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The Second-Order Correction to the Energy and Momentum in Plane Symmetric Gravitational Waves Like Spacetimes

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Abstract: This research provides second-order approximate Noether symmetries of geodetic Lagrangian of time-conformal plane symmetric spacetime. A time-conformal factor is of the form $e^{\epsilon f(t)}$ which perturbs the plane symmetric static spacetime, where ϵ is small a positive parameter that produces perturbation in the spacetime. By considering the perturbation up to second-order in ϵ in plane symmetric spacetime, we find the second order approximate Noether symmetries for the corresponding Lagrangian. Using Noether theorem, the corresponding second order approximate conservation laws are investigated for plane symmetric gravitational waves like spacetimes. This technique tells about the energy content of the gravitational waves.

Keywords: Einstein field equations; time conformal spacetime; approximate conservation of energy

1. Introduction

Gravitational waves are ripples in the fabric of space-time produced by some of the most violent and energetic processes like colliding black holes or closely orbiting black holes and neutron stars (binary pulsars). These waves travel with the speed of light and depend on their sources [1–5]. The study of these waves provide us useful information about their sources (black holes and neutron stars).

In 1905, Henri Poincare [6] proposed that gravitational waves are the outcomes of disturbances or distortions in the fabric of spacetime produced by the accelerated motion of heavy masses like black holes and neutron stars. Einstein in his famous general theory of relativity [7,8] predicted that this distortion (ripples in the spacetime fabric) could travel across the universe stretching and squeezing spacetime as they move through it. The first indirect detection or discovery for gravitational waves was made by Russell A. Hulse and Joseph H. Taylor, Jr. in 1974, and they were awarded a Nobel prize in physics in 1993 [9]. The direct detection of gravitational waves was made by the calibration of advanced LIGO and advanced VIRGO at 2015. It was an important discovery and the three scientists Rainer Weiss, Kip Thorne and Barry Barish were awarded a Nobel prize in 2017 for their work on direct detection of gravitational waves [5,10,11]. Moreover, in Refs. [12,13], the authors studied gravitational waves in de Sitter and asymptotically de Sitter spacetimes. They discussed the energy momentum tensor using the gravitational Poynting vector [14].

The analysis of these gravitational waves is very difficult because of the low amplitude and frequency as its intensity decreases gradually until it reaches the earth with estimated frequency range 10–16 Hz $< f <$ 104 Hz. In fact, these waves are emitted from the sources (accelerated motion of black holes and neutron stars) and carry energy and momentum from the energy and momentum of the sources at the formation of these waves at their early stage. Moreover, the gravitational waves are disturbances in the fabric of spacetime; therefore, the gravitational wave spacetimes would depend upon time t. However, Taub [15] had proved that "A spacetime with plane symmetry with $R_{\mu\nu} = 0$ (gravitational waves spacetime) admits a coordinate system where the line element is independent of time t, that is static"; therefore, it is not possible to find exact plane symmetric gravitational waves spacetime.

Emmy Noether proved in her famous Noether theorem that there is one to one correspondence between Noether symmetries and conservation laws [16]. Conservation of energy and momentum are valid in flat spacetimes, while they are not held in curved spacetimes globally (in general relativity). This paper reflects the same fact that, if the spacetime is flat, then the approximate Noether symmetries do not exist, while they do exist in curved spacetime, which means that the conservation of energy and momentum does not hold in curved spacetimes. The ultimate aim of the approximate Noether symmetry is to search plane symmetric spacetimes which are gravitational wave like spacetimes and also re-scale the energy and momentum in the respective spacetimes.

Approximate Noether symmetry [17–22] techniques have been used frequently by the researchers to re-scale the energy and momentum in gravitational waves like spacetimes for which $R_{\mu\nu} \rightarrow 0$ as $x \rightarrow \infty$. The authors of [19] have already determined it up to the first order in ϵ, in plane symmetric spacetimes. The approximate Noether symmetries and their corresponding approximate conservation laws will answer the question of energy and momentum imparted by the gravitational waves from the sources. It will also answer the question of what would be the approximate gravitational waves (or gravitational waves like) spacetime.

This paper is an extension of the paper [19]. Here, in this article, the second order approximate Noether symmetries of the Lagrangian of second order perturbed plane symmetric spacetimes are explored. Using Noether theorem, the second order approximate conservation laws are also calculated. The paper is arranged in the following order. The first section given in this paper is the Introduction. The tools for calculating the second order approximate Noether symmetries are given in Section 2. The definition and explanation of second-order approximate Noether symmetries for the Lagrangian of time conformal plane symmetric spacetimes are discussed in Section 3. Section 4 contains the main results of the article. Two classes of time conformal plane symmetric spacetimes are given in the same section (Section 4) along with second order approximate Noether symmetries and second order approximate conservation laws. The observations and conclusions are given in Section 5. The system of 19 determining partial differential equations is shifted to the appendix at the end of the paper.

2. Time Conformal Plane Symmetric Spacetime

The general plane symmetric static spacetime takes the form [23]

$$ds_{a1}^2 = e^{\tilde{\nu}(x)}dt^2 - dx^2 - e^{\tilde{\mu}(x)}(dy^2 + dz^2), \tag{1}$$

the Lagrangian corresponding to the spacetime (1) is

$$\mathcal{L}_{a1} = e^{\tilde{\nu}(x)}\dot{t}^2 - \dot{x}^2 - e^{\tilde{\mu}(x)}(\dot{y}^2 + \dot{z}^2), \tag{2}$$

the time conformal spacetime can be defined as

$$ds^2 = e^{\epsilon f(t)} ds_{a1}^2, \tag{3}$$

the corresponding time conformal Lagrangian takes the form

$$\mathcal{L} = e^{\epsilon f(t)} L_{a1}, \tag{4}$$

the second-order perturbed metric is defined as

$$ds^2 = e^{\tilde{\nu}(x)}dt^2 - dx^2 - e^{\tilde{\mu}(x)}(dy^2 + dz^2) + \epsilon f(t)\Big(e^{\tilde{\nu}(x)}dt^2 - dx^2 - e^{\tilde{\mu}(x)}(dy^2 + dz^2)\Big) + \epsilon^2 \frac{f^2(t)}{2}\Big(e^{\tilde{\nu}(x)}dt^2 - dx^2 - e^{\tilde{\mu}(x)}(dy^2 + dz^2)\Big), \quad (5)$$

and the corresponding second-order perturbed Lagrangian is defined as

$$\mathcal{L} = e^{\tilde{\nu}(x)}\dot{t}^2 - \dot{x}^2 - e^{\tilde{\mu}(x)}(\dot{y}^2 + \dot{z}^2) + \epsilon f(t)\Big(e^{\tilde{\nu}(x)}\dot{t}^2 - \dot{x}^2 - e^{\tilde{\mu}(x)}(\dot{y}^2 + \dot{z}^2)\Big) + \epsilon^2 \frac{f^2(t)}{2}\Big(e^{\tilde{\nu}(x)}\dot{t}^2 - \dot{x}^2 - e^{\tilde{\mu}(x)}(\dot{y}^2 + \dot{z}^2)\Big). \quad (6)$$

The Lagrangian given in Equation (6) can be written as

$$\mathcal{L} = \mathcal{L}_{a_1} + \epsilon\mathcal{L}_{a_2} + \epsilon^2\mathcal{L}_{a_3}, \quad (7)$$

where $\mathcal{L}_{a_1}$ is the exact part and $\mathcal{L}_{a_2}$ and $\mathcal{L}_{a_3}$ are the first and the second order approximate parts of the Lagrangian, respectively, which are given below in expanded form:

$$\mathcal{L}_{a_1} = e^{\tilde{\nu}(x)}\dot{t}^2 - \dot{x}^2 - e^{\tilde{\mu}(x)}(\dot{y}^2 + \dot{z}^2), \quad (8)$$

$$\mathcal{L}_{a_2} = f(t)\Big(e^{\tilde{\nu}(x)}\dot{t}^2 - \dot{x}^2 - e^{\tilde{\mu}(x)}(\dot{y}^2 + \dot{z}^2)\Big), \quad (9)$$

$$\mathcal{L}_{a_3} = \frac{f^2(t)}{2}\Big(e^{\tilde{\nu}(x)}\dot{t}^2 - \dot{x}^2 - e^{\tilde{\mu}(x)}(\dot{y}^2 + \dot{z}^2)\Big). \quad (10)$$

3. Second-Order Approximate Noether Symmetry Equation

An approximate Noether symmetries generator $\mathbf{Y}$ is defined as

$$\mathbf{Y}^{[1]}\mathcal{L} + (D\xi)\mathcal{L} = DA, \quad (11)$$

where $\mathbf{Y}^{[1]}$ is known as the first-order prolongation of the second-order approximate Noether symmetry generator $\mathbf{Y} = \mathbf{Y}_{a_1} + \epsilon\mathbf{Y}_{a_2} + \epsilon^2\mathbf{Y}_{a_3}$, and all of them are given in expanded forms:

$$\mathbf{Y}_{a_1} = \xi_{a_1}\frac{\partial}{\partial s} + \eta^0_{a_1}\frac{\partial}{\partial t} + \eta^1_{a_1}\frac{\partial}{\partial x} + \eta^2_{a_1}\frac{\partial}{\partial y} + \eta^3_{a_1}\frac{\partial}{\partial z}, \quad (12)$$

$$\mathbf{Y}_{a_2} = \xi_{a_2}\frac{\partial}{\partial s} + \eta^0_{a_2}\frac{\partial}{\partial t} + \eta^1_{a_2}\frac{\partial}{\partial x} + \eta^2_{a_2}\frac{\partial}{\partial y} + \eta^3_{a_2}\frac{\partial}{\partial z}, \quad (13)$$

$$\mathbf{Y}_{a_3} = \xi_{a_3}\frac{\partial}{\partial s} + \eta^0_{a_3}\frac{\partial}{\partial t} + \eta^1_{a_3}\frac{\partial}{\partial x} + \eta^2_{a_3}\frac{\partial}{\partial y} + \eta^3_{a_3}\frac{\partial}{\partial z}, \quad (14)$$

and the exact parts of Noether symmetry generator are given in Equation (12), whereas Equations (13) and (14) are first and second order approximate parts of the Noether symmetry generator $\mathbf{Y}$. The total differential operator D is defined as

$$D = \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z}. \quad (15)$$

The gauge function in given in Equation (11) is $A = A_{a_1} + \epsilon A_{a_2} + \epsilon^2 A_{a_3}$, where A_{a_1}, A_{a_2} and A_{a_3} are exact, first order and second order approximate parts of the gauge function A, respectively. Equation (11) splits into the following three equations:

$$\epsilon^0: \quad \mathbf{Y}_{a_1}\mathcal{L}_{a_1} + (D\xi_{a_1})L_{a_1} = D\mathcal{A}_{a_1}, \tag{16}$$

$$\epsilon^1: \quad \mathbf{Y}_{a_1}\mathcal{L}_{a_2} + \mathbf{Y}_{a_2}\mathcal{L}_{a_1} + (D\xi_{a_1})L_{a_2} + (D\xi_{a_2})L_{a_1} = D\mathcal{A}_{a_2}, \tag{17}$$

$$\epsilon^2: \quad \mathbf{Y}_{a_1}\mathcal{L}_{a_3} + \mathbf{Y}_{a_2}\mathcal{L}_{a_2} + \mathbf{Y}_{a_3}\mathcal{L}_{a_1} + (D\xi_{a_3})L_{a_1} + (D\xi_{a_2})L_{a_2} + (D\xi_{a_1})L_{a_3} = D\mathcal{A}_{a_3}, \tag{18}$$

where ξ_{a_1}, ξ_{a_2}, ξ_{a_3}, $\eta^i_{a_1}$, $\eta^i_{a_2}$, $\eta^i_{a_3}$, A_{a_1}, A_{a_2}, A_{a_3} all are functions of s, t, x, y and z; moreover, $\dot{\eta}^i_{a_1}$, $\dot{\eta}^i_{a_2}$ and $\dot{\eta}^i_{a_3}$ are functions of s, t, x, y, z, $\dot{t}$, $\dot{x}$, $\dot{y}$, $\dot{z}$, and the "$\cdot$" represent derivative with respect to the arc length parameter "s". The solutions of Equations (16) and (17) are given in Ref. [18]. The solution of Equation (18) is the goal of this research work, which provide us plane symmetric spacetimes which admit time conformal factor up to the second order in ϵ without losing any exact Noether symmetry. The Noether symmetries come with approximate parts. We listed the second order approximate Noether symmetries corresponding to the second order Lagrangians of second order time conformal plane symmetric spacetimes, along with second order approximate conservation laws in Section 4.

4. Second-Order Approximate Noether Symmetries and Conservation Laws along with Second Order Time Conformal Plane Symmetric Spacetimes

4.1. Five Noether Symmetries and the Corresponding Conservation Laws

The following second order perturbed metric of the plane symmetric spacetime is one of the solutions which admit five Noether symmetries without losing any exact Noether symmetry

$$\begin{aligned} ds^2 &= e^{\left(\frac{x}{\alpha}\right)^2}dt^2 - dx^2 - e^{\left(\frac{x}{\alpha}\right)}(dy^2+dz^2) + \frac{\epsilon t}{\alpha}\left(e^{\left(\frac{x}{\alpha}\right)^2}dt^2 - dx^2 - e^{\left(\frac{x}{\alpha}\right)}(dy^2+dz^2)\right) \\ &+ \frac{\epsilon^2 t^2}{2\alpha^2}\left(e^{\left(\frac{x}{\alpha}\right)^2}dt^2 - dx^2 - e^{\left(\frac{x}{\alpha}\right)}(dy^2+dz^2)\right), \quad a \neq 0, 2 \quad and \quad \alpha \neq 0. \end{aligned} \tag{19}$$

The exact Noether symmetries are

$$\mathbf{Z}_0 = \frac{\partial}{\partial s}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial y}, \quad \mathbf{Y}_2 = \frac{\partial}{\partial z}, \quad \mathbf{Y}_3 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$

and the first and second order approximate Noether symmetries are

$$\mathbf{Y_4} = \frac{\partial}{\partial t} + \frac{\epsilon s}{\alpha}\frac{\partial}{\partial s},$$

$$\mathbf{Y_0} = \epsilon\frac{\partial}{\partial t} + \frac{\epsilon^2 s}{\alpha}\frac{\partial}{\partial s}.$$

$\mathbf{Y}_4$ and $\mathbf{Y}_0$ are the first and second order approximate Noether symmetries corresponds to the energy content in the spacetime given in Equation (19). The conservation law (energy) or first integral related to first order approximate symmetry $\mathbf{Y}_4$ and second order approximate symmetry $\mathbf{Y}_0$ are given respectively as

$$E_1 = \left[e^{\left(\frac{x}{\alpha}\right)^2}\dot{t} + \frac{\epsilon}{\alpha}\left(t\dot{t}e^{\left(\frac{x}{\alpha}\right)^2} + \frac{\mathcal{L}s}{2}\right)\right],$$

$$E_2 = \epsilon\left[e^{\left(\frac{x}{\alpha}\right)^2}\dot{t} + \frac{\epsilon}{\alpha}\left(t\dot{t}e^{\left(\frac{x}{\alpha}\right)^2} + \frac{\mathcal{L}s}{2}\right)\right].$$

However, energy of the particle for spacetime (19), for $\epsilon = 0$, is $E_0 = (\frac{x^2}{\alpha^2})\dot{t}$. Thus, the first order correction to the energy of a particle is

$$E_1 = E_0 + \frac{\epsilon t}{\alpha}(E_0 + \frac{\sqrt{1-v^2}}{2}\mathcal{L}), \tag{20}$$

the second order correction to the energy is

$$E_2 = \epsilon\bigg(E_0 + \frac{\epsilon t}{\alpha}(E_0 + \frac{\sqrt{1-v^2}}{2}\mathcal{L})\bigg), \tag{21}$$

and the total energy up to second order in ϵ is

$$E_T = E_1 + E_2 = E_0 + \frac{\epsilon t}{\alpha}(E_0 + \frac{\sqrt{1-v^2}}{2}\mathcal{L}) + \epsilon\bigg(E_0 + \frac{\epsilon t}{\alpha}(E_0 + \frac{\sqrt{1-v^2}}{2}\mathcal{L})\bigg). \tag{22}$$

The approximate Lie algebra of the five Noether symmetries generators is

$$[\mathbf{Y_1}, \mathbf{Y_3}] = \mathbf{Y_2}, \quad [\mathbf{Y_2}, \mathbf{Y_3}] = -\mathbf{Y_1}, \quad [\mathbf{Y_0}, \mathbf{Z_0}] = -\frac{\epsilon^2}{\alpha}\mathbf{Z_0},$$
$$[\mathbf{Y_i}, \mathbf{Y_j}] = 0, \quad [\mathbf{Y_i}, \mathbf{Z_0}] = 0, \quad \textit{otherwise}.$$

4.2. Six Noether Symmetries and the Corresponding Conservation Laws

Metric of the plane symmetric spacetime that admits the second order time conformal perturbation and have five Noether symmetries is

$$\begin{aligned} ds^2 &= \left(\frac{x}{\alpha}\right)^2 dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^a (dy^2 + dz^2) + \frac{\epsilon t}{\alpha}\bigg(\left(\frac{x}{\alpha}\right)^2 dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^a (dy^2+dz^2)\bigg) \\ &+ \frac{\epsilon^2 t^2}{2\alpha^2}\bigg(\left(\frac{x}{\alpha}\right)^2 dt^2 - dx^2 - \left(\frac{x}{\alpha}\right)^a (dy^2+dz^2)\bigg), \quad a \neq 0, 2 \quad \textit{and} \quad \alpha \neq 0. \end{aligned} \tag{23}$$

The exact Noether symmetries are

$$\mathbf{Z}_0 = \frac{\partial}{\partial s}, \quad \mathbf{Y}_1 = \frac{\partial}{\partial y}, \quad \mathbf{Y}_2 = \frac{\partial}{\partial z}, \quad \mathbf{Y}_3 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},$$
$$\mathbf{Z}_1 = s\frac{\partial}{\partial s} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{2-a}{4}y\frac{\partial}{\partial y} + \frac{2-a}{4}z\frac{\partial}{\partial z}.$$

The first order approximate Noether symmetry is given below

$$\mathbf{Y}_4 = \frac{\partial}{\partial t} - \frac{\epsilon}{4\alpha}\bigg(2x\frac{\partial}{\partial x} + (2-a)y\frac{\partial}{\partial y} + (2-a)z\frac{\partial}{\partial z}\bigg),$$

and the second order approximate Noether symmetry is

$$\mathbf{Y}_0 = \epsilon\frac{\partial}{\partial t} - \frac{\epsilon^2}{4\alpha}\bigg(2x\frac{\partial}{\partial x} + (2-a)y\frac{\partial}{\partial y} + (2-a)z\frac{\partial}{\partial z}\bigg).$$

The conservation laws or first integrals related to the first order approximate symmetry $\mathbf{Y}_4$ and second order approximate symmetry $\mathbf{Y}_0$ are

$$E_1 = \left[\left(\frac{x}{\alpha}\right)^2\dot{t} + \frac{\epsilon}{\alpha}\left(t\dot{t}\left(\frac{x}{\alpha}\right)^2 + \frac{x\dot{x}}{2} + (2-a)\frac{y\dot{y}}{4}\left(\frac{x}{\alpha}\right)^a + (2-a)\frac{z\dot{z}}{4}\left(\frac{x}{\alpha}\right)^a\right)\right], \tag{24}$$

$$E_2 = \epsilon \left[\left(\frac{x}{\alpha}\right)^2 \dot{t} + \frac{\epsilon}{\alpha} \left(t\dot{t} \left(\frac{x}{\alpha}\right)^2 + \frac{x\dot{x}}{2} + (2-a)\frac{y\dot{y}}{4}\left(\frac{x}{\alpha}\right)^a + (2-a)\frac{z\dot{z}}{4}\left(\frac{x}{\alpha}\right)^a \right) \right], \tag{25}$$

from the first and second approximate conservation laws, we can write

$$E_2 = \epsilon E_1, \quad E_T = E_1 + E_2 = E_1 + \epsilon E_1, \tag{26}$$

where E_T is the total energy. Furthermore, we conjecture that the third approximation will be of the form

$$E_3 = \epsilon E_2 = \epsilon^2 E_1, \quad E_T = E_1 + E_2 + E_3 = E_1 + \epsilon E_1 + \epsilon^2 E_1, \tag{27}$$

and the nth order approximation will takes the form

$$E_n = \epsilon E_{n-1} = \epsilon^2 E_{n-2} ... = \epsilon^{n-1} E_1, \tag{28}$$
$$E_T = E_1 + E_2 .. + E_n = E_1 + \epsilon E_1 + \epsilon^2 E_1 .. + \epsilon^{n-1} E_1. \tag{29}$$

The approximate Lie algebra related to the six Noether symmetries generators is

$$[\mathbf{Y_1}, \mathbf{Y_3}] = \mathbf{Y_2}, \quad [\mathbf{Y_2}, \mathbf{Y_3}] = -\mathbf{Y_1}, \quad [\mathbf{Y_1}, \mathbf{Z_1}] = \frac{2-a}{4}\mathbf{Y}_1,$$
$$[\mathbf{Y}_2, \mathbf{Z}_1] = \frac{2-a}{4}\mathbf{Y}_2, \quad [\mathbf{Z}_0, \mathbf{Z}_1] = \mathbf{Z}_0, \quad [\mathbf{Y_1}, \mathbf{Y_0}] = \frac{\epsilon^2(a-2)}{4\alpha}\mathbf{Y_1},$$
$$[\mathbf{Y_2}, \mathbf{Y_0}] = \frac{\epsilon^2(a-2)}{4\alpha}\mathbf{Y_2}, \quad [\mathbf{Y_i}, \mathbf{Y_j}] = 0, \quad [\mathbf{Y_i}, \mathbf{Z_j}] = 0, \quad otherwise.$$

5. Conclusions and Observations

Gravitational waves are ripples in the fabric of space-time caused by some of the most violent and energetic processes in the universe, like two black holes or neutrons orbiting each other closely on the fabric of spacetime. These violent and energetic motions produce gravitational waves with very high frequencies which can be detected on earth. These waves are rich sources of information about the black holes and neutron stars. Recently, LIGO and VERGO calibration announced that they directly detected these types of waves [10,11].

In this paper, we find the second order approximation to the energy content imparted by the gravitational wave from the sources (Black hole and Neutron stars). The radiation emitted from the black holes is a part of the energy and mass of the black holes [24]. The information related to the formation of black hole are encoded in the gravitational waves emitted from the black hole and neutron stars.

We find second order approximation to the energy content of the gravitational waves in two classes of time conformal plane symmetric spacetimes. These spacetimes behave like approximate gravitational waves spacetimes. Our calculations for second order approximation to energy and momentum shows that, during the formation and propagation of gravitational waves, the energy and momentum of the black hole are re-scaled continuously. We claim that the exact part of the conservation law is the energy of the source that created the gravitational waves and the approximate part corresponds to the energy imparted by the gravitational waves from the source. We observed the following facts in our calculations:

- Second order approximation appears in the classes of five and six Noether symmetries for the perturbed Lagrangian of plane symmetric spacetimes, which are given in Sections 4.1 and 4.2, respectively.

- Second order approximate Noether symmetries exist for the components of metric of the form $g_{\mu\nu} = e^{\left(\frac{x}{\alpha}\right)^{\gamma}}$ for the five Noether symmetries and $g_{\mu\nu} = \left(\frac{x}{\alpha}\right)^{\gamma}$ for six Noether symmetries, which are evident from the plane symmetric metrics given in Equations (19) and (23).
- Flat spacetimes do not admit second order approximation (second-order perturbation) as well as first-order approximation/first-order perturbation); for example, Minkowski spacetime does not admit approximate Noether symmetries. Therefore, the conservation laws do not hold in general relativity globally and they hold in special relativity.

The last observation confirms that the conservation laws hold in flat spacetime locally as well as globally, while they do not hold in curved spacetimes globally. The reason behind this fact is that the gravity creates curvature in the spacetime, which is an extremely nonlinear phenomenon, and exact measurement of physical quantities is not possible in such a nonlinear phenomenon globally.

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Appendix A

Equation (18) is expanded by using $\mathbf{Y}^{[1]}_{a_1}$, $\mathbf{Y}^{[1]}_{a_2}$, $\mathbf{Y}^{[1]}_{a_3}$, L_{a_1}, L_{a_2}, L_{a_3}, D and A_{a_3}, and we obtain a system of 19 partial differential equations as follows:

$$
\begin{aligned}
&f_t(t)\Big(\eta^0_{a_2}+\eta^0_{a_1}f(t)\Big)+\Big(\eta^1_{a_3}+\eta^1_{a_2}f(t)+\eta^1_{a_1}\frac{f^2(t)}{2}\Big)\tilde{\nu}'(x)+\Big(2\eta^0_{a_3t}+2\eta^0_{a_2t}f(t)+\eta^0_{a_1t}f^2(t)\Big)\\
&-\xi_{a_3s}-\xi_{a_2s}f(t)-\xi_{a_1s}\frac{f^2(t)}{2}=0,\\
&f_t(t)\Big(\eta^0_{a_2}+\eta^0_{a_1}f(t)\Big)+\Big(2\eta^1_{a_3x}+2\eta^1_{a_2x}f(t)+\eta^1_{a_1x}f^2(t)\Big)-\xi_{a_3s}-\xi_{a_2s}f(t)\\
&-\xi_{a_1s}\frac{f^2(t)}{2}=0,\\
&f_t(t)\Big(\eta^0_{a_2}+\eta^0_{a_1}f(t)\Big)+\Big(\eta^1_{a_3}+\eta^1_{a_2}f(t)+\eta^1_{a_1}\frac{f^2(t)}{2}\Big)\tilde{\mu}'(x)+\Big(2\eta^2_{a_3y}+2\eta^2_{a_2y}f(t)+\eta^2_{a_1y}f^2(t)\Big)\\
&-\xi_{a_3s}-\xi_{a_2s}f(t)-\xi_{a_1s}\frac{f^2(t)}{2}=0,\\
&f_t(t)\Big(\eta^0_{a_2}+\eta^0_{a_1}f(t)\Big)+\Big(\eta^1_{a_3}+\eta^1_{a_2}f(t)+\eta^1_{a_1}\frac{f^2(t)}{2}\Big)\tilde{\mu}'(x)+\Big(2\eta^3_{a_3z}+2\eta^3_{a_2z}f(t)+\eta^3_{a_1z}f^2(t)\Big)\\
&-\xi_{a_3s}-\xi_{a_2s}f(t)-\xi_{a_1s}\frac{f^2(t)}{2}=0,\\
&\Big(2\eta^0_{a_3s}+2\eta^0_{a_2s}f(t)+\eta^0_{a_1s}f^2(t)\Big)e^{\tilde{\nu}(x)}-A_{a_3t}=0,\\
&\Big(2\eta^1_{a_3t}+2\eta^1_{a_2t}f(t)+\eta^1_{a_1t}f^2(t)\Big)-\Big(2\eta^0_{a_3x}+2\eta^0_{a_2x}f(t)+\eta^0_{a_1x}f^2(t)\Big)e^{\tilde{\nu}(x)}=0,\\
&\Big(2\eta^0_{a_3y}+2\eta^0_{a_2y}f(t)+\eta^0_{a_1y}f^2(t)\Big)e^{\tilde{\nu}(x)}-\Big(2\eta^2_{a_3t}+2\eta^2_{a_2t}f(t)+\eta^2_{a_1t}f^2(t)\Big)e^{\tilde{\mu}(x)}=0,\\
&\Big(2\eta^0_{a_3z}+2\eta^0_{a_2z}f(t)+\eta^0_{a_1z}f^2(t)\Big)e^{\tilde{\nu}(x)}-\Big(2\eta^3_{a_3t}+2\eta^3_{a_2t}f(t)+\eta^3_{a_1t}f^2(t)\Big)e^{\tilde{\mu}(x)}=0,\\
&2\eta^1_{a_3s}+2\eta^1_{a_2s}f(t)+\eta^1_{a_1s}f^2(t)+A_{a_3x}=0,\\
&\Big(2\eta^2_{a_3s}+2\eta^2_{a_2s}f(t)+\eta^2_{a_1s}f^2(t)\Big)e^{\tilde{\mu}(x)}+A_{a_3y}=0,\\
&\Big(2\eta^2_{a_3z}+2\eta^2_{a_2z}f(t)+\eta^2_{a_1z}f^2(t)\Big)+\Big(2\eta^3_{a_3y}+2\eta^3_{a_2y}f(t)+\eta^3_{a_1y}f^2(t)\Big)=0,\\
&\Big(2\eta^3_{a_3s}+2\eta^3_{a_2s}f(t)+\eta^3_{a_1s}f^2(t)\Big)e^{\tilde{\mu}(x)}+A_{a_3z}=0,\\
&\Big(2\eta^1_{a_3y}+2\eta^1_{a_2y}f(t)+\eta^1_{a_1y}f^2(t)\Big)+\Big(2\eta^2_{a_3x}+2\eta^2_{a_2x}f(t)+\eta^2_{a_1x}f^2(t)\Big)e^{\tilde{\mu}(x)}=0,\\
&\Big(2\eta^1_{a_3z}+2\eta^1_{a_2z}f(t)+\eta^1_{a_1z}f^2(t)\Big)+\Big(2\eta^3_{a_3x}+2\eta^3_{a_2x}f(t)+\eta^3_{a_1x}f^2(t)\Big)e^{\tilde{\mu}(x)}=0,\\
&A_{a_3s}=0,\quad \xi_{a_3t}=0,\quad \xi_{a_3x}=0,\quad \xi_{a_3y}=0,\quad \xi_{a_3z}=0.
\end{aligned}
\tag{30}
$$

References

1. Press, W.H.; Thorne, K.S. Gravitational-wave astronomy. *Ann. Rev. Astron. Astrophys.* **1972**, *10*, 335–374.
2. Price, R.H.; Thorne, K.S. Nonradial pulsation of general relativistic stellar models. II. Properties of the gravitational waves. *Astrophys. J.* **1969**, *155*, 163.
3. Thorne, K.S. Gravitational radiation damping. *Phys. Rev. Lett.* **1968**, *21*, 320.
4. Thorne, K.S.; Hawking, S.W.; Penrose, R. Sources of gravitational waves. *Proc. R. Soc. Lond. A. Math. Phys. Sci.* **1979**, *368*, 9.
5. Thorne, K.S. Gravitational-wave research: Current status and future prospects. *Rev. Mod. Phys.* **1980**, *52*, 285.
6. Poincare, H. The principles of mathematical physics. *Monist* **1905**, *15*, 1–24.
7. Einstein, A. The foundation of the general theory of relativity. *Ann. Phys.* **1916**, *14*, 769–822.
8. Einstein, A.; Rosen, N. On gravitational waves. *J. Frankl. Inst.* **1937**, *223*, 43.
9. Ruthen, R. Catching the wave. *Sci. Am.* **1992**, *266*, 90–101.
10. Abbott, B.P.; Jawahar, S.; Lockerbie, N.A.; Tokmakov, K.V. LIGO Scientific Collaboration and Virgo Collaboration. *Phys. Rev. Lett.* **2016**, *116*, 241102.
11. The LIGO Scientific Collaboration and the Virgo Collaboration. An improved analysis of GW150914 using a fully spin-precessing waveform model. *arXiv* **2016**, arXiv:1606.01210.
12. Nowakowiski, M.; Arraut, I. The fate of a gravitational wave in de Sitter spacetime. *Acta Phys. Polon.* **2010**, *B41*, 911–925.
13. Arraut, I. About the propagation of the gravitational wave in an asymptotically de Sitter spacetime, comparing two point of view. *Mod. Phys. Lett. A* **2013**, *28*, 13500139.
14. Landau, L.D.; Lifshits, L.M. *The Classical Theory of Fields*, 4th ed.; Butterworth-Heinemann: Oxford, UK, 1980.
15. Taub, A.H. Empty spacetimes admiting a three parameter group of motions. *Ann. Math.* **1951**, *53*, 472–490.
16. Noether, E. Invariante Variations probleme. *Nachr. Koing. Gesell. Wissen. Gottinngen Math. Phys.* **1918**, *2*, 235.
17. Ali, F.; Feroze, T. Classification of plane symmetric static spacetimes according to their Noether's symmetries. *Int. J. Theor. Phys.* **2013**, *52*, 3329.
18. Ali, F.; Feroze, T.; Ali, S. Complete classification of spherically symmetric static spacetime via Noether symmetries. *Theor. Math. Phys.* **2015**, *184*, 973.
19. Ali, F.; Feroze, T. Approximate Noether symmetries from geodetic lagrangian for plane symmetric spacetimes. *Int. J. Theor. Phy.* **2015**, *12*, 1550124.
20. Ali, F.; Feroze, T. Complete classification of cylindrically symmetric static spacetime and the corresponding conservation laws. *Mathematics* **2016**, **4**, 50.
21. Jawad, A.; Ali, F.; Umair, S.M.; Abbas, G. Dynamics of particles around time conformal Schwarzschild black hole. *Eur. Phys. J. C* **2016**, *76*, 586.
22. Ali, F.; Mashwani, W.K.; Jan, M.A. Correction to the conservation laws in plane symmetric gravitational wave-like spacetimes. *Mod. Phys. Lett. A* **2018**, *33*, 1850123.
23. Kramer, D.; Stephani, H.; MacCallum, M.A.H.; Herlt, E. *Exact Solution of Einstein Field Equations*; Cambridge University Press: Cambridge, UK, 1980.
24. Iso, S.; Umetsu, H.; Wilczek, F. Anomalies, Hawking radiations, and regularity in rotating black holes. *Phys. Rev. D* **2006**, *74*, 044017.