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Whether an Enormously Large Energy Density of the Quantum Vacuum Is Catastrophic

Vladimir M. Mostepanenko [1,2,3,*] and Galina L. Klimchitskaya [1,2]

[1] Central Astronomical Observatory at Pulkovo of the Russian Academy of Sciences, Saint Petersburg 196140, Russia; g.klimchitskaya@gmail.com
[2] Institute of Physics, Nanotechnology and Telecommunications, Peter the Great Saint Petersburg Polytechnic University, Saint Petersburg 195251, Russia
[3] Kazan Federal University, Kazan 420008, Russia
* Correspondence: vmostepa@gmail.com

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Abstract: The problem of an enormously large energy density of the quantum vacuum is discussed in connection with the concept of renormalization of physical parameters in quantum field theory. Using the method of dimensional regularization, it is recalled that the normal ordering procedure of creation and annihilation operators is equivalent to a renormalization of the cosmological constant leading to its zero and nonzero values in Minkowski space-time and in the standard cosmological model, respectively. It is argued that a frequently discussed gravitational effect, resulting from an enormously large energy density described by the nonrenormalized (bare) cosmological constant, might be nonobservable much like some other bare quantities introduced in the formalism of quantum field theory.

Keywords: quantum vacuum; renormalizations; dark energy; standard cosmological model

1. Introduction

The elaboration of the quantum field theory has raised a number of fundamental problems that remain to be finally resolved. One of them is the problem of the quantum vacuum, which is of crucial importance for physics of elementary particles and cosmology. It has been known that the vacuum stress-energy tensor of quantized fields diverges at large momenta. In the framework of quantum field theory, the special procedure was elaborated which makes the vacuum expectation values of physical observables, such as the energy density and pressure, equal to zero. It is the normal ordering of creation and annihilation operators (note, however, that nonzero vacuum expectation values may occur due to the spontaneous symmetry breaking). This was considered as wholly reasonable as soon as all branches of physics with the only exception of gravitation deal not with the absolute values of energy, but with energy differences. In so doing, all energies are measured from the infinite vacuum energy.

In succeeding years, it was understood, however, that an energy density of quantum vacuum should have incredibly large gravitational effects which are not observed experimentally. It is common to assume that the local quantum field theory is valid up to the Planck energy scale $E_{\rm Pl} \sim 10^{19}$ Gev $\sim 10^{9}$ J. If one makes a cutoff in the divergent vacuum stress-energy tensor at respective Planck momentum $p_{\rm Pl} = E_{\rm Pl}/c$, the obtained energy density $\sim 10^{111}$ J/m^3 would exceed the observed value $\sim 10^{-9}$ J/m^3 associated with an accelerated expansion of the Universe by the factor of 10^{120} [1,2]. Taking into account that the vacuum stress-energy tensor of quantized fields can be described in terms of the cosmological constant [3], the same

discrepancy is obtained between its theoretical and experimental values giving a reason to speak about the "vacuum catastrophe" [4]. At the moment, it is widely believed that this is one of the biggest unsolved problems of modern physics [5].

An assumption that the zero-point oscillations should gravitate similar to real elementary particles could be doubted because there is no direct way to check it out experimentally. In this paper, the problem of vacuum energy is discussed in connection with the concept of renormalizations in quantum field theory. Using the method of dimensional regularization, we demonstrate rigorously that the vacuum stress-energy tensor of quantized fields is proportional to the metrical tensor. On this basis, it is argued that the bare (non-renormalized) value of the cosmological constant might be excluded from theoretical description of the measurement results much as it holds for the bare electron mass and charge in standard quantum electrodynamics.

In Section 2, the divergences in the vacuum stress-energy tensor are discussed in connection with other divergences in quantum field theory. Section 3 contains the dimensional regularization of the vacuum stress-energy tensor. In Section 4, the problem of the vacuum energy density is considered in the context of gravitational theory. Section 5 contains our conclusions and discussion.

The units with $\hbar = c = 1$ are used throughout the paper.

2. Divergences in Quantum Field Theory and the Quantum Vacuum

It is common knowledge that, in renormalizable quantum field theories, the matrix elements of physical observables are identically expressed in terms of bare and real (physical) quantities (for instance, bare and real charge and mass of an electron in quantum electrodynamics). In so doing, the real and bare quantities differ by a formally infinitely large factors. It is necessary to stress that bare (i.e., noninteracting) physical objects are nonobservable because any observation must be accompanied by some interaction [6]. In terms of bare quantities, most of the matrix elements of physical observables are expressed by the divergent integrals and, thus, are infinitely large. However, being expressed in terms of real quantities, these matrix elements become finite and in excellent agreement with the measurement data. Because of this, a removal of divergences by means of going from the bare to real quantities (the so-called renormalization procedure) can be considered as quite satisfactory. As to the status of bare objects, they might be treated as having a little physical importance.

The vacuum stress-energy tensor of quantized fields is characterized by a higher (fourth-order) divergence in comparison with the matrix elements of the normally ordered operators. As discussed in Section 1, the normal ordering procedure makes equal to zero the vacuum expectation values of the main physical observables. (Note that, in nonlinear quantum field theories, the validity of this statement depends on whether or not the vacuum state exists and on the specific form of interaction.) According to the postulate of quantum field theory, the operators of all physical quantities are expressed via the operators of fields in the same way as in classical field theory, and the normal ordering procedure establishes the proper order of operator multiplication [7]. In fact, this postulate is an application of the correspondence principle and the normal ordering procedure makes it unambiguous. As a result, the stress-energy tensor of the zero-point oscillations (the so-called virtual particles) is simply disregarded. For this reason, one may believe that the virtual particles in themselves are not observable and, specifically, do not gravitate.

There are, however, many effects where the zero-point oscillations contribute indirectly as a result of some interaction. In perturbation theory, the pure vacuum diagrams look like the closed loops with no external legs. They are characterized by the fourth-order divergence and give rise to the vacuum energy density. Many diagrams, however, have internal loops and also external legs representing real particles. These diagrams describe experimentally observable quantities, such as the anomalous magnetic moment of an electron or the Lamb shift, whose values are partially determined by the quantum vacuum. One should

mention also the Casimir effect where the spectrum of zero-point oscillations is altered by an interaction with the material boundaries [8–10]. It is important to underline, however, that the finite Casimir energy density and force acting between the boundary surfaces are obtained after subtracting a divergent energy density of the quantum vacuum in an unrestricted space, i.e., disregarding the same quantity as in standard quantum field theory. It cannot be too highly stressed that only this finite and measurable energy density is a source of gravitational interaction [11–13]. The Casimir-like energy density also arises in spaces with nontrivial topology due to identification conditions imposed on the quantized fields [14–16]. Note that there is an approach which describes the Casimir effect as relativistic quantum forces between charges and currents in the material boundaries without reference to zero-point energies [17]. It seems, however, that along these lines it would be difficult to describe the Casimir energy density arising in topologically nontrivial spaces in cosmology because the identification conditions are not caused by a matter and do not involve any charges and currents.

The question arises of whether it is possible to treat the normal ordering procedure in terms of renormalization. Before answering it in the next section, we briefly recall an explicit form of the vacuum stress-energy tensor of quantized fields. For this purpose, one degree of freedom of a quantized field can be modelled by the real scalar field $\varphi(x)$ of mass m having the Lagrangian density

$$L(x) = \frac{1}{2}\left[\partial_k \varphi(x)\partial^k \varphi(x) - m^2\varphi^2(x)\right] \tag{1}$$

and the stress-energy tensor

$$T_{ij}(x) = \partial_i \varphi(x)\partial_j \varphi(x) - L(x)g_{ij}. \tag{2}$$

Here, $i, j, k = 0, 1, 2, 3$, and the metrical tensor is $g_{ij} = \mathrm{diag}(1, -1, -1, -1)$.

The field operator is presented in the form

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\boldsymbol{p}}{\sqrt{2\omega(\boldsymbol{p})}} \left(e^{-ipx} c_{\boldsymbol{p}} + e^{ipx} c^+_{\boldsymbol{p}}\right), \tag{3}$$

where $\omega(\boldsymbol{p}) = (m^2 + \boldsymbol{p}^2)^{1/2}$, $px = \omega t - \boldsymbol{px}$ and $c_{\boldsymbol{p}}, c^+_{\boldsymbol{p}}$ are the annihilation and creation operators satisfying the standard commutation conditions. The vacuum state is defined by

$$c_{\boldsymbol{p}}|0\rangle = 0, \qquad \langle 0|c^+_{\boldsymbol{p}} = 0. \tag{4}$$

Substituting Equations (1) and (3) in Equation (2) for $i = j = 0$ and using Equation (4), one finds

$$\langle 0|T_{00}(x)|0\rangle = \frac{1}{2(2\pi)^3} \int d^3\boldsymbol{p}\, \omega(\boldsymbol{p}). \tag{5}$$

In a similar way, from Equation (2) at $i = j = \mu = 1, 2, 3$, we obtain the common result for all diagonal components

$$\langle 0|T_{\mu\mu}(x)|0\rangle = \frac{1}{2(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{\omega(\boldsymbol{p})} p^2_\mu. \tag{6}$$

It is seen that the quantities (5) and (6) diverge as p^4 at high momenta. This is because the stress-energy tensor was not normally ordered with respect to the creation and annihilation operators; otherwise, the zero results would be obtained. As to the nondiagonal components of the vacuum stress-energy tensor, they are equal to zero,

$$\langle 0|T_{0\mu}(x)|0\rangle = 0, \tag{7}$$

whether or not the normal ordering procedure is used.

Equations (5)–(7) can be identically rewritten in the form of one equation

$$\langle 0|T_{\mu\mu}(x)|0\rangle = \frac{1}{2(2\pi)^3}\int d^3p\frac{p_ip_j}{\omega(\boldsymbol{p})}. \tag{8}$$

Equation (8) is in fact applicable to one degree of freedom of any bosonic or fermionic field [2]. In the latter case, the commutation relations for creation and annihilation operators should be replaced with the anticommutation ones and the sign minus appears in front of the right-hand side in Equation (8). Let us have P bosonic fields $\varphi_1, \ldots, \varphi_P$ with masses $m_1, \ldots, m_P$ and $g_1, \ldots, g_P$ degrees of freedom each and Q fermionic fields $\psi_1, \ldots, \psi_Q$ with masses $M_1, \ldots, M_Q$ and $h_1, \ldots, h_Q$ degrees of freedom, respectively. In this case, the vacuum stress-energy tensor takes the form

$$\langle 0|T^{\rm tot}_{ij}(x)|0\rangle = \frac{1}{2(2\pi)^3}\int d^3p\, p_ip_j\left(\sum_{l=1}^{P}\frac{g_l}{\sqrt{m_l^2+\boldsymbol{p}^2}}-\sum_{l=1}^{Q}\frac{h_l}{\sqrt{M_l^2+\boldsymbol{p}^2}}\right). \tag{9}$$

In spite of the fact that this quantity contains both positive and negative contributions, it remains divergent and turns into zero only in the case of exact supersymmetry that is not supported by the experimental data. In the next sections, we discuss the possibility to remove this infinity by means of renormalization.

3. Geometric Structure of the Vacuum Stress-Energy Tensor

It is a common assumption based on the relativistic covariance that the vacuum stress-energy tensor is proportional to a constant times the metrical tensor. In such a manner, the vacuum energy density can be identified up to a multiple with the cosmological constant that might be a subject of renormalization similar to some other parameters in quantum field theory. Taking into account the fundamental importance of this issue, it seems useful from at least a pedagogical point of view to perform an immediate analysis of the geometrical structure of Equations (6)–(9). This can be made by the method of dimensional regularization [18].

We start with generalization of Equations (5) and (6) to the case of N-dimensional space-time having $N-1$ spatial dimensions. From Equation (5), one obtains

$$\langle 0|T_{00}(x)|0\rangle_N = \frac{1}{2(2\pi)^{N-1}}\int d^{N-1}p\,\sqrt{m^2+\boldsymbol{p}^2}, \tag{10}$$

where the volume element and the second power of momentum are given by

$$d^{N-1}p = p^{N-2}dpd\Omega_{N-2}, \qquad \boldsymbol{p}^2 = p^2 = \sum_{\mu=1}^{N-1}p_\mu^2 \tag{11}$$

and the surface element of a unit sphere in $(N-1)$-dimensional space is expressed in spherical coordinates as

$$d\Omega_{N-2} = d\varphi\sin\theta_1d\theta_1\sin^2\theta_2d\theta_2\ldots\sin^{N-3}\theta_{N-3}d\theta_{N-3}. \tag{12}$$

Taking into consideration that all quantities (6) with different μ are equal in value due to the isotropy of space, the generalization of Equation (6) to N-dimensional case takes the form

$$\langle 0|T_{\mu\mu}(x)|0\rangle_N = \frac{1}{2(2\pi)^{N-1}(N-1)} \int \frac{p^2 d^{N-1}p}{\sqrt{m^2+p^2}} \delta_{\mu\mu}, \tag{13}$$

where $\delta_{\mu\mu} = 1$ are the diagonal elements of Kronecker's symbol.

Integrating in Equations (10) and (13) with the help of Equations (11) and (12) and the equality

$$\int d\Omega_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}, \tag{14}$$

where $\Gamma(z)$ is the gamma function, we arrive at

$$\begin{aligned} \langle 0|T_{00}(x)|0\rangle_N &= D_N \int_0^\infty dp p^{N-2} \sqrt{m^2+p^2}, \\ \langle 0|T_{\mu\mu}(x)|0\rangle_N &= \frac{D_N}{N-1} \int_0^\infty dp \frac{p^N}{\sqrt{m^2+p^2}} \delta_{\mu\mu}. \end{aligned} \tag{15}$$

Here, the factor D_N is defined as

$$D_N = \frac{1}{2^{N-1}\pi^{\frac{N-1}{2}}\Gamma\left(\frac{N-1}{2}\right)}. \tag{16}$$

It is convenient to introduce in Equation (15) the dimensionless integration variable $y = p/m$ and obtain

$$\begin{aligned} \langle 0|T_{00}(x)|0\rangle_N &= D_N m^N \int_0^\infty dy y^{N-2} \sqrt{1+y^2}, \\ \langle 0|T_{\mu\mu}(x)|0\rangle_N &= \frac{D_N m^N}{N-1} \int_0^\infty dy \frac{y^N}{\sqrt{1+y^2}} \delta_{\mu\mu}. \end{aligned} \tag{17}$$

Now, we put $N = 4 + 2\varepsilon$, where ε is, in general, a complex parameter. In this case, Equation (17) is rewritten as

$$\begin{aligned} \langle 0|T_{00}(x)|0\rangle_{4+2\varepsilon} &= D_{4+2\varepsilon} m^4 \left(\frac{m}{m_f}\right)^{2\varepsilon} \int_0^\infty dy y^{2+2\varepsilon} \sqrt{1+y^2}, \\ \langle 0|T_{\mu\mu}(x)|0\rangle_{4+2\varepsilon} &= \frac{D_{4+2\varepsilon} m^4}{3+2\varepsilon} \left(\frac{m}{m_f}\right)^{2\varepsilon} \int_0^\infty dy \frac{y^{4+2\varepsilon}}{\sqrt{1+y^2}} \delta_{\mu\mu}, \end{aligned} \tag{18}$$

where m_f is a fictitious mass introduced in order to make the dimension of quantities (18), written in the space-time of $4 + 2\varepsilon$ dimensions, the same as for $N = 4$. It is significant that, whereas the quantities (17) are divergent, the quantities (18) converge in terms of the boundary values of distributions when $\mathrm{Im}\varepsilon \neq 0$.

The integrals in the quantities (18) can be calculated in terms of the beta function [19]

$$\int_0^\infty \frac{y^{2x-1} dy}{(1+y^2)^{x+t}} = \frac{1}{2} \mathrm{B}(x,t) \tag{19}$$

with the result

$$\langle 0|T_{00}(x)|0\rangle_{4+2\varepsilon} = \frac{1}{2}D_{4+2\varepsilon}m^4\left(\frac{m}{m_f}\right)^{2\varepsilon}\mathrm{B}\left(\frac{3+2\varepsilon}{2},-2-\varepsilon\right),$$
$$\langle 0|T_{\mu\mu}(x)|0\rangle_{4+2\varepsilon} = \frac{D_{4+2\varepsilon}m^4}{2(3+2\varepsilon)}\left(\frac{m}{m_f}\right)^{2\varepsilon}\mathrm{B}\left(\frac{5+2\varepsilon}{2},-2-\varepsilon\right)\delta_{\mu\mu}. \tag{20}$$

Then, using the equality [19]

$$\mathrm{B}(x,t) = \frac{\Gamma(x)\Gamma(t)}{\Gamma(x+t)}, \tag{21}$$

we finally find

$$\langle 0|T_{00}(x)|0\rangle_{4+2\varepsilon} = -\frac{m^4}{2^{5+2\varepsilon}\pi^{2+\varepsilon}}\left(\frac{m}{m_f}\right)^{2\varepsilon}\Gamma(-2-\varepsilon),$$
$$\langle 0|T_{\mu\mu}(x)|0\rangle_{4+2\varepsilon} = \frac{m^4}{2^{5+2\varepsilon}\pi^{2+\varepsilon}}\left(\frac{m}{m_f}\right)^{2\varepsilon}\Gamma(-2-\varepsilon)\delta_{\mu\mu}. \tag{22}$$

From Equation (22), it is seen that

$$\langle 0|T_{\mu\mu}(x)|0\rangle_{4+2\varepsilon} = -\langle 0|T_{00}(x)|0\rangle_{4+2\varepsilon} \tag{23}$$

for any ε and, thus, the regularized vacuum stress-energy tensor is really proportional to g_{ij}.

Using elementary properties of the gamma function, one obtains

$$\Gamma(-2-\varepsilon) = \frac{1}{2}\left[-\frac{1}{\varepsilon}+\frac{3}{2}-\gamma+O(\varepsilon)\right], \tag{24}$$

where $\gamma = 0.5772\ldots$ is the Euler constant. Then, expanding the quantities (22) in powers of ε and preserving only those terms which do not vanish in the limit $\varepsilon \to 0$, we arrive at

$$\langle 0|T_{ij}(x)|0\rangle_{4+2\varepsilon} = \frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon}-\frac{3-2\gamma}{2}+\ln\frac{m^2}{4\pi m_f^2}\right)g_{ij}. \tag{25}$$

Applying the result (25) to each degree of freedom of P bosonic and Q fermionic fields, we can write

$$\langle 0|T_{ij}^{\mathrm{tot}}(x)|0\rangle_{4+2\varepsilon} = I_\varepsilon\, g_{ij}, \tag{26}$$

where the divergent in the limiting case $\varepsilon \to 0$ constant on the right-hand side is equal to

$$I_\varepsilon = \frac{1}{64\pi^2}\left[\sum_{l=1}^{P} g_l m_l^4\left(\frac{1}{\varepsilon}-\frac{3-2\gamma}{2}+\ln\frac{m_l^2}{4\pi m_f^2}\right)-\sum_{l=1}^{Q} h_l M_l^4\left(\frac{1}{\varepsilon}-\frac{3-2\gamma}{2}+\ln\frac{M_l^2}{4\pi m_f^2}\right)\right]. \tag{27}$$

In flat Minkowski space-time, the normal ordering procedure thus implies a transition from an infinitely large in the limit $\varepsilon \to 0$ quantity (27) to zero. In the next section, we consider the case of curved space-time and show that such a transition is equivalent to zero renormalized value of the cosmological constant.

4. The Quantum Vacuum and Gravitation

Having no quantum theory of gravity, it is reasonable to consider quantized fields on the background of curved space-time. This is some kind of a semi-classical approach that is analogous to quantum electrodynamics in the external field and can be considered as a one-loop approximation to the future quantum gravity [20,21].

It is easily seen [21,22] that, in the quasi-Euclidean Friedmann Universe, which is a part of the standard cosmological model, the main divergent term in the vacuum expectation value of the stress-energy tensor of quantized fields has the same form as in Minkowski space-time, i.e., is given by Equations (26) and (27). Therefore, the self-consistent Einstein equations, determining the space-time metric, can be written as

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda_{\varepsilon}^{(b)}g_{ij} = -8\pi G\left[I_{\varepsilon}g_{ij} + \langle T_{ij}\rangle + T_{ij}^{(m)}\right]. \tag{28}$$

Here, R_{ij} and R are the Ricci tensor and the scalar curvature, $\Lambda_{\varepsilon}^{(b)}$ is the bare (nonrenormalized) cosmological constant, G is the gravitational constant, $\langle T_{ij}\rangle$ is a remaining part of the vacuum stress-energy tensor after separating the leading divergent term, and $T_{ij}^{(m)}$ is the stress-energy tensor of classical background matter. Note that, in curved space-time, $\langle T_{ij}\rangle$ may contain the lower-order divergent terms as well as contributions from the vacuum polarization and creation of particles due to nonstationarity of the gravitational background. As a result, the physical gravitational constant differs from the bare one by a divergent factor [20,21].

Based on Equation (28), one can define the physical (renormalized) cosmological constant by the equality

$$\Lambda^{(\mathrm{ren})} = \Lambda_{\varepsilon}^{(b)} + 8\pi G I_{\varepsilon}. \tag{29}$$

In doing so, only $\Lambda^{(\mathrm{ren})}$ enters the self-consistent Equation (28). The value of $\Lambda^{(\mathrm{ren})}$ should be determined experimentally. Specifically, in Minkowski space-time, one should put $\Lambda^{(\mathrm{ren})} = 0$, whereas, in the standard cosmological model, the value of $\Lambda^{(\mathrm{ren})}$ was found from an acceleration rate of the Universe expansion (see Section 1). Under this approach, an infinitely large (bare) value of the cosmological constant might be considered as of little physical importance similar to other bare parameters of quantum field theory. It would be, of course, desirable to find the value of $\Lambda^{(\mathrm{ren})}$, or at least to justify its smallness, theoretically. This is, however, presently impossible just as we cannot calculate the value of the electric charge of an electron but only to measure it.

5. Discussion

In the foregoing, we have adduced several arguments in favor of the viewpoint that a divergent or enormously large value of the stress-energy tensor of the quantum vacuum might constitute not as serious problem as it is often believed. In the framework of this approach, it is recalled that the vacuum stress-energy tensor is proportional to the metrical tensor and, thus, the normal ordering procedure, which helps to obtain zero vacuum expectation values of physical observables in flat space-time, is equivalent to the renormalization of the cosmological constant. Unlike a widely believed opinion that an enormously large vacuum energy density should produce large gravitational effect, we argue that the zero-point energy is not directly observable and, in particular, does not gravitate.

6. Conclusions

To conclude, the renormalized (observable) value of the cosmological constant and, thus, the physical vacuum energy density should be determined experimentally. It is zero in flat space-time and takes a nonzero value found from the acceleration of the Universe expansion in cosmology. As to the question about possible gravitational effects of an enormously large energy density related to the nonrenormalized (bare) cosmological constant, it may be considered more philological than physical. The point is that the bare parameters in quantum field theory are not directly observable, and the bare cosmological constant should not be an exception to this rule.

It is anticipated that future investigations of the nature of dark energy will shed new light on the fundamental problem of quantum vacuum.

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