where $1 \leq j \leq i$. Set $R_{t+j} = \{x_{(t+j)+(m+1)}, x_{(t+j)+2(m+1)}, ..., x_{q(m+1)+(t+j)}\}$ and $R_{m+j+3} = \{x_{t+j}, x_{(t+j)+q(m+1)}\}$. Then, it is easy to show that $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+t+1\}$ given by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

(6) If $n = q(m+1) + m$, take $R_1, R_2, ..., R_{m+t+1}$ similar to the case $n = q(m+1) + (m-1)$. Then, $\pi = \{R_i : i = 1, 2, ..., m+t+1\} \cup R_{m+t+2}$, where $R_{m+t+2} = \{x_{q(m+1)+m}\}$ is a partition of $V(C_n^{m+1})$ and the function $c : V(C_n^m) \longrightarrow \{1, 2, ..., m+t+2\}$ given by $c(x_i) = j$ for any $x_i \in R_j$ is a locating coloring of C_n^m.

□

As a consequence of Lemmas 3, 4, 5 and 6, we have the following.

Theorem 4. *If $m = 2t \geq 4$ or $m = 2t-1 \geq 5$, then $m+3 \leq \chi_L(C_n^m) \leq m+t+2$.*

In view of Lemma 2, the lower bound of the above inequality is sharp. Next, we will show that the upper bound is also sharp.

Theorem 5. *[11] Let n and m be positive integers such that $n \geq 2m$. If $n = q(m+1) + r$, $q > 0$ and $0 \leq r \leq m$, then $\chi(C_n^m) = m+1+\lceil \frac{r}{q} \rceil$.*

The following two theorems give the exact values of $\chi_L(C_n^m)$, which illustrate the sharpness of the upper bounds in Theorem 4.

Theorem 6. *If $m = 2t \geq 4$, and $n = 2(m+1) + m$, then $\chi_L(C_n^m) = m+t+2$.*

Proof. By Theorem 5, $\chi(C_n^m) = m+t+1$. Hence $\chi_L(C_n^m) \geq m+t+1$. Suppose that $\chi_L(C_n^m) = m+t+1$ and c is a locating coloring of C_n^m and $\pi = \{R_i : i = 1, ..., m+t+1\}$ is the partition of $V(C_n^m)$ into color classes resulting from c. Then, $|R_i| \leq 2$ for all i, otherwise there exists j such that $|R_j| \geq 3$ and hence R_j has two adjacent vertices. Thus, $|R_i| = 2$ for all i. Now, let $V(C_n^m) = \{x_1, x_2, ..., x_n\}$ and $x_i \in R_i$ for $i = 1, 2, ..., m+1$. Set $T = \{x_{m+2}, x_{m+3}, ..., x_{2m+1}\}$ and $S = \{x_{2m+3}, x_{2m+4}, ..., x_n\}$. Clearly, $T \cap R_i \neq \phi$ and $S \cap R_i \neq \phi$ for $m+2 \leq i \leq m+t+1$. Since $|R_{m+1}| = 2$ and $T \subseteq \cup_{i \neq m+1} R_i$, there exists $x_j \in R_{m+1}$ for some $j \in \{2m+2\} \cup S$. However, $N(x_j) \cap R_i \neq \phi$ for any $i \neq m+1$. Thus, $d(x_j, x_i) = 1$ for all $i \neq m+1$ and hence $c_\pi(x_j) = c_\pi(x_{m+1})$, a contradiction. By using Theorem 6, we have $\chi_L(C_n^m) = m+t+2$. □

Theorem 7. *If $m = 2t-1 \geq 3$, and $n = 2(m+1) + m$, then $\chi_L(C_n^m) = m+t+2$.*

Proof. From Theorem 5, $\chi(C_n^m) = m+t+1$. Hence, $\chi_L(C_n^m) \geq m+t+1$. Assume that $\chi_L(C_n^m) = m+t+1$. Let c be a locating coloring of C_n^m and $\pi = \{R_i : i = 1, ..., m+t+1\}$ be the partition of $V(C_n^m)$ into color classes resulting from c such that $|R_1| \leq |R_2| \leq ... \leq |R_{m+t+1}|$. Since $|R_i| \leq 2$ for all i, we have $|R_1| = 1$ and $|R_i| = 2, i \geq 2$. Let $V(C_n^m) = \{x_1, x_2, ..., x_n\}$ and $x_i \in R_i$ for $i = 1, 2, ..., m+2$. Set $T = \{x_{m+3}, x_{m+4}, ..., x_{2m+1}\}$, $S = \{x_{2m+3}, x_{2m+4}, ..., x_n\}$ and $S_k = S \cup \{x_{2m+2}, x_{2m+1}, ..., x_{2m-k+3}\}$, where $1 \leq k \leq t$. Then, $T \cap R_i \neq \phi$ for all $i \geq m+3$ and $S \cap R_i \neq \phi$ for all $i \geq m+2$, while $S_k \cap R_i \neq \phi$ for all $i \geq m-k+2$. Now, note that $N(x_{m+1}) = T \cup (\{x_1, x_2, ..., x_{m+2}\} \backslash \{x_{m+1}\})$, $N(x_{m-k}) = (T \backslash \{x_{2m+1}, x_{2m}, ..., x_{(2m+1)-k}\}) \cup (\{x_{n-k}, x_{n-k+1}, ..., x_n, x_1, x_2, ..., x_{m+2}\} \backslash \{x_{m-k}\})$ for all $0 \leq k \leq m-2$, $N(x_{n-k}) = (S_{k+1} \backslash \{x_{n-k}\}) \cup \{x_1, x_2, ..., x_{m-k}\}$ for all $0 \leq k \leq t$.

Since $S \cap R_i \neq \phi, i \geq m+2$, there exist t vertices of S belong to $\cup_{i=m+2}^{m+t+1} R_i$ and all other vertices of S belong to $\cup_{i=2}^{m+1} R_i$. Thus, we have the following cases:

(1) If $x_n \in R_{m+1}$, then $c_\pi(x_n) = c_\pi(x_{m+1})$.

(2) If $x_n \in R_l, l \geq m+2$ and $x_{n-1} \in R_k$, where $k = m$ or $m+1$. Then, $(T \backslash \{x_{2m+1}\}) \cap R_i \neq \phi$ for all $i \geq m+3$ and $i \neq l$. However, $k = m$, which gives $N(x_k) = (\{x_n, x_1, x_2, ..., x_{m+2}\} \backslash \{x_m\}) \cup (T \backslash \{x_{2m+1}\})$ and $k = m+1$, which gives $N(x_k) = (\{x_1, x_2, ..., x_{m+2}\} \backslash \{x_{m+1}\}) \cup T$, while $N(x_{n-1}) = \{x_1, x_2, ..., x_{m-1}\} \cup (S_2 \backslash \{x_{n-1}\})$. Thus, $N(x_j) \cap R_i \neq \phi$ whenever $j = k$, or $n-1$ and $i \neq k$. Thus, $c_\pi(x_{n-1}) = c_\pi(x_k)$.

(3) If $x_n, x_{n-1}, ..., x_{n-r+1} \in \cup_{j=1}^{r} R_{l_j}$, where $2 \leq r \leq t-1$, $l_j \geq m+2$ and $x_{n-r} \in R_k$, where $m-r+1 \leq k \leq m+1$. Then, $(T \backslash \{x_{2m+1}, x_{2m}, ..., x_{2m-r+2}\}) \cap R_i \neq \phi$ for all $i \geq m+3$ and $i \notin \cup_{j=1}^{r} l_j$. Thus, $N(x_j) \cap R_i \neq \phi$ whenever $j = k$, or $n-r$ and $i \neq k$ Then, $c_\pi(x_{n-r}) = c_\pi(x_k)$.

Therefore, $\chi_L(C_n^m) \geq m+t+2$. By Theorem 6, we get $\chi_L(C_n^m) = m+t+2$. □

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