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On Sliced Spaces: Global Hyperbolicity Revisited

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Received: 24 January 2019; Accepted: 26 February 2019; Published: 1 March 2019

Abstract: We give a topological condition for a generic sliced space to be globally hyperbolic without any hypothesis on lapse function, shift function, and spatial metric.

Keywords: sliced space; product topology; Alexandrov topology; global hyperbolicity

1. Preliminaries

The definition of a sliced space, which one can read in Reference [1], is a continuation of a study in References [2] and [3] on systems of Einstein equations.

Let $V = M \times I$, where M is an n-dimensional smooth manifold, and I is an interval of the real line, $\mathbb{R}$. We equip V with a $n+1$-dimensional Lorentz metric g, which splits in the following way:

$$g = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j,$$

where $\theta^0 = dt$, $\theta^i = dx^i + \beta^i dt$, $N = N(t, x^i)$ is the *lapse function*, $\beta^i(t, x^j)$ is the shift function and $M_t = M \times \{t\}$, spatial slices of V, are spacelike submanifolds equipped with the time-dependent spatial metric $g_t = g_{ij}dx^i dx^j$. Such product space V is called a sliced space.

Throughout the paper, we consider $I = \mathbb{R}$.

The author in Reference [1] considered sliced spaces with uniformly bounded lapse, shift, and spatial metric; by this hypothesis, it is ensured that parameter t measures up to a positive factor bounded (below and above) the time along the normals to spacelike slices M_t, the g_t norm of the shift vector β is uniformly bounded by a number, and the time-dependent metric $g_{ij}dx^i dx^j$ is uniformly bounded (below and above) for all $t \in I(= \mathbb{R})$, respectively.

Given the above hypothesis, in the same article, the following theorem was proved.

Theorem 1 (Cotsakis)**.** *Let (V, g) be a sliced space with uniformly bounded lapse N, shift β and spatial metric g_t. Then, the following are equivalent:*

1. *(M_0, γ) a complete Riemannian manifold.*
2. *Spacetime (V, g) is globally hyperbolic.*

In this article, we review global hyperbolicity of sliced spaces in terms of the product topology defined on space $M \times \mathbb{R}$ for some finite dimensional smooth manifold M.

2. Strong Causality of Sliced Spaces

Let $(V = M \times \mathbb{R}, g)$ be a sliced space. Consider product topology T_P on V. Since M is finite-dimensional, a base for T_P consists of all sets of form $A \times B$, where $A \in T_M$ and $B \in T_{\mathbb{R}}$.

Here, T_M denotes the natural topology of manifold M where, for an appropriate Riemann metric h, it has a base consisting of open balls $B^h_\epsilon(x)$, and $T_{\mathbb{R}}$ is the usual topology on the real line, with a base consisting of open intervals (a, b). For trivial topological reasons, we can restrict our discussion on T_P to basic-open sets $B^h_\epsilon(x) \times (a, b)$, which can intuitively be called "open cylinders" in V.

We remind that the Alexandrov topology T_A (see Reference [4]) has a base consisting of open sets of the form $< x, y >= I^+(x) \cap I^-(y)$, where $I^+(x) = \{z \in V : x \ll z\}$ and $I^-(y) = \{z \in V : z \ll y\}$, where $\ll$ is the chronological order defined as $x \ll y$ iff there exists a future-oriented timelike curve joining x with y. By $J^+(x)$, one denotes the topological closure of $I^+(x)$, and by $J^-(y)$ that one of $I^-(y)$.

We use the definition of global hyperbolicity from Reference [4], where one can read about global causality conditions in more detail, as well as characterizations for strong causality. In particular, a spacetime is strongly causal iff it possesses no closed timelike curves, and global hyperbolicity is an important causal condition in a spacetime related to major problems such as spacetime singularities and cosmic cencorship.

Definition 1. *A spacetime is globally hyperbolic iff it is strongly causal and the "causal diamonds" $J^+(x) \cap J^-(y)$ are compact.*

We prove the following theorem:

Theorem 2. *Let (V, g) be a Hausdorff sliced space. Then, the following are equivalent.*

1. *V is strongly causal.*
2. *$T_A \equiv T_P$.*
3. *T_A is Hausdorff.*

Proof. Here, 2. implies 3. is obvious and that 3. implies 1. can be found in Reference [4].

For 1. implies 2., we consider two events $X, Y \in V$, such that $X \neq Y$; we note that each $X \in V$ has two coordinates, say (x_1, x_2), where $x_1 \in M$ and $x_2 \in \mathbb{R}$. Obviously, $X \in M_x = M \times \{x\}$ and $Y \in M_y = M \times \{y\}$. Then, $< X, Y >= I^+(X) \cap I^-(Y) \in T_A$. Let also $A \in M_a = M \times \{a\}$, where $a < x$ ($<$ is the natural order on $\mathbb{R}$) and $B \in M_b = M \times \{b\}$, where $y < b$. Consider some $\epsilon > 0$, such that $B^h_\epsilon(A) \in M$. Obviously, $B^h_\epsilon(A) \times (a, b) \in T_P$ and, for $\epsilon > 0$ sufficiently large enough, $< X, Y >\subset B^h_\epsilon(A) \times (a, b)$. Thus, $< X, Y >\in T_P$.

For 2. implies 1., we consider $\epsilon > 0$, such that $B^h_\epsilon(A) \in T_M$, so that $B^h_\epsilon(A) \times (a, b) = B \in T_P$. We let strong causality hold at an event P and consider $P \in B \in T_P$. We show that there exists $< X, Y >\in T_A$, such that $P \in< X, Y >\subset B$. Now, consider a simple region R in $< X, Y >$ which contains P and $P \in Q$, where Q is a causally convex-open subset of R. Thus, we have $U, V \in Q$, such that $P \in< U, V >\subset Q$. Finally, $P \in< U, V >\subset Q \subset B$, and this completes the proof. □

3. Global Hyperbolicity of Sliced Spaces, Revisited

For the following theorem, we use Nash's result that refers to finite-dimensional manifolds (see Reference [5]).

Theorem 3. *Let (V, g) be a Hausdorff sliced space, where $V = M \times R$, M is an n-dimensional manifold and g the $n + 1$ Lorentz metric in V. Then, (V, g) is globally hyperbolic iff $T_P = T_A$, in V.*

Proof. Given the proof of Theorem 2, strong causality in V holds iff $T_P = T_A$ and, given Nash's theorem, the closure of $B^h_\epsilon(x) \times (a, b)$ is compact. □

We note that neither in Theorem 2 nor in Theorem 3 did we make any hypothesis on the lapse function, shift function, or spatial metric.

Author Contributions: Conceptualization, K.P. and N.K.; methodology, K.P., N.K. and B.K.P.; investigation, K.P. and N.K.; resources, K.P.; writing—original draft preparation, K.P.; writing—review and editing, K.P.; supervision, K.P. and B.K.P.; project administration, K.P. and N.K.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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