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Connection Problem for Sums of Finite Products of Legendre and Laguerre Polynomials

Taekyun Kim [1], Kyung-Won Hwang [2,*], Dae San Kim [3] and Dmitry V. Dolgy [4]

1 Department of Mathematics, Kwangwoon University, Seoul 01897, Korea; tkkim@kw.ac.kr
2 Department of Mathematics, Dong-A University, Busan 49315, Korea
3 Department of Mathematics, Sogang University, Seoul 04107, Korea; dskim@sogang.ac.kr
4 Institute of National Science, Far Eastern Federal University, Vladivostok 690950, Russia; dvdolgy@gmail.com
* Correspondence: khwang@dau.ac.kr

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Abstract: The purpose of this paper is to represent sums of finite products of Legendre and Laguerre polynomials in terms of several orthogonal polynomials. Indeed, by explicit computations we express each of them as linear combinations of Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials, some of which involve terminating hypergeometric functions ${}_1F_1$ and ${}_2F_1$.

Keywords: Legendre polynomials; Laguerre polynomials; generalized Laguerre polynomials; Gegenbauer polynomials; hypergeometric functions ${}_1F_1$ and ${}_2F_1$

1. Preliminaries

Here, after fixing some notations that will be needed throughout this paper, we will review briefly some basic facts about orthogonal polynomials relevant to our discussion. As general references on orthogonal polynomials, we recommend the reader to refer to [1,2].

As is well known, the falling factorial sequence $(x)_n$ and the rising factorial sequence $\langle x\rangle_n$ are respectively defined by

$$(x)_n = x(x-1)\dots(x-n+1), \quad (n\geq 1), (x)_0 = 1, \tag{1}$$

$$\langle x\rangle_n = x(x+1)\dots(x+n-1), \quad (n\geq 1), \langle x\rangle_0 = 1. \tag{2}$$

The two factorial sequences are related by

$$(-1)^n(x)_n = \langle -x\rangle_n, \quad (-1)^n\langle x\rangle_n = (-x)_n. \tag{3}$$

$$\frac{(2n-2s)!}{(n-s)!} = \frac{2^{2n-2s}(-1)^s\left\langle \frac{1}{2}\right\rangle_n}{\left\langle \frac{1}{2}-n\right\rangle_s}, \quad (n\geq s\geq 0). \tag{4}$$

$$\frac{(2n+2s)!}{(n+s)!} = 2^{2n+2s}\left\langle \frac{1}{2}\right\rangle_n \left\langle n+\frac{1}{2}\right\rangle_s, \quad (n,s\geq 0). \tag{5}$$

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}, \quad (n\geq 0), \tag{6}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n, \quad \frac{\Gamma(x+n)}{\Gamma(x)} = \langle x\rangle_n, \quad (n\geq 0), \tag{7}$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad (\mathrm{Re}\,x, \mathrm{Re}\,y > 0), \tag{8}$$

where $\Gamma(x)$ and $B(x,y)$ denote respectively the gamma and beta functions.

The hypergeometric function is defined by

$${}_pF_q = (a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{\langle a_1 \rangle_n \ldots \langle a_p \rangle_n}{\langle b_1 \rangle_n \ldots \langle b_q \rangle_n} \frac{x^n}{n!}. \tag{9}$$

Now, we are ready to recall some relevant facts about Legendre polynomials $P_n(x)$, Laguerre polynomials $L_n(x)$, Hermite polynomials $H_n(x)$, generalized (extended) Laguerre polynomials $L_n^{\alpha}(x)$, Gegenbauer polynomials $C_n^{(\lambda)}(x)$, and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. All the facts stated here can also be found in [3–8].Interested readers may refer to [1,2,9–13] for full accounts of orthogonal polynomials and also to [14,15] for papers discussing relevant orthogonal polynomials.

The above-mentioned orthogonal polynomials are given, in terms of generating functions, by

$$F(t,x) = (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{10}$$

$$G(t,x) = (1-t)^{-1} exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n, \tag{11}$$

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{12}$$

$$(1-t)^{-\alpha-1} exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n, \tag{13}$$

$$\frac{1}{(1-2xt+t^2)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n, \quad (\lambda > -\frac{1}{2}, \lambda \neq 0, |t| < 1, |x| \leq 1), \tag{14}$$

$$\begin{aligned} &\frac{\alpha+\beta}{R(1-t++R)^{\alpha}(1+t+R)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n, \\ &(R = \sqrt{1-2xt+t^2}, \alpha, \beta > -1). \end{aligned} \tag{15}$$

In terms of explicit expressions, those orthogonal polynomials are given explicitly as follows:

$$\begin{aligned} P_n(x) &= {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) \\ &= \frac{1}{2^n} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n}{l} \binom{2n-2l}{n} x^{n-2l}, \end{aligned} \tag{16}$$

$$\begin{aligned} L_n(x) &= {}_1F_1(-n; 1; x) \\ &= \sum_{l=0}^{n} (-1)^{n-l} \binom{n}{l} \frac{1}{(n-l)!} x^{n-l}, \end{aligned} \tag{17}$$

$$H_n(x) = n! \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l}{l!(n-2l)!} (2x)^{n-2l}, \tag{18}$$

$$\begin{aligned} L_n^{\alpha}(x) &= \frac{\langle \alpha+1 \rangle_n}{n!} {}_1F_1(-n; \alpha+1; x) \\ &= \sum_{l=0}^{n} \frac{(-1)^l \binom{n+\alpha}{n-l}}{l!} x^l, \end{aligned} \tag{19}$$

$$\begin{aligned} C_n^{\lambda}(x) &= \binom{n+2\lambda-1}{n} {}_2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\right) \\ &= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)k!(n-2k)!}(2x)^{n-2k}, \end{aligned} \tag{20}$$

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{\langle \alpha+1 \rangle_n}{n!} {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right) \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}. \end{aligned} \tag{21}$$

For Legendre, Gegenbauer and Jacobi polynomials, we have Rodrigues' formulas, and for Hermite and generalized Laguerre polynomials, we have Rodrigues-type formulas.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \tag{22}$$

$$L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}), \tag{23}$$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n, \tag{24}$$

$$(1-x^2)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) = \frac{(-2)^n}{n!} \frac{\langle \lambda \rangle_n}{\langle n+2\lambda \rangle_n} \frac{d^n}{dx^n}(1-x^2)^{n+\lambda-\frac{1}{2}}, \tag{25}$$

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}(1-x)^{n+\alpha}(1+x)^{n+\beta}. \tag{26}$$

The orthogonal polynomials in Equations (22)–(26) satisfy the following orthogonality relations with respect to various weight functions.

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = 2^n n! \sqrt{\pi}\delta_{m,n}, \tag{27}$$

$$\int_0^{\infty} x^{\alpha} e^{-x} L_n^{\alpha}(x) L_m^{\alpha}(x)dx = \frac{1}{n!}\Gamma(\alpha+n+1)\delta_{m,n}, \tag{28}$$

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{m,n}, \tag{29}$$

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)dx = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}\delta_{m,n}, \tag{30}$$

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\delta_{m,n}. \tag{31}$$

2. Introduction

In this paper, we will consider two sums of finite products

$$\gamma_{n,r}(x) = \sum_{i_1+\cdots+i_{2r+1}=n} P_{i_1}(x)P_{i_2}(x)\ldots P_{i_{2r+1}}(x), \quad (n, r \geq 0), \tag{32}$$

in terms of Legendre polynomials and

$$\varepsilon_{n,r}(x) = \sum_{i_1+\cdots+i_{r+1}=n} L_{i_1}\left(\frac{x}{r+1}\right) L_{i_2}\left(\frac{x}{r+1}\right)\ldots L_{i_{r+1}}\left(\frac{x}{r+1}\right), \quad (n, r \geq 0), \tag{33}$$

in terms of Laguerre polynomials. We represent each of them as linear combinations of Hermite, extended Laguerre, Legendre, Gegenbauer, and Jacobi polynomials (see Theorems 1 and 2). It is amusing to note here that, for some of these expressions, the coefficients involve certain terminating hypergeometric functions ${}_2F_1$ and ${}_1F_1$. These representations are obtained by carrying out explicit computations with the help of Propositions 1 and 2. We observe here that the formulas in Proposition 1 can be derived from the orthogonalities in Equation (27)–(31), Rodrigues' and Rodrigues-type formulas in Equation (22)–(26), and integration by parts.

Our study of such representation problems can be justified by the following. Firstly, the present research can be viewed as a generalization of the classical connection problems. Indeed, the classical connection problems are concerned with determining the coefficients in the expansion of a product of two polynomials in terms of any given sequence of polynomials (see [1,2]).

Secondly, studying such kinds of sums of finite products of special polynomials can be well justified also by the following example. Let us put

$$\alpha_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x) B_{m-k}(x), \ (m \geq 2),$$

where $B_n(x)$ are the Bernoulli polynomials. Then we can express $\alpha_m(x)$ as linear combinations of Bernoulli polynomials, for example from the Fourier series expansion of the function closely related to that. Indeed, we can show that

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} B_{2k}(x) B_{2m-2k}(x) + \frac{2}{2m-1} B_1(x) B_{2m-1}(x) \\ =&\frac{1}{m} \sum_{k=1}^{m} \frac{1}{2k} \binom{2m}{2k} B_{2k} B_{2m-2k}(x) + \frac{1}{m} H_{2m-1} B_{2m}(x) + \frac{2}{2m-1} B_{2m-1} B_1(x), \end{aligned} \tag{34}$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers.

Further, some simple modification of this gives us the famous Faber-Pandharipande-Zagier identity and a slightly different variant of the Miki's identity by letting respectively $x = \frac{1}{2}$ and $x = 0$ in (34). We note here that all the other known derivations of F-P-Z and Miki's identity are quite involved, while our proof of Miki's and Faber-Pandharipande-Zagier identities follow from the polynomial identity (34), which in turn follows immediately the Fourier series expansion of $\alpha_m(x)$. Indeed, Miki makes use of a formula for the Fermat quotient $\frac{a^p - a}{p}$ modulo p^2, Shiratani-Yokoyama employs p-adic analysis, Gessel's proof is based on two different expressions for Stirling numbers of the second kind $S_2(n,k)$, and Dunne-Schubert exploits the asymptotic expansion of some special polynomials coming from the quantum field theory computations. For some details on these, we let the reader refer to the introduction in [16] and the papers therein.

The next two theorems are the main results of this paper.

Theorem 1. *For any nonnegative integers n and r, we have the following representation.*

$$\begin{aligned} &\sum_{i_1+i_2+\cdots+i_{2r+1}=n} P_{i_1}(x) P_{i_2}(x) \ldots P_{i_{2r+1}}(x) \\ &= \frac{2^r (n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \sum_{j=0}^{[\frac{n}{2}]} \frac{{}_1F_1(-j; \frac{1}{2}-n-r; -1)}{j!(n-2j)!} H_{n-2j}(x) \end{aligned} \tag{35}$$

$$\begin{aligned} &= \frac{1}{(2r-1)!!2^{n+r}} \sum_{k=0}^{n} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \\ &\times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l (2n+2r-2l)! \Gamma(n-2l+\alpha+1)}{l!(n+r-l)!(n-k-2l)!} L_k^{\alpha}(x) \end{aligned} \tag{36}$$

$$\begin{aligned}
&= \frac{2^{r-1}(n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \\
&\times \sum_{j=0}^{[\frac{n}{2}]} \frac{(2n+1-4j)\,{}_2F_1(-j, j-n-\frac{1}{2}; \frac{1}{2}-n-r; 1)}{j!(n-j+\frac{1}{2})_{n-j+1}} P_{n-2j}(x)
\end{aligned} \tag{37}$$

$$\begin{aligned}
&= \frac{2^{r}\Gamma(\lambda)(n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \\
&\times \sum_{j=0}^{[\frac{n}{2}]} \frac{(n+\lambda-2j)\,{}_2F_1(-j, j-n-r; \frac{1}{2}-n-r; 1)}{\Gamma(n+\lambda-j+1)j!} C_{n-2j}^{(\lambda)}(x)
\end{aligned} \tag{38}$$

$$\begin{aligned}
&= \frac{(-1)^n}{(2r-1)!!2^{n+r}} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha+\beta+1)(-2)^k}{\Gamma(2k+\alpha+\beta+1)} \\
&\times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l(2n+2r-2l)!}{l!(n+r-l)!(n-k-2l)!} \\
&\times {}_2F_1(2l+k-n, k+\beta+1; 2k+\alpha+\beta+2; 2) P_k^{(\alpha,\beta)}(x).
\end{aligned} \tag{39}$$

Here $(2r-1)!!$ is the double factorial given by

$$(2r-1)!! = (2r-1)(2r-3)\dots 1, \quad (r \geq 1), (-1)!! = 1. \tag{40}$$

Remark 1. *An alternative expression for (36) is given by*

$$\begin{aligned}
\gamma_{n,r}(x) &= \frac{1}{\Gamma(\alpha+1)(2r-1)!!2^{n+r}} \\
&\times \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l(2n+2r-2l)!\Gamma(n-2l+\alpha+1)}{l!(n+r-l)!(n-2l)!} \sum_{k=0}^{n-2l} \frac{\langle 2l-n\rangle_k}{\langle \alpha+1\rangle_k} L_k^{\alpha}(x).
\end{aligned} \tag{41}$$

Theorem 2. *For any nonnegative integers n and r, we have the following representation.*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} L_{i_1}\left(\frac{x}{r+1}\right) L_{i_2}\left(\frac{x}{r+1}\right) \dots L_{i_{r+1}}\left(\frac{x}{r+1}\right)$$

$$= (n+r)! \sum_{k=0}^{n} \frac{(-\frac{1}{2})^k}{k!} \sum_{j=0}^{[\frac{n-k}{2}]} \frac{(\frac{1}{4})^j}{j!(n-k-2j)!(r+k+2j)!} H_k(x) \tag{42}$$

$$= (n+r)! \sum_{k=0}^{n} \frac{{}_2F_1(k-n, k+\alpha+1; r+k+1; 1)}{(n-k)!(r+k)!} L_k^{\alpha}(x) \tag{43}$$

$$\begin{aligned}
&=(n+r)! \sum_{k=0}^{n} 2^{k+1}(2k+1) \\
&\times \sum_{j=0}^{[\frac{n-k}{2}]} \frac{(k+j+1)!}{j!(n-k-2j)!(r+k+2j)!(2k+2j+2)!} P_k(x)
\end{aligned} \tag{44}$$

$$\begin{aligned}
&=(n+r)!\Gamma(\lambda) \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k (k+\lambda) \\
&\times \sum_{j=0}^{[\frac{n-k}{2}]} \frac{(\frac{1}{4})^j}{j!(n-k-2j)!(r+k+2j)!\Gamma(k+j+\lambda+1)} C_k^{(\lambda)}(x)
\end{aligned} \tag{45}$$

$$
\begin{aligned}
&=(n+r)!\sum_{k=0}^{n}\frac{\Gamma(k+\alpha+\beta+1)(-2)^k}{\Gamma(2k+\alpha+\beta+1)}\\
&\times\sum_{l=0}^{n-k}\frac{{}_2F_1(k-n+l,k+\beta+1;2k+\alpha+\beta+2;2)}{l!(n+r-l)!(n-k-l)!}P_k^{(\alpha,\beta)}(x).
\end{aligned}
\tag{46}
$$

Remark 2. *An alternative expression for* (42) *is as follows:*

$$
\varepsilon_{n,r}(x)=(n+r)!\sum_{j=0}^{[\frac{n}{2}]}\frac{(\frac{1}{4})^j}{j!(n-2j)!(r+2j)!}\sum_{k=0}^{n-2j}\frac{(\frac{1}{2})^k\langle 2j-n\rangle_k}{k!\,\langle r+2j+1\rangle_k}H_k(x). \tag{47}
$$

Before we move on to the next section, we would like to mention some of the related previous works. In [16–18], sums of finite products of Bernoulli, Euler and Genocchi polynomials were represented as linear combinations of Bernoulli polynomials. These were derived from the Fourier series expansions for the functions closely related to those sums of finite products. In addition, in [9] the same had been done for sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials.

On the other hand, in terms of all kinds of Chebyshev polynomials, sums of finite products of Chebyshev polynomials of the second, third and fourth kinds and of Fibonacci, Legendre and Laguerre polynomials were expressed in [11,12,19]. Further, by the orthogonal polynomials in Equations (16), and (18)–(21), sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials were represented in [13].

Finally, the reader may want to see [20,21] for some other aspects of Legendre and Laguerre polynomials.

3. Proof of Theorem 1

We will first state Propositions 1 and 2 that will be needed in showing Theorems 1 and 2.

The results in the next proposition can be derived from the orthogonalities in (27)–(31), Rodrigues' and Rodrigues-type formulas in (22)–(26), and integration by parts, as we mentioned earlier. The facts (a), (b), (c), (d) and (e) in Proposition 1 are respectively from (3.7) of [5], (2.3) of [7] (see also (2.4) of [3]), (2.3) of [6], (2.3) of [4] and (2.7) of [8].

Proposition 1. *For any polynomial $q(x)\in\mathbb{R}[x]$ of degree n, the following hold.*

(a)

$$
q(x)=\sum_{k=0}^{n}C_{k,1}H_k(x),\ \text{where}\quad C_{k,1}=\frac{(-1)^k}{2^k k!\sqrt{\pi}}\int_{-\infty}^{\infty}q(x)\frac{d^k}{dx^k}e^{-x^2}dx,
$$

(b)

$$
q(x)=\sum_{k=0}^{n}C_{k,2}L_k^{\alpha}(x),\ \text{where}\quad C_{k,2}=\frac{1}{\Gamma(\alpha+k+1)}\int_{0}^{\infty}q(x)\frac{d^k}{dx^k}(e^{-x}x^{k+\alpha})dx,
$$

(c)

$$
q(x)=\sum_{k=0}^{n}C_{k,3}P_k(x),\ \text{where}\quad C_{k,3}=\frac{2k+1}{2^{k+1}k!}\int_{-1}^{1}q(x)\frac{d^k}{dx^k}(x^2-1)^k dx,
$$

(d)

$$
q(x)=\sum_{k=0}^{n}C_{k,4}C_k^{(\lambda)}(x),\ \text{where}
$$

$$
C_{k,4}=\frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}\int_{-1}^{1}q(x)\frac{d^k}{dx^k}(1-x^2)^{k+\lambda-\frac{1}{2}}dx,
$$

(e)

$$q(x) = \sum_{k=0}^{n} C_{k,5} P_k^{(\alpha,\beta)}(x), \text{ where}$$

$$C_{k,5} = \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \times \int_{-1}^{1} q(x) \frac{d^k}{dx^k}(1-x)^{k+\alpha}(1+x)^{k+\beta} dx.$$

Proposition 2. *The following proposition was stated in [16].*
For any nonnegative integers m and k, the following identities hold.

(a)

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \begin{cases} 0, & \text{if } m \equiv 1 \ (mod\ 2), \\ \frac{m!\sqrt{\pi}}{(\frac{m}{2})!2^m}, & \text{if } m \equiv 0 \ (mod\ 2), \end{cases}$$

(b)

$$\int_{-1}^{1} x^m (1-x^2)^k dx = \begin{cases} 0, & \text{if } m \equiv 1 \ (mod\ 2), \\ \frac{2^{2k+2} k! m! (k+\frac{m}{2}+1)!}{(\frac{m}{2})!(2k+m+2)!}, & \text{if } m \equiv 0 \ (mod\ 2), \end{cases}$$
$$= 2^{2k+1} k! \sum_{s=0}^{m} \binom{m}{s} 2^s (-1)^{m-s} \frac{(k+s)!}{(2k+s+1)!},$$

(c)

$$\int_{-1}^{1} x^m (1-x^2)^{k+\lambda-\frac{1}{2}} dx = \begin{cases} 0, & \text{if } m \equiv 1 \ (mod\ 2), \\ \frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{m}{2}+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m}{2}+1)}, & \text{if } m \equiv 0 \ (mod\ 2), \end{cases}$$

(d)

$$\int_{-1}^{1} x^m (1-x)^{k+\alpha}(1+x)^{k+\beta} dx = 2^{2k+\alpha+\beta+1} \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} 2^s \times \frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)}.$$

Differentiation of (10) gives us the following lemma.

Lemma 1. *For any nonnegative integers n and r, we have the following identity.*

$$\sum_{i_1+i_2+\dots i_{2r+1}=n} P_{i_1}(x), P_{i_2}(x), \dots, P_{i_{2r+1}}(x) = \frac{1}{(2r-1)!!} P_{n+r}^{(r)}(x), \tag{48}$$

where the sum is over all nonnegative integers $i_1, i_2, \dots, i_{2r+1}$, with $i_1 + i_2 + \dots i_{2r+1} = n$.
By taking rth derivative of (16), we have

$$P_n^{(r)}(x) = \frac{1}{2^n} \sum_{l=0}^{[\frac{n-r}{2}]} (-1)^l \binom{n}{l} \binom{2n-2l}{n} (n-2l)_r \, x^{n-2l-r}. \tag{49}$$

Actually, we need the following particular case of (49).

$$P_{n+r}^{(r+k)}(x) = \frac{1}{2^{n+r}} \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r}{l} \binom{2n+2r-2l}{n+r} \times (n+r-2l)_{r+k} \, x^{n-2l-k}. \tag{50}$$

Here we are going to show (35), (36) and (38), leaving the other two (37) and (39) as exercises.

With $\gamma_{n,r}(x)$ as in (32), let us put

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,1} H_k(x). \tag{51}$$

Then, from (a) of Proposition 1, (48), (50), and by integrating by parts k times, we have

$$\begin{aligned}
C_{k,1} &= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \gamma_{n,r}(x) \frac{d^k}{dx^k} e^{-x^2} dx \\
&= \frac{(-1)^k}{2^k k! \sqrt{\pi}(2r-1)!!} \int_{-\infty}^{\infty} P_{n+r}^{(r)}(x) \frac{d^k}{dx^k} e^{-x^2} dx \\
&= \frac{1}{2^k k! \sqrt{\pi}(2r-1)!!} \int_{-\infty}^{\infty} P_{n+r}^{(r+k)}(x) e^{-x^2} dx \\
&= \frac{1}{2^{k+n+r} k! \sqrt{\pi}(2r-1)!!} \\
&\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r}{l} \binom{2n+2r-2l}{n+r} (n+r-2l)_{r+k} \\
&\quad \times \int_{-\infty}^{\infty} x^{n-2l-k} e^{-x^2} dx.
\end{aligned} \tag{52}$$

From (52) and making use of (a) of Proposition 2, we obtain

$$\begin{aligned}
C_{k,1} &= \frac{1}{2^{k+n+r} k! \sqrt{\pi}(2r-1)!!} \\
&\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r}{l} \binom{2n+2r-2l}{n+r} (n+r-2l)_{r+k} \\
&\quad \times \begin{cases} 0, & \text{if} \quad k \not\equiv n \ (mod\ 2), \\ \frac{(n-k-2l)!\sqrt{\pi}}{2^{n-k-2l}(\frac{n-k}{2}-l)!}, & \text{if} \quad k \equiv n \ (mod\ 2). \end{cases}
\end{aligned} \tag{53}$$

Now, from (51) and (53) and after some simplifications,

$$\begin{aligned}
\gamma_{n,r}(x) &= \frac{1}{2^{2n+r}(2r-1)!!} \sum_{\substack{0 \le k \le n \\ k \equiv n \ (mod\ 2)}} \frac{1}{k!} \\
&\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-4)^l (2n+2r-2l)!}{l!(n+r-l)!(\frac{n-k}{2}-l)!} H_k(x) \\
&= \frac{1}{2^{2n+r}(2r-1)!!} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{j!(n-2j)!} H_{n-2j}(x) \\
&\quad \times \sum_{l=0}^{j} \frac{(-4)^l (j)_l (2n+2r-2l)!}{l!(n+r-l)!} \\
&= \frac{2^r (n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{j!(n-2j)!} H_{n-2j}(x) \\
&\quad \times \sum_{l=0}^{j} \frac{(-1)^l \langle -j \rangle_l}{l! \left\langle \frac{1}{2}-n-r \right\rangle_l} \\
&= \frac{2^r (n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \sum_{j=0}^{[\frac{n}{2}]} \frac{{}_1F_1(-j; \frac{1}{2}-n-r; -1)}{j!(n-2j)!} H_{n-2j}(x).
\end{aligned} \tag{54}$$

This shows (35) in Theorem 1.

Next, we put

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,2}\, L_k^{\alpha}(x). \tag{55}$$

Then, from (b) of Proposition 1, (48), (50) and integration by parts k times, we get

$$\begin{aligned} C_{k,2} &= \frac{(-1)^k}{\Gamma(\alpha+k+1)(2r-1)!!} \int_0^\infty P_{n+r}^{(r+k)}(x) e^{-x} x^{k+\alpha} dx \\ &= \frac{(-1)^k}{\Gamma(\alpha+k+1)(2r-1)!!\,2^{n+r}} \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r}{l} \binom{2n+2r-2l}{n+r} \\ &\quad \times (n+r-2l)_{r+k}\, \Gamma(n-2l+\alpha+1) \\ &= \frac{(-1)^k}{\Gamma(\alpha+k+1)(2r-1)!!\,2^{n+r}} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l (2n+2r-2l)!\Gamma(n-2l+\alpha+1)}{l!(n+r-l)!(n-k-2l)!}. \end{aligned} \tag{56}$$

Combining (55) and (56), and changing order of summation, we immediately have

$$\begin{aligned} \gamma_{n,r}(x) &= \frac{1}{(2r-1)!!\,2^{n+r}} \sum_{k=0}^{n} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l (2n+2r-2l)!\,\Gamma(n-2l+\alpha+1)}{l!(n+r-l)!(n-k-2l)!} L_k^{\alpha}(x) \\ &= \frac{1}{\Gamma(\alpha+1)(2r-1)!!\,2^{n+r}} \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l (2n+2r-2l)!\Gamma(n-2l+\alpha+1)}{l!(n+r-l)!(n-2l)!} \\ &\quad \times \sum_{k=0}^{n-2l} \frac{\langle 2l-n\rangle_k}{\langle \alpha+1\rangle_k} L_k^{\alpha}(x). \end{aligned} \tag{57}$$

This yields (36) in Theorem 1.

Finally, we let

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\lambda)}(x) \tag{58}$$

Then, from (d) of Proposition 1, (48), (50), integration by parts k times and making use of (c) of Proposition 2, we have

$$\begin{aligned} C_{k,4} &= \frac{(k+\lambda)\,\Gamma(\lambda)(-1)^k}{(-2)^k \sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})(2r-1)!!} \\ &\quad \times \int_{-1}^{1} P_{n+r}^{(r+k)}(x)(1-x^2)^{k+\lambda-\frac{1}{2}} dx \\ &= \frac{(k+\lambda)\Gamma(\lambda)}{2^{k+n+r}\sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})(2r-1)!!} \\ &\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} (-1)^l \binom{n+r}{l} \binom{2n+2r-2l}{n+r} (n+r-2l)_{r+k} \\ &\quad \times \begin{cases} 0, & \text{if} \quad k \not\equiv n \ (mod\ 2), \\ \frac{\Gamma(k+\lambda+\frac{1}{2})\,\Gamma(\frac{n-k+1}{2}-l)}{\Gamma(\frac{n+k}{2}+\lambda-l+1)}, & \text{if} \quad k \equiv n \ (mod\ 2). \end{cases} \end{aligned} \tag{59}$$

From (58) *and* (59)*, exploiting* (3)*,* (4)*,* (6) *and* (7)*, and after some simplifications, we finally derive*

$$
\begin{aligned}
\gamma_{n,r}(x) &= \frac{\Gamma(\lambda)}{\sqrt{\pi}(2r-1)!!2^{n+r}} \sum_{\substack{0\le k\le n \\ k\equiv n\ (mod\ 2)}} \frac{(k+\lambda)}{2^k} \\
&\quad \times \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(-1)^l (2n+2r-2l)!\,\Gamma(\frac{n-k+1}{2}-l)}{l!(n+r-l)!(n-k-2l)!\Gamma(\frac{k+n}{2}+\lambda-l+1)} C_k^{(\lambda)}(x) \\
&= \frac{\Gamma(\lambda)}{\sqrt{\pi}(2r-1)!!2^{n+r}} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n-2j+\lambda)}{2^{n-2j}} \\
&\quad \times \sum_{l=0}^{j} \frac{(-1)^l (2n+2r-2l)!\,\Gamma(j-l+\frac{1}{2})}{l!\,(n+r-l)!\,(2j-2l)!\,\Gamma(n+\lambda-j-l+1)} C_{n-2j}^{(\lambda)}(x) \\
&= \frac{\Gamma(\lambda)}{(2r-1)!!\,2^{2n+r}} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n-2j+\lambda)}{\Gamma(n+\lambda-j+1)} \\
&\quad \times \sum_{l=0}^{j} \frac{(-4)^l (2n+2r-2l)!\,(n+\lambda-j)_l}{l!\,(n+r-l)!\,(j-l)!} C_{n-2j}^{(\lambda)}(x) \\
&= \frac{2^r\,\Gamma(\lambda)(n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n-2j+\lambda)}{\Gamma(n+\lambda-j+1)\,j!} \\
&\quad \times \sum_{l=0}^{j} \frac{\langle -j\rangle_l\ \langle j-n-r\rangle_l}{l!\,\left\langle \frac{1}{2}-n-r\right\rangle_l} C_{n-2j}^{(\lambda)}(x) \\
&= \frac{2^r\,\Gamma(\lambda)(n+r-\frac{1}{2})_{n+r}}{(2r-1)!!} \\
&\quad \times \sum_{j=0}^{[\frac{n}{2}]} \frac{(n-2j+\lambda)\,{}_2F_1(-j, j-n-r; \frac{1}{2}-n-r; 1)}{\Gamma(n+\lambda-j+1)\,j!} C_{n-2j}^{(\lambda)}(x).
\end{aligned} \tag{60}
$$

This completes the proof for (38) *in Theorem 1.*

4. Proof of Theorem 2

The proofs for (42), (43) and (45) are left to the reader as an exercise and we will show only (44) and (46) in Theorem 2.

The following lemma is important for our discussion in this section and can be derived by differentiating (11).

Lemma 2. *Let n, r be nonnegative integers. Then we have the following identity.*

$$
\sum_{i_1+i_2+\cdots+i_{r+1}=n} L_{i_1}\left(\frac{x}{r+1}\right) L_{i_2}\left(\frac{x}{r+1}\right) \ldots L_{i_{r+1}}\left(\frac{x}{r+1}\right) = (-1)^r L_{n+r}^{(r)}(x), \tag{61}
$$

where the sum runs over all nonnegative integers $i_1, i_2, \ldots, i_{r+1}$, with $i_1+i_2+\cdots+i_{r+1}=n$.

From (17)*, it is immediate to see that the rth derivative of $L_n(x)$ is given by*

$$
L_n^{(r)}(x) = \sum_{l=0}^{n-r} (-1)^{n-l} \binom{n}{l} \frac{1}{(n-l-r)!}\, x^{n-l-r}. \tag{62}
$$

In particular, we have

$$L_{n+r}^{(r+k)}(x) = \sum_{l=0}^{n-k} (-1)^{n+r-l} \binom{n+r}{l} \frac{1}{(n-k-l)!}\, x^{n-k-l}. \tag{63}$$

With $\varepsilon_{n,r}(x)$ as in (33), let us set

$$\varepsilon_{n,r}(x) = \sum_{k=0}^{n} C_{k,3} P_k(x). \tag{64}$$

Then, from (c) of Proposition 1, (61), (63), by integration by parts k times and using (b) of Proposition 2, we get

$$\begin{aligned}
C_{k,3} &= \frac{(2k+1)(-1)^{r+k}}{2^{k+1}k!} \int_{-1}^{1} L_{n+r}^{(r+k)}(x)(x^2-1)^k dx \\
&= \frac{(-1)^{n+k}(2k+1)(n+r)!}{2^{k+1}\, k!} \sum_{l=0}^{n-k} \frac{(-1)^l}{l!\,(n+r-l)!\,(n-k-l)!} \\
&\quad \times \begin{cases} 0, & \text{if} \quad l \not\equiv n-k \ (mod\ 2) \\ \frac{2^{2k+2}\, k!\,(n-k-l)!(\frac{n+k-l}{2}+1)!}{(\frac{n-k-l}{2})!(n+k-l+2)!}, & \text{if} \quad l \equiv n-k \ (mod\ 2) \end{cases} \\
&= (-1)^{n+k}(2k+1)2^{k+1}(n+r)! \\
&\quad \times \sum_{\substack{0 \le l \le n-k \\ l \equiv n-k \ (mod\ 2)}} \frac{(-1)^l\,(\frac{n+k-l}{2}+1)!}{l!\,(n+r-l)!\,(\frac{n-k-l}{2})!\,(n+k-l+2)!} \\
&= (n+r)!\,(2k+1)\,2^{k+1} \\
&\quad \times \sum_{j=0}^{[\frac{n-k}{2}]} \frac{(k+j+1)!}{j!\,(n-k-2j)!\,(r+k+2j)!\,(2k+2j+2)!}.
\end{aligned} \tag{65}$$

By combining (64) and (65) we get the following result.

$$\begin{aligned}
\varepsilon_{n,r}(x) = (n+r)! &\sum_{k=0}^{n} (2k+1)\,2^{k+1} \\
&\times \sum_{j=0}^{[\frac{n-k}{2}]} \frac{(k+j+1)!}{j!\,(n-k-2j)!\,(r+k+2j)!\,(2k+2j+2)!} P_k(x).
\end{aligned} \tag{66}$$

This completes the proof for (44).

Finally, we put

$$\varepsilon_{n,r}(x) = \sum_{k=0}^{n} C_{k,5}\, P_k^{(\alpha,\beta)}(x). \tag{67}$$

Then, from (e) of Proposition 1, (61), (63), integration by parts k times and exploiting (d) of Proposition 2, we have

$$\begin{aligned}
C_{k,5} &= \frac{(-1)^r\,(2k+\alpha+\beta+1)\,\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\quad \times \int_{-1}^{1} L_{n+r}^{(r+k)}(x)(1-x)^{k+\alpha}(1+x)^{k+\beta}dx \\
&= \frac{(-1)^r(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\quad \times \sum_{l=0}^{n-k}(-1)^{n+r-l}\binom{n+r}{l}\frac{1}{(n-k-l)!} \\
&\quad \times \int_{-1}^{1} x^{n-k-l}(1-x)^{k+\alpha}(1+x)^{k+\beta}dx \\
&= \frac{(n+r)!\,(-2)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta+k+1)} \\
&\quad \times \sum_{l=0}^{n-k}\frac{1}{l!\,(n+r-l)!}\sum_{s=0}^{n-k-l}\frac{(-2)^s\,\Gamma(k+\beta+s+1)}{s!\,(n-k-l-s)!\,\Gamma(2k+\alpha+\beta+s+2)} \\
&= \frac{(n+r)!\,(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \\
&\quad \times \sum_{l=0}^{n-k}\frac{1}{l!\,(n+r-l)!\,(n-k-l)!}\sum_{s=0}^{n-k-l}\frac{2^s\,\langle k+l-n\rangle_s\,\langle k+\beta+1\rangle_s}{s!\,\langle 2k+\alpha+\beta+2\rangle_s} \\
&= \frac{(n+r)!\,(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \\
&\quad \times \sum_{l=0}^{n-k}\frac{{}_2F_1(k+l-n,k+\beta+1;2k+\alpha+\beta+2;2)}{l!\,(n+r-l)!\,(n-k-l)!}
\end{aligned} \tag{68}$$

We now obtain

$$\begin{aligned}
\varepsilon_{n,r}(x) &= (n+r)!\,\sum_{k=0}^{n}\frac{(-2)^k\,\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)} \\
&\quad \times \sum_{l=0}^{n-k}\frac{{}_2F_1(k+l-n,k+\beta+1;2k+\alpha+\beta+2;2)}{l!\,(n+r-l)!\,(n-k-l)!}P_k^{(\alpha,\beta)}(x).
\end{aligned} \tag{69}$$

This verifies (46) in Theorem 2.

5. Conclusions

Let $\gamma_{m,r}(x)$, $\varepsilon_{m,r}(x)$, and $\alpha_m(x)$ denote the following sums of finite products given by

$$\gamma_{n,r}(x) = \sum_{i_1+\cdots+i_{2r+1}=n} P_{i_1}(x)P_{i_2}(x)\ldots P_{i_{2r+1}}(x),$$

$$\varepsilon_{n,r}(x) = \sum_{i_1+\cdots+i_{r+1}=n} L_{i_1}\left(\frac{x}{r+1}\right)L_{i_2}\left(\frac{x}{r+1}\right)\ldots L_{i_{r+1}}\left(\frac{x}{r+1}\right),$$

$$\alpha_m(x) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(x)B_{m-k}(x),\ (m\geq 2),$$

where $P_n(x)$, $L_n(x)$, $B_n(x)$, $(n\geq 0)$ are respectively Legendre, Laguerre and Bernoulli polynomials. In this paper, we studied sums of finite products of Legendre polynomials $\gamma_{m,r}(x)$ and those of Laguerre polynomials $\varepsilon_{m,r}(x)$, and expressed them as linear combinations of the orthogonal polynomials $H_n(x), L_n^{\alpha}(x), P_n(x), C_n^{(\lambda)}(x)$, and $P_n^{(\alpha,\beta)}(x)$. These have been done by carrying out explicit computations. In recent years, we have obtained similar results for many other special polynomials.

For example, we considered sums of finite products of Bernoulli, Euler and Genocchi polynomials and represented them in terms of Bernoulli polynomials. In addition, as for Chebyshev polynomials of the second, third, and fourth kinds, and Fibonacci, Legendre and Laguerre polynomials, we expressed them not only in terms of Bernoulli polynomials but also of Chebyshev polynomials of all kinds and Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials.

We gave twofold justification for studying such sums of finite products of special polynomials. Firstly, it can be viewed as a generalization of the classical connection problem in which one wants to determine the connection coefficients in the expansion of a product of two polynomials in terms of any given sequence of polynomials. Secondly, from the representation of $\alpha_m(x)$ in terms of Bernoulli polynomials we can derive the famous Faber-Pandharipande-Zagier identity and a slightly different variant of the Miki's identity. We emphasized that these identities had been obtained by several different methods which are quite involved and not elementary, while our previous method used only elementary Fourier series expansions.

Along the same line of the present paper, we would like to continue to work on representing sums of finite products of some special polynomials in terms of various kinds of special polynomials and to find interesting applications of them in mathematics, science and engineering areas.

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References

1. Andrews, G.E.; Askey, R.; Roy, R. *Special Functions, Encyclopedia of Mathematics and Its Applications*; Cambridge University Press: Cambridge, UK, 1999.
2. Beals, R.; Wong, R. Special functions and orthogonal polynomials. In *Cambridge Studies in Advanced Mathematics*; Cambridge University Press: Cambridge, UK, 2016.
3. Kim, D.S.; Kim, T.; Dolgy, D.V. Some identities on Laguerre polynomials in connection with Bernoulli and Euler numbers. *Discret. Dyn. Nat. Soc.* **2012**, *2012*, 619197. [CrossRef]
4. Kim, D.S.; Kim, T.; Rim, S. Some identities involving Gegenbauer polynomials. *Adv. Differ. Equ.* **2012**, *2012*, 219. [CrossRef]
5. Kim, D.S.; Kim, T.; Rim, S.; Lee, S.H. Hermite polynomials and their applications associated with Bernoulli and Euler numbers. *Discret. Dyn. Nat. Soc.* **2012**, *2012*, 974632. [CrossRef]
6. Kim, D.S.; Rim, S.; Kim, T. Some identities on Bernoulli and Euler polynomials arising from orthogonality of Legendre polynomials. *J. Inequal. Appl.* **2012**, *2012*, 227. [CrossRef]
7. Kim, T.; Kim, D.S. Extended Laguerre polynomials associated with Hermite, Bernoulli, and Euler numbers and polynomials. *Abstr. Appl. Anal.* **2012**, *2012*, 957350. [CrossRef]
8. Kim, T.; Kim, D.S.; Dolgy, D.V. Some identities on Bernoulli and Hermite polynomials associated with Jacobi polynomials. *Discret. Dyn. Nat. Soc.* **2012**, *2012*, 584643. [CrossRef]
9. Kim, T.; Kim, D.S.; Dolgy, D.V.; Park, J.-W. Sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials. *J. Inequal. Appl.* **2018**, *2018*, 148. [CrossRef] [PubMed]
10. Kim, T.; Kim, D.S.; Dolgy, D.V.; Park, J.-W. Sums of finite products of Legendre and Laguerre polynomials. *Adv. Differ. Equ.* **2018**, *2018*, *277*. [CrossRef]
11. Kim, T.; Kim, D.S.; Dolgy, D.V.; Ryoo, C.S. Representing sums of finite products of Chebyshev polynomials of third and fourth kinds by Chebyshev polynomials. *Symmetry* **2018**, *10*, 258. [CrossRef]
12. Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. Sums of finite products of Legendre and Laguerre polynomials by Chebyshev polynomials. *Adv. Stud. Contemp. Math.* **2018**, *28*, 551–565.
13. Kim, T.; Kim, D.S.; Kwon, J.; Dolgy, D.V. Expressing sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials by several orthogonal polynomials. *Mathematics* **2018**, *6*, 210. [CrossRef]

14. Cesarano, C.; Fornaro, C. A note on two-variable Chebyshev polynomials. *Georgian Math. J.* **2017**, *24*, 339–349. [CrossRef]
15. Cesarano, C.; Fornaro, C.; Vazquez, L. A note on a special class of Hermite polynomials. *Int. J. Pure Appl. Math.* **2015**, *98*, 261–273. [CrossRef]
16. Kim, T.; Kim, D.S.; Jang, L.C.; Jang, G. Sums of finite products of Genocchi functions. *Adv. Differ. Equ.* **2017**; *2017*, 268. [CrossRef]
17. Agarwal, R.P.; Kim, D.S.; Kim, T.; Kwon, J. Sums of finite products of Bernoulli functions. *Adv. Differ. Equ.* **2017**, *2017*, 237. [CrossRef]
18. Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. Sums of finite products of Euler functions. In *Advances in Real and Complex Analysis with Applications*; Trends in Mathematics; Springer: Berlin, Germany, 2017, pp. 243–260.
19. Kim, T.; Dolgy, D.V.; Kim, D.S. Representing sums of finite products of Chebyshev polynomials of second kind and Fibonacci polynomials in terms of Chebyshev polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2018**, *28*, 321–335.
20. Araci, S.; Acikgoz, M.; Bagdasaryan, A.; Sen, E. The Legendre polynomials associated with Bernoulli, Euler, Hermite and Bernstein polynomials. *Turk. J. Anal. Number Theory* **2013**, *1*, 1–3. [CrossRef]
21. Khan, W.A.; Araci, S.; Acikgoz, M. A new class of Laguerre-based Apostol type polynomials. *Cogent Math.* **2016**, *3*, 1243839. [CrossRef]