symmetry

Article

The Indirect-Utility Criterion for Ranking Opportunity Sets over Time

María Dolores García-Sanz * **and José Carlos Rodríguez Alcantud**

BORDA Research Unit and Multidisciplinary Institute of Enterprise (IME), University of Salamanca, E37007 Salamanca, Spain; jcr@usal.es

* Correspondence: dgarcia@usal.es; Tel.: +34-663064001

Received: 31 January 2019; Accepted: 13 February 2019; Published: 15 February 2019

Abstract: A preference defined on a set of alternatives can be extended to a preference on the subsets of alternatives (named opportunity sets) in different ways. We specifically consider the application of the indirect-utility (IU) criterion in various stages, when both the alternatives and the preferences can change over time. In other words, we maintain the symmetry over time as far as criteria are concerned, but neither in the preferences, nor in the alternatives. We characterize this criterion by three testable axioms. Our study bears comparison with Krause (Economic Theory, 2008) for the two-period model.

Keywords: opportunity sets; ranking; preference; indirect-utility (IU); lexicographical order

---

1. Introduction

In many decision problems, the final choice is made after a sequential process. A selection of a subset of the alternatives from the universal set of options precedes the final decision. For example, a hiring committee usually discards some of the candidates before making the final decision with a comprehensive inspection of a part of all submitted CVs. Another simple example where an agent ranks subsets of alternatives, already cited by [1], concerns menus or restaurants. The customer eventually orders a meal, but she has first to select a menu from which she will later choose her meal. Some other contexts where these problems appear include voting situations (the selection of a committee), matchings and assignments (admission of sets of students in colleges, hiring of several workers, etc.), or coalition formation (an agent assigns values to the feasible subsets of colleagues to form a coalition).

In this framework, our paper concerns the problem of ranking subsets of alternatives, also called “opportunity sets”.

The standard solution to this model considers an agent’s preference ordering on the set of alternatives and then extends it to a ranking of the collection of its nonempty subsets. The rules governing this process consist of axioms that are deemed suitable in the respective context. Furthermore, in all of these situations, the elements to rank are opportunity sets (or menus of feasible alternatives).

It is in this context that we examine the classical “indirect-utility criterion” (IU criterion). This rule is suitable when the quality of the final choice is all that matters, that is to say, those subsets with better top elements are preferred.

Our approach in this work is in line with the fundamental trend that determines the ranking of the subsets by only looking at their best alternatives. In this sense, our model adheres to the principle of limited rationality, so that the agent concentrates on some focal alternatives: the subset formed by the best elements. Indeed, this model is the germ of the indirect-utility approach characterized in [1].

Moreover, agents often make decisions over time. The successive sets of alternatives and criteria may either be fixed or evolve. When the set of options does not vary, we can order the subsets by looking at different attributes and apply them separately instead of sequentially to the same subsets (see, e.g., [2]). However, here, we are concerned with choices in a finite number of stages, and we do not discard the possibility of changes.

We define a ranking for tuples of ordered subsets (elements of a direct product), the i^{th} element being extracted from the set of alternatives available in the i^{th} period. This overall ranking is defined using the indirect-utility criterion (put succinctly, $A \succcurlyeq B \Leftrightarrow \max(A) R \max(B)$ for any subsets of alternatives A, B) in each period and with a lexicographic order. Fishburn [3] (see also [4]) characterized lexicographic preferences. Fishburn considered a condition that restricts tradeoffs among different factors, which had also been used in the same sense by [5]. Nevertheless, we do not aim at characterization of lexicographic preferences, but we focus on a specific criterion with a lexicographic nature instead; nor are we concerned with ranking opportunity sets by the lexicographic application of different criteria, but with ranking opportunity sets in different stages by an overall IU criterion associated with the corresponding preference on the alternatives in each period. This unique criterion has a preference for an earlier period over a later one.

This question has already been studied in [6] for the case of two periods. This setting ("present" vs. "future") is sufficient for creating a role for discounting. The IU criterion is applied to the alternatives in the first period, and only in the case of ties at the top, to the second. Krause characterized this criterion using five axioms, although three of them are dominance-style axioms. The notation there is slightly different from ours, and it is supposed that for all the subsets, "it is natural" to assume its equivalence with a two-element subset (the subset containing the best elements of the set in each period), because he was concerned only with the indirect utility context. For this reason, Krause only expressed the axioms for the two-element sets and used a complete quasi-ordering defined over the set of alternatives.

By contrast, we characterize our criterion with only three axioms, that are different in nature to the three axioms that [6] uses dominance, time discounting, and neutrality. Moreover in our model, the preference relation on the sets of alternatives is not fixed. This feature is in line with [1], and clearly separates our approximation from the analysis in [6].

Relatedly, we recall that the IU criterion is not universally accepted for ranking opportunity sets in a freedom of choice framework. In order to represent the value of freedom of choice, the criterion based on the number of alternatives of the subsets is an extreme case (see, for example, [7]), although there are of course other approaches [8]. A variety of criteria provide a compromise solution between extreme positions; a survey of the responses to the problem is [9]. However, more recently, a criterion using the IU and a lexicographic ranking that takes into account the number of best elements in each subset is characterized in [10,11]. Hence, not only is it important to have best elements in a set, but also to have more than others. In the case of ties, the "second best" elements are tie-breakers, and so on.

The rest of the article is organized as follows. In Section 2, we introduce the notation and some preliminary concepts. Section 3 gathers the axioms and the main results, and Section 4 is devoted to some final conclusions.

2. Notation and Preliminaries

We denote by X a finite set of objects or alternatives, and by 2^X the set of nonempty subsets of X. A binary relation on X, $R \subseteq X \times X$, captures the preference relation of an agent. The notation xRy is a shorthand for $(x, y) \in R$, and we interpret xRy if and only if the element $x \in X$ is considered at least as good as the element $y \in X$. Then, P and I respectively stand for the asymmetric and the symmetric parts of R. Moreover, R is a complete preorder if it verifies reflexivity, transitivity, and completeness.

We deal with the problem of ranking opportunity sets, that is ranking subsets of the set of alternatives X. More precisely, we investigate the criterion that ranks these subsets on the basis of their best elements with respect to an underlying ranking of alternatives. The next definitions fix these concepts.

Definition 1. *Given a total preorder R on a finite set X, a best element for R is an element $x \in X$ such that* xRx' for all $x' \in X$.

Due to the finiteness assumption, the best element exists, but it may not be unique. Uniqueness is guaranteed when the ordering of X is linear.

Definition 2. *Let X be a nonempty set of alternatives and R a preference ordering on X. The indirect-utility criterion $\succcurlyeq_U$ on textcolorblack2^X associated with R is defined by letting $A \succcurlyeq_U B \Leftrightarrow \max(A) R \max(B)$, for each $A, B \in 2^X$. Here, $\max(X')$ stands for a best element of a subset $X' \subseteq X$.*

Observe that the properties of R ensure that this criterion is well defined: either $\max(A) R \max(B)$ is true whatever the choice of elements in $\max(A), \max(B)$ or it is always false.

Conversely, if we have a complete preorder $\succcurlyeq$ on *textcolorblack*2^X, then we can naturally define a complete preorder R on X as $xRy \Leftrightarrow \{x\} \succcurlyeq \{y\}$.

Moreover, it is usual for a decision maker to make choices across time and to have different sets of alternatives or to use different criteria in each moment. Citing from [6], "in reality agents typically make a sequence of choices over time from a corresponding sequence of opportunity sets, rather than a single once-and-for-all choice from a single opportunity set". In order to study this situation, we consider $\{(X_i, R_i),\ i = 1, \ldots, n\}$, where X_i is the finite and nonempty set of alternatives available in time i, and R_i is a complete preorder defined on X_i. The space of alternatives is $\mathcal{D} = 2^{X_1} \times \ldots \times 2^{X_n}$, and $\succcurlyeq$ is a complete preorder defined on $\mathcal{D}$. When we want to pinpoint period i, the sequence $(A_1, \ldots, A_n) \in \mathcal{D}$ is denoted by (A_i, A_{-i}). Similarly, $\mathcal{D}_{-i} = 2^{X_1} \times \ldots \times 2^{X_{i-1}} \times 2^{X_{i+1}} \times \ldots \times 2^{X_n}$.

Our goal is to study the relationship among $R_1, R_2, \ldots, R_n$ and $\succcurlyeq$ in a remarkable case. The next section is devoted to this model.

3. The Indirect-Utility Criterion over Time

We now formalize the criterion consisting of the lexicographical application of the indirect-utility criterion on $\mathcal{D}$ associated with the complete preorders $R_1, \ldots, R_n$. They are respectively defined on $X_1, \ldots, X_n$, the successive sets of alternatives. As in the previous section, for each subset $X_i' \subseteq X_i$, the operator max gives a selection $\max(X_i') \in X_i'$ of a best element in X_i'.

Definition 3. *The indirect-utility criterion over time is defined by: for any $A_i, B_i \in 2^{X_i}$ ($i = 1, \ldots, n$), we denote $(A_1, \ldots, A_n) \succcurlyeq_U (B_1, \ldots, B_n)$ if and only if:*

- *either $\max(A_i) I_i \max(B_i)$ for all $i = 1, \ldots, n$, or*
- *there exists $j \in N = \{1, \ldots, n\}$ such that $\max(A_i) I_i \max(B_i)$ for all $i = 1, \ldots, j-1$ and $\max(A_j) P_j \max(B_j)$.*

The fact that the R_i's are complete preorders assures that the choice of best elements by max does not affect the definition of $\succcurlyeq_U$. This criterion satisfies Axioms 1 and 2 below, which have important implications on the structure under inspection. They refer to sequences that only differ in the opportunity set that is offered in one of the periods.

Axiom 1 (A1)**.** *For any pair $(A_i, A_{-i}), (B_i, A_{-i}) \in \mathcal{D}$ of sequences that differ only in state i, if $(B_i, A_{-i}) \succcurlyeq (A_i, A_{-i})$, then it must be the case that $(B_i, A'_{-i}) \succcurlyeq (A_i, A'_{-i})$ for all $A'_{-i} \in \mathcal{D}_{-i}$.*

Axiom 1 is similar in spirit to independence-style axioms in the literature. The intuition underlying our axiom is as follows. Suppose that with respect to fixed situation, replacing the opportunity set A_i in period i by another opportunity set B_i produces a sequence that is not worse off. Then in any situation where A_i appears in period i, replacing it with the opportunity set B_i produces a sequence that is not worse off either.

Our next axiom extends a condition from [1] to an intertemporal setting. Kreps [1] provides a characterization of the indirect-utility criterion on the set of subsets of alternatives in a model where the preference ordering on the set of alternatives is not fixed. He used an extended-robustness condition that establishes that adding less interesting alternatives to the possible choices of the DM does not make him worse off. In our framework, we state the following variation:

Axiom 2 (A2). *Extended robustness.*
If $(A_i, A_{-i}) \in \mathcal{D}$ *and* $A'_i \subseteq X_i$*, then:*

$$(A_i, A_{-i}) \succcurlyeq (A'_i, A_{-i}) \Rightarrow (A_i, A_{-i}) \sim (A_i \cup A'_i, A_{-i})$$

Now, we proceed to establish a proposition that is crucial for the subsequent analysis. It supposes a key difference with the approximation in [6].

Proposition 1. *Let* $\succcurlyeq$ *be a complete preorder on* $\mathcal{D}$ *that satisfies Axioms A1 and A2. Then, for any* $(A_1, \dots, A_n) \in \mathcal{D}$*, there exist* $a_1 \in A_1, \dots, a_n \in A_n$ *such that* $(A_1, \dots, A_n) \sim (\{a_1\}, \dots, \{a_n\})$*. Moreover, each* $a_i \in A_i$ *satisfies that* $(\{a_i\}, A'_{-i}) \succcurlyeq (\{a'_i\}, A'_{-i})$ *for all* $A'_j \subseteq X_j,\ j \in N \setminus \{i\}$*, and for all* $a'_i \in A_i$*.*

Proof. Let us select an element $(A_1, \dots, A_n) \in \mathcal{D}$ and let us suppose $A_1 = \{a^1_1, a^1_2, \dots, a^1_{p_1}\}$. We proceed by induction. Because $\succcurlyeq$ is complete, we can suppose without loss of generality that:

$$(\{a^1_1\}, A_{-1}) \succcurlyeq (\{a^1_2\}, A_{-1}) \succcurlyeq \dots \succcurlyeq (\{a^1_{p_1}\}, A_{-1})$$

The application of Axiom A2 and then transitivity yields:

$$(\{a^1_1\}, A_{-1}) \sim (\{a^1_1, a^1_2\}, A_{-1}) \sim \dots \sim (\{a^1_1, \dots, a^1_{p_1}\}, A_{-1})$$

therefore

$$(\{a^1_1\}, A_{-1}) \sim (A_1, \dots, A_n)$$

Let us now suppose (induction hypothesis) that:

$$(\{a^1_1\}, \{a^2_1\}, \dots, \{a^{i-1}_1\}, A_i, \dots, A_n) \sim (A_1, \dots, A_n)$$

and $A_i = \{a^i_1, \dots, a^i_{p_i}\}$.
We can consider:

$$(\{a^1_1\}, \dots, \{a^i_1\}, A_{i+1}, \dots, A_n) \succcurlyeq (\{a^1_1\}, \dots, \{a^i_2\}, A_{i+1}, \dots, A_n) \succcurlyeq$$

$$\succcurlyeq \dots \succcurlyeq (\{a^1_1\}, \dots, \{a^i_{p_i}\}, A_{i+1}, \dots, A_n)$$

Applying transitivity and then Axiom A2, we obtain:

$$(\{a^1_1\}, \{a^2_1\}, \dots, \{a^i_1\}, A_{i+1}, \dots, A_n) \sim (A_1, \dots, A_n).$$

We conclude $(A_1, \dots, A_n) \sim (\{a^1_1\}, \dots, \{a^n_1\})$ by the recursive argument.
This means that the elements $a_1 = a^1_1, \dots, a_n = a^n_1$ satisfy our first claim.

The second claim has been established above for the first period and the sets $A_2, \dots, A_n$. Axiom A1 extends it for any subsets $A'_2 \subseteq X_2, \dots, A'_n \subseteq X_n$.

Proceeding in the same way for each $A_i = \{a^i_1, \dots, a^i_{p_i}\}$, the proof is concluded. □

In addition to the properties above, the ranking given in Definition 3 satisfies the next axiom (impatience or A3).

Axiom 3 (A3). *For any pair $(A_i, A_{-i}), (A'_i, A_{-i}) \in \mathcal{D}$ with an only state being different,*

$$(A_i, A_{-i}) \succ (A'_i, A_{-i}) \Rightarrow (A_1, \ldots, A_i, \ldots, A_n) \succ (A_1, \ldots, A_{i-1}, A'_i, A'_{i+1}, \ldots, A'_n)$$

for all $A'_{i+1} \subseteq X_{i+1}, \ldots, A'_n \subseteq X_n$.

The interpretation of A3 is as follows. Suppose that with respect to an initial situation in $\mathcal{D}$, the opportunity set in moment i changes (from A_i to A'_i), and as a consequence, the decision-maker is worse off. Then she has a strict preference for the initial sequence when the opportunity sets that she faces after that period change too. Axiom 3 expresses a dictatorship of any period over its future. This is exactly the spirit of a lexicographic criterion: it is meant to enforce a very strong preference for immediate rewards. This behavior is therefore incompatible with the idea that future benefits can compensate losses at the present as in the well-known discounted utility model (see e.g., [12] for a technical explanation of its basic features). Albeit discounted sums of utilities yield analytical clarity, they are not universally accepted in Economics. Chichilnisky [13] (Section 2) is a lucid critical account of the pros and cons of this criterion in growth theory or sustainable development. Observe in addition that neither Axiom 3 nor the indirect-utility criterion impose a dictatorship by any period. Formal expressions of dictatorial behaviors appear e.g., in [13,14] in an intertemporal framework with an infinite horizon, or in [15] in social welfare judgements in a finite society.

We are now ready to characterize the indirect-utility criterion over time:

Theorem 1. *A complete preorder $\succcurlyeq$ on $\mathcal{D}$ satisfies A1, A2, and A3 if and only if there exist complete preorders R_i on X_i ($i = 1, \ldots, n$) such that $\succcurlyeq = \succcurlyeq_U$, i.e., $\succcurlyeq$ is the indirect-utility criterion associated with $R_1, \ldots, R_n$.*

Proof. We have already mentioned that the sufficient condition is easy to check, i.e., the ranking $\succcurlyeq_U$ in Definition 3 satisfies Axioms A1, A2, and A3.

Let us now prove that if $\succcurlyeq$ is a complete preorder on $\mathcal{D}$ satisfying Axioms A1, A2, and A3, then there exist complete preorders R_i on $X_i, i = 1, \ldots, n$, and such that $\succcurlyeq = \succcurlyeq_U$.

Given $\succcurlyeq$ on $\mathcal{D}$, we define R_i on X_i in the following way: for all $x_i, y_i \in X_i$,

$$x_i R_i y_i \Leftrightarrow (\{x_i\}, A_{-i}) \succcurlyeq (\{y_i\}, A_{-i}) \text{ for every } A_{-i} \in \mathcal{D}_{-i}$$

By virtue of Axiom A1, R_i can be defined by the alternative expression $x_i R_i y_i$ if and only if there exists $A_{-i} \in \mathcal{D}_{-i}$ such that $(\{x_i\}, A_{-i}) \succcurlyeq (\{y_i\}, A_{-i})$.

Now, let us denote by $\succcurlyeq_U$ the total preorder on $\mathcal{D}$ associated with these $R_1, \ldots, R_n$ according to Definition 3. We proceed to prove that $\succcurlyeq_U = \succcurlyeq$.

Proposition 1 establishes that for each $(A_1, \ldots, A_n) \in \mathcal{D}$, there exist elements $a_i \in A_i$, for all $i = 1, \ldots, n$ such that:

$$\begin{aligned} &(A_1, \ldots, A_n) \sim (\{a_1\}, \ldots, \{a_n\}) \\ &(\{a_i\}, A_{-i}) \succcurlyeq (\{a'_i\}, A_{-i}),\ \forall a'_i \in A_i,\ \forall A_j \subseteq X_j, j \in N \setminus \{i\} \end{aligned} \tag{1}$$

Let us prove that it is also true that $(A_1, \ldots, A_n) \sim_U (\{a_1\}, \ldots, \{a_n\})$.

Indeed, we know that $\succcurlyeq_U$ satisfies Axioms A1 and A2; thus, there must exist $\bar{a}_i \in A_i, i = 1, \ldots, n$, such that $(A_1, \ldots, A_n) \sim_U (\{\bar{a}_1\}, \ldots, \{\bar{a}_n\})$ and:

$$(\{\bar{a}_i\}, A_{-i}) \succcurlyeq_U (\{a'_i\}, A_{-i}),\ \forall a'_i \in A_i \text{ and } \forall A_j \subseteq X_j,\ j \in N \setminus \{i\}$$

Taking $i = 1$ and $a'_1 = a_1$

$$(\{\bar{a}_1\}, A_{-1}) \succcurlyeq_U (\{a_1\}, A_{-1}),\ \forall A_j \subseteq X_j,\ j \in N \setminus \{1\}$$

Applying the definition of $\succcurlyeq_U$, we have that $\bar{a}_1 P_1 a_1$ or $\bar{a}_1 I_1 a_1$. However, $\bar{a}_1 P_1 a_1 \Leftrightarrow (\{\bar{a}_1\}, A_{-1}) \succ (\{a_1\}, A_{-1})$, which is impossible because of (1), and therefore, $\bar{a}_1 I_1 a_1$, which leads to $(\{\bar{a}_1\}, A_{-1}) \sim_U (\{a_1\}, A_{-1})$.

A similar argument applied to:

$$(\{\bar{a}_1\}, \{\bar{a}_2\}, A_3, \ldots, A_n) \succcurlyeq_U (\{\bar{a}_1\}, \{a_2\}, A_3, \ldots, A_n)$$

leads to $(\{\bar{a}_1\}, \{\bar{a}_2\}, A_3, \ldots, A_n) \sim_U (\{\bar{a}_1\}, \{a_2\}, A_3, \ldots, A_n)$.

An induction argument and the transitivity of $\sim_U$ justifies the assertion.

Let us now consider the elements $(A_1, \ldots, A_n) \sim (\{a_1\}, \ldots, \{a_n\})$ and $(A'_1, \ldots, A'_n) \sim (\{a'_1\}, \ldots, \{a'_n\})$ such that $(A_1, \ldots, A_n) \succcurlyeq_U (A'_1, \ldots, A'_n)$. We must prove that $(A_1, \ldots, A_n) \succcurlyeq (A'_1, \ldots, A'_n)$.

There are two possibilities:

1. There exists $i \in N$ such that $a_j I_j a'_j$ for all $j = 1, \ldots, i-1$ and $a_i P_i a'_i$.

   An application of the definition of R_i produces $(\{a_i\}, A_{-i}) \succ (\{a'_i\}, A_{-i})$ for all $A_j \subseteq X_j$, $j \in N \setminus \{i\}$, and in particular:

   $$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a_1\}, \ldots, \{a'_i\}, \ldots, \{a_n\}).$$

   Axiom A3 leads to:

   $$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a_1\}, \ldots, \{a'_i\}, A'_{i+1}, \ldots, A'_n)$$

   for all $A'_j \subseteq X_j, j = i+1, \ldots, n$.

   Taking $A'_j = \{a'_j\}, \forall j = i+1, \ldots, n$, we conclude:

   $$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a_1\}, \ldots, \{a_{i-1}\}, \{a'_i\}, \ldots, \{a'_n\}).$$

   From $a_j I_j a'_j$ for all $j = 1, \ldots, i-1$, and the definition of $R_1, \ldots, R_{i-1}$, we have that:

   $$(\{a_1\}, \ldots, \{a_{i-1}\}, \{a'_i\}, \ldots, \{a'_n\}) \sim (\{a'_1\}, \ldots, \{a'_n\})$$

   and transitivity leads to:

   $$(\{a_1\}, \ldots, \{a_n\}) \succ (\{a'_1\}, \ldots, \{a'_n\}).$$

2. For all $i = 1, \ldots, n$, $a_i I_i a'_i$.

   In this case, we have $(\{a_1\}, \ldots \{a_n\}) \sim (\{a'_1\}, \ldots, \{a'_n\})$, thus, we conclude.

To end the proof, we have to justify that for all $(A_1, \ldots, A_n), (A'_1, \ldots, A'_n) \in \mathcal{D}$ such that $(A_1, \ldots, A_n) \succcurlyeq (A'_1, \ldots, A'_n)$, it must be the case that $(A_1, \ldots, A_n) \succcurlyeq_U (A'_1, \ldots, A'_n)$.

Assume that $(\{a'_1\}, \ldots, \{a'_n\}) \succ_U (\{a_1\}, \ldots, \{a_n\})$ by way of contradiction, then we have that there exists $i \in \{1, \ldots, n\}$ such that $a'_j I_j a_j$ for all $j = 1, \ldots, i-1$ and $a'_i P_i a_i$. Therefore:

$$(\{a'_1\}, \ldots, \{a'_i\}, A_{i+1}, \ldots, A_n) \succ (\{a'_1\}, \ldots, \{a'_{i-1}\}, \{a_i\}, A_{i+1}, \ldots, A_n)$$

for all $A_j \subseteq X_j,\ j = i+1, \ldots, n$.

In particular:

$$(\{a'_1\}, \ldots, \{a'_n\}) \succ (\{a'_1\}, \ldots, \{a_i\}, \ldots, \{a'_n\})$$

and Axiom A3 ensures:

$$(\{a'_1\},\ldots,\{a'_n\}) \succ (\{a'_1\},\ldots,\{a'_{i-1}\},\{a_i\},\ldots,\{a_n\})$$

However, $a'_j I_j a_j$ for all $j = 1,\ldots,i-1$, leads to:

$$(\{a'_1\},\ldots,\{a'_{i-1}\},\{a_i\},\ldots,\{a_n\}) \sim (\{a_1\},\ldots,\{a_n\})$$

and then:

$$(\{a'_1\},\ldots,\{a'_n\}) \succ (\{a_1\},\ldots,\{a_n\})$$

against the hypothesis. □

Remark 1. *As a particular case, we obtain the result in [6].*

In order to justify that our characterization is tight, now we present three examples that prove the independence of the axioms in Theorem 1. We begin with a complete preorder satisfying A1 and A3, but not A2.

Example 1. *Given* $A = (A_1,\ldots,A_n), B = (B_1,\ldots,B_n) \in 2^{X_1} \times \ldots \times 2^{X_n}$:

$$A \succcurlyeq B \Leftrightarrow \begin{cases} (i)\ |A_i| = |B_i|, \quad \forall i = 1,\ldots,n, \text{ or} \\ (ii)\ \exists j \in \{1,\ldots,n\} \text{ such that } |A_i| = |B_i|\ \forall i = 1,\ldots,j-1 \\ \ \text{and } |A_j| > |B_j| \end{cases}$$

Then, $\succcurlyeq$ satisfies A1 and A3, but not A2. ◁

The next example gives a complete preorder that contradicts A1, but satisfies A2 and A3.

Example 2. *Let us consider a complete preorder R defined on X and the ranking on $\mathcal{D} = 2^X \times \ldots \times 2^X$ defined in such a way that for any $A = (A_1,\ldots,A_n), B = (B_1,\ldots,B_n) \in \mathcal{D}$:*

(i) $A \succ B \Leftrightarrow \exists j \in \{1,\ldots,n\}$ *such that* $\max\left(\cup_{k=1}^{i} A_k\right) I \left(\cup_{k=1}^{i} B_k\right)$,
$\forall i = 1,\ldots,j-1$, *and* $\max(A_1 \cup \ldots \cup A_j) P \max(B_1 \cup \ldots \cup B_j)$
(ii) $A \sim B \Leftrightarrow \max(A_1 \cup \ldots \cup A_i) I \max(B_1 \cup \ldots B_i)$ *for all* $i = 1,\ldots,n$.

It is immediate to prove that this ranking satisfies A3.
A2 is satisfied because if:

$$(A_1,\ldots,A_i,\ldots,A_n) \succcurlyeq (A_1,\ldots,A'_i,\ldots,A_n)$$

we have that:

$$\max(A_1 \cup \ldots \cup A_i) R \max(A_1 \cup \ldots \cup A'_i)$$

and then:

$$\max(A_1 \cup \ldots \cup A_i) I \max(A_1 \cup \ldots \cup A_i \cup A'_i)$$

which leads to:

$$(A_1,\ldots,A_i,\ldots,A_n) \sim (A_1,\ldots,A_i \cup A'_i,\ldots,A_n)$$

In order to prove that A1 is not satisfied, let us take $a_1, a_2, b_1, b_2 \in X$ such that $a_1 P b_2 P b_1 P a_2$. We have:

$$(\{a_1\},\{a_2\},A_3,\ldots,A_n) \succcurlyeq (\{a_1\},\{b_2\},A_3,\ldots,A_n)$$

but:

$$(\{b_1\},\{b_2\},A_3,\ldots,A_n) \succ (\{b_1\},\{a_2\},A_3,\ldots,A_n)$$

because $\max(\{b_1\} \cup \{b_2\}) = b_2 P \max(\{b_1\} \cup \{a_2\}) = b_1$. $\triangleleft$

Finally, Example 3 is very simple. It satisfies A1 and A2, but not A3.

Example 3. *Let us fix a complete preorder R defined on X and then define the associated ranking* $\succcurlyeq$ *on* $2^X \times \ldots \times 2^X$ *as:*

$$(A_1, \ldots, A_n) \succcurlyeq (B_1, \ldots, B_n) \Leftrightarrow \max(A_1 \cup \ldots \cup A_n) R \max(B_1 \cup \ldots \cup B_n)$$

4. Conclusions

We have approached the problem of ranking opportunity sets (i.e., subsets of a finite set of alternatives) over time. Situations where criteria and/or sets of alternatives change along several periods, and the decision-maker has to choose time after time, abound in real life. In this framework, we have considered different complete preorders defined on the corresponding sets of alternatives for a finite number of periods. Then, the indirect-utility criterion ranks the subsets in each stage (the decision-makers are supposed to focus on the best elements) and only evaluates a period when the previous ones give ties. Formally speaking, this is a lexicographic ordering.

We have characterized this natural criterion with three testable axioms. The complete preorders that justify this rule are endogenously derived from these principles. This feature distinguishes our approach inspired by [6], whose focus is on the economic interest of the results rather than technical efficiency.Krause's approach is normative: a combination of his discounting axiom with other standard and plausible axioms in the literature (dominance and neutrality) leads to a lexicographic order of opportunity sets.

The possibility of extending these arguments to some other criteria of ranking opportunity sets applied over time deserves further consideration in the future. It may also be possible to dwell on the extended problem with an infinitely long horizon (see [16] for the motivation and related literature).

Author Contributions: Formal analysis, M.D.G.-S. and J.C.R.A.; writing, original draft, M.D.G.-S.; writing, review and editing, J.C.R.A.

Funding: This research received no external funding.

Acknowledgments: The authors are grateful to the anonymous referees for their insightful comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kreps, D.M. A representation theorem for preference for flexibility. *Econometrica* **1979**, *47*, 565–577. [CrossRef]
2. García-Sanz, M.D.; Alcantud, J.C.R. Sequential rationalization of multivalued choice. *Math. Soc. Sci.* **2015**, *74*, 29–33. [CrossRef]
3. Fishburn, P.C. Axioms for lexicographic preferences. *Rev. Econ. Stud.* **1975**, *42*, 415–419. [CrossRef]
4. Mitra, M.; Sen, D. An alternative proof of Fishburn's axiomatization of lexicographic preferences. *Econ. Lett.* **2014**, *124*, 168–170. [CrossRef]
5. Petri, H.; Voorneveld, M. Characterizing lexicographic preferences. *J. Math. Econ.* **2016**, *63*, 54–61. [CrossRef]
6. Krause, A. Ranking opportunity sets in a simple intertemporal framework. *Econ. Theor.* **2008**, *35*, 147–154. [CrossRef]
7. Pattanaik, P.; Xu, Y. On ranking opportunity sets in economic environments. *J. Econ. Theor.* **2000**, *93*, 48–71. [CrossRef]
8. Alcantud, J.C.R.; Arlegi, P. An axiomatic analysis of ranking sets under simple categorization. *SERIEs* **2012**, *3*, 227–245. [CrossRef]
9. Barberá, S.; Bossert, W.; Pattanaik, P.K. Ranking preferences to a set of objects. In *Handbook of Utility Theory*; Barberá, S., Hammond, P., Seidl, C., Eds.; Kluwer Academic Press: Boston, MA, USA, 2004; Volume II, pp. 893–977.

10. Vázquez, C. Ranking opportunity sets on the basis of similarities of preferences: A proposal. *Math. Soc. Sci.* **2014**, *67*, 23–26. [CrossRef]
11. Lauwers, L.; Potoms, T.; Vázquez, C. Addendum to: 'Ranking opportunity sets on the basis of similarities of preferences: A proposal'. *Math. Soc. Sci.* **2017**, *88*, 1–2. [CrossRef]
12. Alcantud, J.C.R.; Muñoz Torrecillas, M.J. Intertemporal choice of fuzzy soft sets. *Symmetry* **2018**, *10*, 371. [CrossRef]
13. Chichilnisky, G. An axiomatic approach to sustainable development. *Soc. Choice Welf.* **1996**, *13*, 231–257. [CrossRef]
14. Alcantud, J.C.R. Liberal approaches to ranking infinite utility streams: When can we avoid interference? *Soc. Choice Welf.* **2013**, *41*, 381–396. [CrossRef]
15. Mariotti, M.; Veneziani, R. On the impossibility of complete non-interference in Paretian social judgements. *J. Econ. Theor.* **2013**, *148*, 1689–1699. [CrossRef]
16. Alcantud, J.C.R.; García-Sanz, M.D. Paretian evaluation of infinite utility streams: An egalitarian criterion. *Econ. Lett.* **2010**, *106*, 209–211. [CrossRef]