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Absolute Quantum Theory (after Chang, Lewis, Minic and Takeuchi), and a Road to Quantum Deletion

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Abstract: In a recent paper, Chang et al. have proposed studying "quantum $\mathbb{F}_{un}$": the $q \mapsto 1$ limit of modal quantum theories over finite fields $\mathbb{F}_q$, motivated by the fact that such limit theories can be naturally interpreted in classical quantum theory. In this letter, we first make a number of rectifications of statements made in that paper. For instance, we show that quantum theory over $\mathbb{F}_1$ does have a natural analogon of an inner product, and so orthogonality is a well-defined notion, contrary to what was claimed in Chang et al. Starting from that formalism, we introduce time evolution operators and observables in quantum $\mathbb{F}_{un}$, and we determine the corresponding unitary group. Next, we obtain a typical no-cloning result in the general realm of quantum $\mathbb{F}_{un}$. Finally, we obtain a no-deletion result as well. Remarkably, we show that we can perform quantum deletion by *almost unitary operators*, with a probability tending to 1. Although we develop the construction in quantum $\mathbb{F}_{un}$, it is also valid in any other quantum theory (and thus also in classical quantum theory in complex Hilbert spaces).

Keywords: København quantum theory; modal quantum theories; field with one element; no-cloning; quantum deletion; almost unitary operator

1. Quantum $\mathbb{F}_{un}$, and $\mathbb{F}_{1^\ell}$

In many papers, alternative quantum theories have been proposed for classical quantum theory (in complex Hilbert spaces, following the København interpretation). For instance, there is a number of papers on "modal quantum theories" (MQTs), which consider similar theories over finite fields (see e.g., [1–5]). Whether the motivation is that these simply serve as toy models for the classical theory, or that they maybe come closer to physical reality, is arguable. But that the fundamental results, such as no-cloning, can also be obtained in MQTs, makes the latter interesting in their own right.

In the last ten years, there has been an increasing interest in the *field with one element*; this nonexisting object is contained in every field, and its geometric theory (in a broad sense of the word: algebraic geometry, incidence geometry, ...) is an "absolute theory" which is present in any geometric theory over a field. We refer to the monograph [6] for a thorough introduction. A very simple and equally important manifestation of "$\mathbb{F}_1$" is the following. Consider the class of all combinatorial projective spaces $\mathbb{P}^n(\mathbb{F}_q)$ over finite fields $\mathbb{F}_q$ [7,8]; each such space has an automorphism group $\mathbf{P\Gamma L}_{n+1}(\mathbb{F}_q)$ of nonsingular semilinear transformations. Each such space has

(A) $q+1$ points per line,
(B) any two different points are contained in precisely one line, and
(C) any two different intersecting lines are contained in one axiomatic projective plane of order q.

Axiomatic projective planes are characterized by three simple properties:

(1) property (B);
(2) any two different lines intersect in precisely one point and

(3) there exist four points with no three of them on the same line.

Each such plane has an *order*: A positive integer c such that each line contains $c+1$ points and each point is on $c+1$ lines. If we imagine that c goes to one, we end up with a set of points in which each line has two points, and for which (1) and (2) hold. It is easy to see that the set must have three points, and that we obtain the geometry of a triangle. So (3) does not hold anymore. Turning back to the combinatorial geometry of $\mathbb{P}^n(\mathbb{F}_q)$ and letting q go to 1 (so that we shrink $\mathbb{F}_q$ to a "field with one element," $\mathbb{F}_1$), we end up with a limit geometry in which

(A′) each line has 2 points;
(B′) any two different points are contained in one unique line;
(C′) each two different intersecting lines are contained in a unique triangle.

Obviously, (C′) follows from (A′) and (B′). And it also clear that $\mathbb{P}^n(\mathbb{F}_1)$ is a complete graph. Observe that its number of points is $n+1$. The picture only becomes complete after the observation that indeed, a complete graph on $n+1$ points is a subgeometry of any combinatorial projective space $\mathbb{P}^n(k)$, where k is a field (or even a division ring, cf. Section 2), and that the group which is induced by $\mathbf{P\Gamma L}_{n+1}(k)$ on such a subgeometry, is isomorphic to the full symmetric group on $n+1$ letters. The latter is precisely the full combinatorial automorphism group of $\mathbb{P}^n(\mathbb{F}_1)$.

In the inspiring paper [3], the authors propose to apply the same formalism on the level of quantum theory, so as to interpret phenomena in modal quantum theories in classical quantum theory over the complex numbers. So [3] bids for a transition from the finite MQTs to classical quantum theory (which we abbreviate by "AQT," referring to "actual quantum theory" as in [5]) through the limit $q \mapsto 1$.

$$\mathbf{MQT}_q \xrightarrow{q \mapsto 1} \mathbf{AQT}. \tag{1}$$

This idea is the starting point of the present note.

In [9], we have introduced a general approach to quantum theories—in the København setting—over so-called division rings (we will recall the basics in the next section); this approach unifies all known quantum theories in this setting, but it also argues that even over the complex numbers, there are very interesting alternate quantum theories to the classical one. If we want the combinatorics of projective wave space available, we also argued in [9] that the approach of general quantum theories (GQTs) is the most general possible, since combinatorial projective spaces in dimension at least 3 are always coordinatized over division rings (by [10]). Many other results are obtained: For instance, we have showed that no-cloning holds in every GQT. We also showed how to use the "quantum kernel," a singular object which arises from the equation which defines the Hermitian form which replaces the inner product in these theories, in both the new and classical theories (e.g., on the level of quantum codes).

A different, more general, formulation of the diagram (1) could also be:

> "Can one describe a quantum theory which "sees" (fundamental aspects in) all (actual, modal, general) quantum theories?"

In [9] we have introduced such an "absolute quantum theory" in characteristic 0: The minimal standard model, which is defined over the rationals $\mathbb{Q}$. The philosophy of minimal models fits very well in the contents of [3], and the present paper.

1.1. A Virtual Deletion Machine in all Quantum Theories

The principle of superposition is a fundamental property in quantum mechanics; if two evolving states $|s\rangle_1$ and $|s\rangle_2$ solve the Schrödinger equation, then an arbitrary linear combination $a|s\rangle_1 + b|s\rangle_2$ is also a solution. The famous no-cloning result of Wootters and Zurek [11] and Dieks [12] has been obtained as an implication of the superposition principle, and so has the no-deletion principle of Pati and Braunstein [13]. In general quantum theories, the author has shown that both no-theorems still hold, and superposition remains to be a key in the proofs [9].

As $\mathbb{F}_1$-theory lacks addition on the algebraic level (see Section 3), a major basic question is whether similar no-cloning and no-deletion results will still hold in quantum $\mathbb{F}_{un}$. And whether the diagram (1) remains to have a meaning in the context of such more advanced questions. In the theory of Chang et al. [3], such questions make no sense, since they have no unitary operators available, but we do. And as we will see, the lack of flexibility due to not having addition at hand, will be compensated by the fact that the unitary groups in quantum $\mathbb{F}_{un}$ are of a restricted type. In the end, we will obtain the no-cloning and no-deletion theorems in quantum $\mathbb{F}_{un}$.

On the other hand, after introducing *almost unitary operators* (which are allowed to be singular), we obtain a quantum deletion theory which deletes one copy of any two given state rays with a probability tending to 1. The diagram (1) does apply to this result, so that we virtually obtain deletion in classical quantum theory.

1.2. Overview

In this letter, we first make a number of rectifications of statements made in the interesting recent note [3]. For instance, we show that quantum theory over $\mathbb{F}_1$ *does* have a natural analogon of an inner product, and so orthogonality is a well-defined notion, contrary to what is claimed in [3]. A general and widespread misconception in modal quantum theory papers is the common belief that such theories do not allow inproducts (see for instance [5]). As explained in [9]—see also the next section—this is not true: Even in the setting of general quantum theories, one has natural generalized versions of inproducts available, and in many cases (such as in the case of general quantum theories over algebraically closed fields in characteristic 0), the theory comes with a Born rule as well. Starting from that new formalism, we introduce time evolution operators and observables in quantum $\mathbb{F}_{un}$, and we determine the corresponding unitary group. Finally, we develop a no-cloning and no-deleting theory in quantum $\mathbb{F}_{un}$.

In the next section we tersely review the viewpoint of general quantum theory. The following two Sections 3 and 4 prepare in some detail the theory of quantum $\mathbb{F}_{un}$. This includes the completion of what is described in [3], but also other aspects which are needed to understand the setting. Section 5 contains a small dictionary which compares some basic aspects of actual, modal, general and absolute quantum theories.

With that dictionary in mind, quantum information theorists might want to skip Section 4, and focus on Sections 6 and 7, which form the core of this paper on the level of physical applications.

2. A Quick Review of General Quantum Theory

In this section, a division ring is a field in which multiplication not necessarily is commutative. Example: the quaternions. If one constructs a projective space from a left or right vector space over a division ring in the usual way, then one obtains a space of which the underlying combinatorial incidence geometry (which one defines by taking the points, lines, planes, etc. of the space, endowed with the natural symmetrized containment relation) still is an axiomatic projective space (in the sense of Veblen and Young [10]), and division rings are the most general algebraic objects with this property [10]: the paper [10] shows that axiomatic combinatorial projective spaces of dimension at least three, are projective spaces coming from vector spaces over division rings.

2.1. $(\sigma, 1)$-Hermitian Forms

Let k be a division ring. An anti-automorphism of k is a map $\gamma : k \mapsto k$ such that γ is bijective; for any $u, v \in k$, we have $\gamma(u+v) = \gamma(u) + \gamma(v)$; and for any $a, b \in k$, we have $\gamma(ab) = \gamma(b)\gamma(a)$.

If k is a commutative field, then anti-automorphisms and automorphisms coincide. Note that the fields $\mathbb{Q}$ and $\mathbb{R}$ do not admit nontrivial automorphisms.

2.2. Hermitian Forms

Suppose that k is a division ring, and suppose σ is an anti-automorphism of k. Let V be a right vector space over k. A σ-sesquilinear form on V is a map $\nu : V \times V \mapsto k$ for which we have the following properties:

- for all $a, b, c, d \in V$ we have that $\nu(a+b, c+d) = \nu(a,c) + \nu(b,c) + \nu(a,d) + \nu(b,d)$;
- for all $a, b \in V$ and $\alpha, \beta \in k$, we have that $\nu(a\alpha, b\beta) = \sigma(\alpha)\nu(a,b)\beta$.

We have that ν is reflexive if and only if there exists an $\epsilon \in k$ such that for all $a, b \in V$, we have

$$\nu(b,a) = \sigma\Big(\nu(a,b)\Big)\epsilon. \tag{2}$$

Such sesquilinear forms are called (σ,ϵ)-Hermitian. If $\epsilon = 1$ and $\sigma^2 = \mathrm{id} \neq \sigma$, then we speak of a Hermitian form.

The standard inner product (and in fact any inner product) in a classical Hilbert space over $\mathbb{C}$ is a Hermitian form.

2.3. Standard $(\sigma, 1)$-Hermitian Forms

If k is a division ring with involution σ, the standard $(\sigma,1)$-Hermitian form on the right vector space $V(d,k)$, is given by

$$\Big\langle x \Big| y \Big\rangle \ := \ x_1^\sigma y_1 + \cdots + x_d^\sigma y_d, \tag{3}$$

where $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$.

In the case that $\sigma = \mathrm{id}$, we obtain a form which is usually called symmetric; it is not a proper Hermitian form, but still comes in handy in some situations (for example in cases of field reduction: "real Hilbert spaces" have often been considered in quantum theory; see e.g., [14–16]).

2.4. The Unitary Group $\mathbf{U}(V,\varphi)$

An automorphism of a $(\sigma,1)$-Hermitian form φ on the k-vector space V, is a bijective linear operator $\omega : V \mapsto V$ which preserves φ, that is, for which

$$\varphi(\omega(x), \omega(y)) = \varphi(x,y) \tag{4}$$

for all $(x,y) \in V \times V$. The group of all such automorphisms is called the unitary group, and denoted $\mathbf{U}(V,\varphi)$.

Example. Let $k = \mathbb{C}$, σ be complex conjugation, and $V = V(n,\mathbb{C})$. Then $\mathbf{U}(V,\varphi) = \mathbf{GU}_n(\mathbb{C}) = \mathbf{U}(n)$.

2.5. GQT

If we speak of "division ring with involution," we mean a division ring with an involutory anti-automorphism.

> From now on, we propose to depict a physical quantum system by a general Hilbert space $\mathcal{H} = \Big((V(\omega,k),+,\cdot),\langle\cdot,\cdot\rangle\Big)$, with k a division ring with involutionσ, and$\Big\langle \cdot \Big| \cdot \Big\rangle$a$(\sigma,1)$-Hermitian form.

If we speak of "standard GQT," we mean that given σ, the general Hilbert space comes with the standard $(\sigma,1)$-Hermitian form. Also, as some fields such as the reals and the rational numbers do not admit nontrivial involutions, they only can describe "improper" quantum systems. By extension of quantum theories (which is described in [9]), this is no problem (as often has been the case when switching between AQT over $\mathbb{C}$ and $\mathbb{R}$).

In this paper, we will only focus on the standard Hermitian forms, to keep the analogy with AQT as clear as possible. We also refer to Section 5 for an overview of some basic notions in the different quantum theories.

3. The Formalism over $\mathbb{F}_1$

3.1. $\mathbb{F}_1$ *and* $\mathbb{F}_{1^2}$

As in [3], we define $\mathbb{F}_1$ to be the set $\{0,1\}$ endowed with the obvious multiplication, once we agree that 0 is the absorbing element. So $0 \cdot 1 = 1 \cdot 0 = 0 \cdot 0 = 0$, and $1 \cdot 1 = 1$. We adopt the traditional $\mathbb{F}_1$-Mantra that there is no addition [17].

Define the quadratic extension of $\mathbb{F}_1$, and denote it by $\mathbb{F}_{1^2}$, as the set $\{0\} \cup \mu_2$, where μ_2 is the group of two elements, multiplicatively written as $\{1,a\}$. So $a \cdot 1 = 1 \cdot a = a$ and $a^2 = 1$. Again, 0 is the absorbing element. We define $\mathbb{F}_{1^\ell}$, for any positive integer ℓ, in a similar manner (replacing μ_2 by the cyclic group μ_ℓ of order ℓ).

3.2. Vector Space over $\mathbb{F}_{1^\ell}$*, and the Affine Viewpoint: Frames*

Fix a dimension m (positive integer different from 0). In this section, our state space will be the vector space $V = V(m, \mathbb{F}_{1^\ell})$. We will take on the viewpoint of our recent paper [17], and also, we will make no distinction at this point between vector spaces over $\mathbb{F}_1$ and its extensions, and affine spaces.

Let S be $\mathbb{F}_1$, or $\mathbb{F}_{1^2}$. The affine frame $\mathcal{A}(m,S)$ is defined as the set

$$\{(s_1, \ldots, s_m) \mid s_i \in S\}. \tag{5}$$

Let $(0, \ldots, 0) =: \omega$. In the case $S = \mathbb{F}_{1^\ell}$, we will use the same definitions.

Besides ω, the frame points with exactly one nonzero entry play a special role; those were the points used in [3] in the case $S = \mathbb{F}_1$. We call such points simple points. As we showed in [17], and as was conjectured by others (see the details in [17]), we need to include the extra points to set up a natural connection with the "functor-of-points viewpoint."

It is very important to note that we cannot add the points in a frame, as the underlying set S only is endowed with multiplication.

In the rest of this section, we solely work over $\mathbb{F}_{1^2}$ to fix ideas.

3.3. The Standard Form

As we will later see, in Section 4, $\mathbb{F}_{1^2}$ does not allow a nontrivial involution. On the other hand, there is a natural standard $(1,1)$-Hermitian form, being:

$$\left\langle \overline{x} \middle| \overline{y} \right\rangle := x_1 y_1 + \cdots + x_m y_m = \overline{x}^T \cdot \overline{y}. \tag{6}$$

Now the $\mathbb{F}_1$-Mantra adds an extra rule to the formalism: in (6), $\left\langle \overline{x} \middle| \overline{y} \right\rangle$ only has a meaning if at most one nonzero occurs in the summation in (6).

In the next subsection, we will look at the implications for orthogonality relations.

3.4. Orthogonality

If $\overline{x} = (x_1, \ldots, x_m)$ is a point in a vector space over $\mathbb{F}_{1^\ell}$, then by $\mathtt{supp}(\overline{x})$ we denote the set of indices j for which $x_j \neq 0$ ("support" of $\overline{x}$). By $\mathtt{supp}^c(\overline{x})$, we denote the complement $\{1, \ldots, m\} \setminus \mathtt{supp}(\overline{x})$. Then vectors $\overline{u}$ and $\overline{v}$ are orthogonal if and only if

$$\mathtt{supp}(\overline{u}) \subseteq \mathtt{supp}^c(\overline{v}). \tag{7}$$

If we now define $\overline{x}^{\perp}$ as

$$\left\{\overline{y} \mid \left\langle \overline{x} \middle| \overline{y} \right\rangle = 0\right\}, \tag{8}$$

then the dimension of these spaces obviously (only) depends on $|\mathtt{supp}(\overline{x})|$—in vector spaces over a field such as $\mathbb{C}$, this dependence is not present. Over $\mathbb{F}_{1^\ell}$ we have that $\overline{x}^{\perp}$ is a vector space of dimension $|\mathtt{supp}^c(\overline{x})|$; for example, if $\overline{x}$ is a simple point, then $\overline{x}^{\perp}$ is a hyperplane (dimension $m-1$, so classical behavior), and if $\overline{x}$ has maximal support m, then $\overline{x}^{\perp}$ is 0-dimensional, only consisting of the zero vector.

3.5. Time Evolution and Hermitian Operators

Before defining time evolution and Hermitian operators, we need to know what the linear operators of $V(m, \mathbb{F}_{1^2})$ are. As $\mathbb{F}_1$-theory, and in particular the vector spaces/frames as defined in Section 3.2, does not allow addition, we can only work with matrices with entries in $\mathbb{F}_{1^2}$ which have at most one nonzero entry in each row and column. If we want these operators to be invertible, we ask that every row and column has precisely one nonzero entry. This means that every such $(m \times m)$-matrix has the structure of a permutation matrix, but the nonzero entries vary over μ_2. As the set of all $(m \times m)$-permutation matrices forms a group which is isomorphic to the symmetric group $\mathbf{S}_m$ on m letters, we can thus write that

$$\mathbf{GL}(m, \mathbb{F}_{1^2}) := \mu_2 \wr \mathbf{S}_m, \ \ \text{a generalized symmetric group } S(2, m); \tag{9}$$

such groups are also known as "signed symmetric groups."

An element $A \in \mathbf{GL}(m, \mathbb{F}_{1^2})$ preserves the standard form if and only if $A^T A = \mathrm{id}_m$. Every permutation matrix satisfies this identity, and since the entries of elements of $\mathbf{GL}(m, \mathbb{F}_{1^2})$ are contained in $\mathbb{F}_{1^2}$, this property remains to be true; so we obtain that

$$\mathbf{U}(m, \mathbb{F}_{1^2}) = \mathbf{GL}(m, \mathbb{F}_{1^2}). \tag{10}$$

This is not true for general extensions $\mathbb{F}_{1^\ell}$, but by accident, the unitary groups over $\mathbb{F}_{1^2}$ are maximally large.

If we now adapt the notion of observable, we obtain that $H \in \mathbf{GL}(m, \mathbb{F}_{1^2})$ defines an observable if and only if $H^T = H$, that is, if and only if

$$H^2 = \mathrm{id}_m. \tag{11}$$

Again, the outcome is incidental because we are working over $\mathbb{F}_{1^2}$!

4. The Bigger Picture

Now that we have introduced the basics of a København quantum theory over $\mathbb{F}_1$, it is necessary to extend the theory to arbitrary extensions of $\mathbb{F}_1$. This consideration yields extra elbow room for theory and applications, as we will see.

4.1. The Frobenius Maps

Let $\overline{\mathbb{F}_1}$ be the algebraic closure of $\mathbb{F}_1$; it consists of all complex roots of unity plus an element 0, endowed with the natural multiplication; see [17,18]. For every positive integer ℓ, we have that $\mathbb{F}_{1^\ell} \leq \overline{\mathbb{F}_1}$. Elements of $\mathbb{F}_{1^\ell}$ are characterized by the fact that they are precisely the solutions of the equation

$$x^{\ell+1} = x. \tag{12}$$

Compare this to the analogous situation for the algebraic closure $\overline{\mathbb{F}_q}$ of the finite field $\mathbb{F}_q$; in this case, the Frobenius map $\mathtt{Fr}^q : v \mapsto v^q$ singles out the elements of $\mathbb{F}_q$.

Following [17,18], we call the map $\mathtt{Fr}_1^{\ell+1} : \overline{\mathbb{F}_1} \mapsto \overline{\mathbb{F}_1} : u \mapsto u^{\ell+1}$ the absolute (or $\mathbb{F}_1$-) Frobenius endomorphism of degree $\ell + 1$. We use the same name if the domain of $\mathtt{Fr}_1^{\ell+1}$ is reduced.

4.2. $\mathtt{Aut}(\mathbb{F}_{1^\ell})$

An automorphism of $\mathbb{F}_{1^\ell}$ is a permutation φ of $\mathbb{F}_{1^\ell}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all a, b. Note that $\varphi(0) = 0$ and $\varphi(1) = 1$. The set of all automorphisms of $\mathbb{F}_{1^\ell}$ is denoted by $\mathtt{Aut}(\mathbb{F}_{1^\ell})$, and is a group if we endow it with the group law "composition of maps."

Note that all automorphisms of $\mathbb{F}_{1^\ell}$ are given by maps $u \mapsto u^m$, with $(\ell, m) = 1$ (as they correspond to automorphisms of the cyclic group μ_ℓ). It follows that $\mathtt{Aut}(\mathbb{F}_{1^\ell})$ is isomorphic to the group of multiplicative units in the ring $\mathbb{Z}/\ell\mathbb{Z}$.

4.3. Involutions of the Fields $\mathbb{F}_{1^\ell}$

The involutory automorphism "complex conjugation" plays a crucial role in actual quantum theory over $\mathbb{C}$ (for instance, to define the standard inner product, orthogonality, etc.), and in GQT, an analogous role is played by the involutory (anti-)automorphisms. We need to understand such maps in the context of quantum theories over $\mathbb{F}_{1^\ell}$.

The following lemma classifies involutions of extension of $\mathbb{F}_1$.

Lemma 1. *The map $\mathtt{Fr}_1^{r+1}$ defines a nontrivial involutory automorphism of $\mathbb{F}_{1^m}$ if and only if the following conditions are satisfied:*

SUB *m divides $r(r+2)$;*
NTRIV *m does not divide r.*

Proof. Let $\sigma : v \mapsto v^{r+1}$ be a nontrivial involutory automorphism of $\mathbb{F}_{1^m}$. Then

$$u^{(r+1)(r+1)} = u \tag{13}$$

for all $u \in \mathbb{F}_{1^m}$. If we pass to $\overline{\mathbb{F}_1}$, then all solutions of (13) are precisely given by the elements of $\mathbb{F}_{1^{r(r+2)}}$ (see Section 4.1), so (SUB) holds. On the other hand, the fixed field of σ in $\mathbb{F}_{1^{r(r+2)}}$ is $\mathbb{F}_{1^r}$, so since we assume σ to be nontrivial, m cannot divide r (NTRIV): m divides r if and only if μ_m is a subgroup of μ_r if and only if $\mathbb{F}_{1^m}$ is a subfield of $\mathbb{F}_{1^r}$.

Finally, for σ to be an automorphism of $\mathbb{F}_{1^m}$, we need to invoke the necessary and sufficient condition (AUT): $(r + 1, m) = 1$; this property follows from (SUB). □

Example 1. *The map $\mathtt{Fr}_1^{r+1} : v \mapsto v^{r+1}$ induces involutions in the following natural cases: $\mathbb{F}_{1^{r+2}}$, $\mathbb{F}_{1^{2r}}$ (in case r is even), $\mathbb{F}_{1^{2(r+2)}}$ (in case r is even), and $\mathbb{F}_{1^{r(r+2)}}$. The latter example might seem more natural to some, if one replaces r by $\ell - 1$; we then get that the absolute Frobenius $\mathtt{Fr}_1^\ell$ is an involutory automorphism of $\mathbb{F}_{1^{\ell^2-1}}$ (which has size ℓ^2), with fixed field $\mathbb{F}_{1^{\ell-1}}$ (which has size ℓ). This strongly resembles the case of finite fields when ℓ is a prime power.*

4.4. Standard Examples of Absolute Quantum Theories

In this subsection, we describe an additional absolute quantum theory with standard inproduct, based on our knowledge of Lemma 1 (and the examples following the lemma). In a similar way, one can describe all absolute quantum theories.

So we consider the field $\mathbb{F}_{1^{r(r+2)}}$ and its absolute Frobenius automorphism $\mathtt{Fr}_1^{r+1} : v \mapsto v^{r+1}$. By Lemma 1 we know that it is involutory, and the fixed field is $\mathbb{F}_{1^r}$. We represent our states in the state space $V(n, \mathbb{F}_{1^{r(r+2)}})$, with standard inproduct

$$\left\langle \overline{x} \middle| \overline{y} \right\rangle = x_1^{r+1} y_1 + \cdots + x_n^{r+1} y_n. \tag{14}$$

Observables are Hermitian operators H satisfying

$$H = \left[H^T \right]^{\mathtt{Fr}_1^{r+1}}; \tag{15}$$

the underlying structure is that of an involutory permutation matrix, and for each nonzero entry h_{ij}, we must have that

$$h_{ij} = h_{ji}^{r+1}. \tag{16}$$

Note that $h_{ij} = h_{ji}^{r+1}$ if and only if $h_{ji} = h_{ij}^{r+1}$ since $\mathtt{Fr}_1^{r+1}$ is an involution.

Also note that all symmetric matrices in $\mathbf{GL}(n, \mathbb{F}_{1^{r(r+2)}})$ with only entries in $\mathbb{F}_{1^r}$ satisfy this condition.

Unitary operators are given by operators U for which

$$\left[U^T \right]^{\mathtt{Fr}_1^{r+1}} U = \mathrm{id}_m; \tag{17}$$

every permutation matrix satisfies this identity, and for U defined over $\mathbb{F}_{1^{r(r+2)}}$, we must have that each nonzero entry a satisfies $a^{r+2} = 1$, that is, $a \in \mathbb{F}_{1^{r+2}}$.

Theorem 1 (Unitaries and observables)**.** *With $\sigma = \mathtt{Fr}_1^{r+1}$, $\mathbf{U}(m, \mathbb{F}_{1^{r(r+2)}})$ is given by the wreath product $\mu_{1^{r+2}} \wr \mathbf{S}_m$, the generalized symmetric group $S(r+2, m)$. And observables are given by $(m \times m)$-matrices with precisely one nonzero element in each row and column, for which $h_{ij} = h_{ji}^{r+1}$ for each nonzero entry h_{ij}.*

The unitaries and observables now look very different than those in Section 3.5!

4.5. Orthogonality

If we work in quantum theories over extensions of type $\mathbb{F}_{1^\ell}$, everything we have observed in Section 3.4 remains valid.

5. Dictionary

In the instructive table below (Table 1), we compare actual quantum theory, modal quantum theory, general quantum theories and absolute quantum theory. For the latter, we only have plugged in the last example of the previous section. In the case of modal quantum theory, we present its "completed version" which I described in [9]. Also, k denotes the algebraic structure over which we coordinatize Hilbert spaces; σ is an involutory automorphism of k; $k_\sigma := \{a \mid a^\sigma = a\}$ plus induced field structure of k; and the last column provides the standard "inner product" defined by σ. Note that k_σ plays an important role in the formulation of Born's rule; for instance, in actual quantum theory over $\mathbb{C}$, we consider a quantum system described by the wave function $|\xi\rangle$, and suppose $|\phi_i\rangle$ is an eigenvector of an orthogonal base of eigenvectors of an observable H; also, let λ_i be the corresponding eigenvalue. If we write $\langle \phi_i | \xi \rangle = c + id$, with $c, d \in \mathbb{R}$, then the probability of the measurement λ_i is

$$\left| \langle \phi_i | \xi \rangle \right|^2 = c^2 + d^2. \tag{18}$$

As we mentioned in [9], one can formulate this rule in many other GQTs.

Table 1. A quantum dictionary.

Quantum Theory	k	σ	k_σ	Standard form $\left\langle \overline{x} \middle| \overline{y} \right\rangle$
Actual quantum theory	$\mathbb{C}$	$v \mapsto \overline{v}$	$\mathbb{R}$	$\overline{x_1}y_1 + \cdots + \overline{x_m}y_m$
Modal quantum theory	$\mathbb{F}_{q^2}$	$v \mapsto v^q$	$\mathbb{F}_q$	$x_1^q y_1 + \cdots + x_m^q y_m$
General quantum theory	division ring with involution σ	σ	k_σ	$x_1^\sigma y_1 + \cdots + x_m^\sigma y_m$
Absolute quantum theory	$\mathbb{F}_{1^{r(r+2)}}$	$v \mapsto v^{r+1}$	$\mathbb{F}_{1^r}$	$x_1^{r+1} y_1 + \cdots + x_m^{r+1} y_m$

6. One Cannot Clone an Unknown State in Absolute Quantum Theory

A result of Wootters and Zurek [11,19], and independently Dieks [12], states that one cannot "clone" an unknown state. Formally, one wants to solve the next equation:

$$U \cdot \Big(|\phi\rangle_A \otimes |e\rangle_B\Big) = |\phi\rangle_A \otimes |\phi\rangle_B \tag{19}$$

where $|\phi\rangle_A$ is an unknown state in a complex Hilbert space H_A and $|\phi\rangle_B$ is the clone in the Hilbert space H_B (which is a copy of H_A), where $|e\rangle_B$ is an unknown blank state in H_B, and U is a unitary operator.

Since $|\phi\rangle_A$ is arbitrary, we can replace it by a linear combination

$$\alpha|\phi\rangle_A + \beta|\phi'\rangle_A, \tag{20}$$

and then the unitarity of U (or better: the linearity) easily leads to a contradiction. We refer to the discussion in Section 1.1 for additional remarks.

In [5], the authors have shown that similarly, one cannot clone an unknown state in modal quantum theory over prime fields. In [9], we obtained the general result that one cannot clone an unknown state in a general quantum theory over any division ring. Due to the degree of generality, the proof of that result is slightly more subtle than the complex or modal case.

Over $\mathbb{F}_{1^\ell}$, one cannot use mixed states such as (20), since we cannot add states. So we need a (slightly) different approach. We will consider (19) with only pure states, and take U to be in $\mathbf{U}(m, \mathbb{F}_{1^\ell})$. We will also suppose that $|e\rangle_B$ is an unknown, but fixed, blank state. As we will see, the particular nature of unitary operators over $\mathbb{F}_{1^\ell}$ already prevents the fact that the blank state can be randomly chosen.

First of all, note that the following identity should hold for any state $|\phi\rangle_A$ and any $\alpha \in \mathbb{F}_{1^\ell}$:

$$U\Big(\alpha|\phi\rangle_A \otimes |e\rangle_B\Big) = \alpha\Big(|\phi\rangle_A \otimes |\phi\rangle_B\Big) = \alpha|\phi\rangle_A \otimes \alpha|\phi\rangle_B, \tag{21}$$

so that $\alpha^2 = \alpha$ for all α, which is already false if $\ell \neq 1$, even for simple states.

In what follows, we will therefore work on the projective level, to see what the influence of factors is in this context.

We first work with an unknown simple state $|\phi\rangle_A$. As U must have the structure of a permutation matrix, the fact that $|\phi\rangle_A \otimes |\phi\rangle_B$ is simple, implies that $|e\rangle_B$ also must be simple. But then if we consider a state $|\phi'\rangle_A$ which is not simple, obviously the identity (19) cannot work, due again to the permutation matrix structure of U.

Still, as states are only determined up to factors, the natural question arises whether we can clone, projectively, the simple state rays. This is in fact very easy: As we have seen, since we are only cloning simple states, $|e\rangle_B$ must be a simple state itself. On the other hand, since we are working in a projective space, there are only m different simple states if we assume that H_A has dimension m over $\mathbb{F}_{1^\ell}$; in fact, if H_A would have dimension m over $\mathbb{F}_1$, we would obtain essentially the same points. Obviously, we can find a permutation matrix U in $\mathbf{U}(m, \mathbb{F}_{1^\ell})$ which maps $|\phi\rangle_A \otimes |e\rangle_B$ to $|\phi\rangle_A \otimes |\phi\rangle_B$ with $|\phi\rangle_A$ varying through the set of simple states, so that we obtain a "simple cloning" result.

Interpreting this result on the level of the classical case (so over $\mathbb{C}$), we obtain the well-known understanding that orthogonal states indeed can be cloned (see for instance Wootters and Zurek [11,19]) (note that in the initial vectorial case, the simple cloning result also works if one assumes the simple states one is considering to be orthogonal). So in the philosophy of Chang et al. [3], we obtain a new instance of the formalism

$$\mathbf{MQT}_q \xrightarrow{q \mapsto 1} \mathbf{AQT}. \tag{22}$$

7. Quantum Deletion in the Absolute and Actual Context

In [13], Pati and Braunstein obtain a no-deleting result in actual quantum theory, which was later shown to hold in all GQTs, in [9]. Formally, one now wants to solve the next equation:

$$U \cdot \Big(|\phi\rangle_A \otimes |\phi\rangle_B \Big) = |\phi\rangle_A \otimes |e\rangle_B, \tag{23}$$

where $|\phi\rangle_A$ is an unknown state in a complex Hilbert space H_A and $|\phi\rangle_B$ is the copy of $|\phi\rangle_A$ in the Hilbert space H_B (which is a copy of H_A), where $|e\rangle_B$ is an unknown blank state in H_B, and U is a unitary operator.

The simplest proof of the fact that no such U can exist, seems to be the following: Simply observe that if U is as above, then U^{-1} is a cloning operator in the sense of the previous section, so that we can finish the proof by using that section.

Still, it might be interesting to consider the problem in a more general context, and to allow "singular unitary operators." In fact, *because* we can show that quantum deletion is not possible, simply by inverting U, this very fact suggests that the initial definition of quantum deletion of [13] might not be the correct one: We propose to break the symmetry between the current notions of "cloning operators" and "deleting operators," by allowing the latter operators to be singular (while at the same time exhibiting unitary properties). In some sense, the singularity property is more natural than the nonsingularity: Once a singular operator deletes a copy of some wave state, the process can not be reversed.

Call an operator U (seen as an $(m \times m)$-matrix) almost unitary if every nonsingular submatrix which is constructed by deleting columns and rows with the same column –and row indices, is unitary. Many other alternative definitions could be formulated. In any case, if an almost unitary operator is nonsingular, it is unitary, and every unitary operator is almost unitary.

Now we consider the Equation (23) in absolute quantum theory, and allow U to be almost unitary (in the absolute context). In exactly the same way as in the previous section, we find that $\alpha^2 = \alpha$ for each $\alpha \in \mathbb{F}_{1^\ell}$. So again, we look at the more natural projective situation.

Observe that if $|\phi\rangle_A$ is a simple state, then $|e\rangle_B$ necessarily is simple. We suppose without loss of generality, that the first entry of $|e\rangle_B$ is 1, and that the others are 0. Now define U, an $(m^2 \times m^2)$-matrix, as follows:

- $U_{11} = U_{(m+1)(m+1)} = \cdots = U_{(m^2-m+1)(m^2-m+1)} = 1$;
- all other entries are 0.

Then obviously, U is almost unitary over $\mathbb{F}_{1^\ell}$, $\mathbb{C}$, and any other division ring/field.

Now consider any state $|\phi\rangle_A = (a_1 \ldots a_m)^T$ with first entry $a_1 \neq 0$. Then, with $\mathbf{0}$ denoting the $\Big((m-1) \times 1\Big)$-zero matrix,

$$U \cdot \Big(|\phi\rangle_A \otimes |\phi\rangle_B \Big) = a_1 \cdot \begin{pmatrix} a_1 \\ \mathbf{0} \\ a_2 \\ \mathbf{0} \\ \vdots \\ a_m \\ \mathbf{0} \end{pmatrix} = a_1 \cdot |\phi\rangle_A \otimes \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix} = a_1 \cdot |\phi\rangle_A \otimes |e\rangle_B, \tag{24}$$

and since we work projectively, this means that U indeed quantum deletes one copy from every such $|\phi\rangle_A \otimes |\phi\rangle_B$.

If $a_1 = 0$, then U maps $|\phi\rangle_A \otimes |\phi\rangle_B$ to the zero element of the vector space—that is, its action on the projective state points with $a_1 = 0$ is not defined. As the condition $a_1 \neq 0$ defines an affine subspace of the same dimension as the projective space, and as our considerations do not use the fact that we are working over $\mathbb{F}_{1^\ell}$, the formalism is also true for actual quantum theory, and also for every GQT. So as in [3], considerations over $\mathbb{F}_{1^\ell}$ lead to a classical result.

We have obtained the following result, which we state separately for the three types of quantum theories (actual, general, absolute).

Theorem 2 (Quantum deletion by almost unitary operators—actual/classical)**.** *There exists an almost unitary operator $U : \mathcal{H}_A \otimes \mathcal{H}_B \mapsto \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are copies of the same Hilbert space over $\mathbb{C}$, and a blank state $|e\rangle_B$, such that U quantum deletes one copy in each (projective) state space point $|\phi\rangle_A \otimes |\phi\rangle_B$ for which the first coordinate is not zero.*

More generally, we have the general formulation:

Theorem 3 (Quantum deletion by almost unitary operators—general)**.** *There exists an almost unitary operator $U : \mathcal{H}_A \otimes \mathcal{H}_B \mapsto \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are copies of the same Hilbert space over any division ring with involution, and a blank state $|e\rangle_B$, such that U quantum deletes one copy in each (projective) state space point $|\phi\rangle_A \otimes |\phi\rangle_B$ for which the first coordinate is not zero.*

Finally, we have the "absolute version."

Theorem 4 (Quantum deletion by almost unitary operators—absolute)**.** *There exists an almost unitary operator $U : \mathcal{H}_A \otimes \mathcal{H}_B \mapsto \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are copies of the same Hilbert space over $\mathbb{F}_{1^\ell}$, and a blank state $|e\rangle_B$, such that U quantum deletes one copy in each (projective) state space point $|\phi\rangle_A \otimes |\phi\rangle_B$ for which the first coordinate is not zero.*

7.1. Deleting Probability

In the final part of this section, we calculate the portion of projective states which is effectively deleted (relative to the entire set of states). As we will see, the ratio of such states relative to all states tends to 1, which justifies the term "virtual deletion operators."

We handle the finite case (that is, Hilbert spaces over finite fields and extensions of $\mathbb{F}_1$), and the infinite case (Hilbert spaces over other division rings, and in particular $\mathbb{C}$), in separate subsections.

7.1.1. Finite Case

Consider an $\mathbb{F}_1$-extension $\mathbb{F}_{1^\ell}$, or a finite field $\mathbb{F}_q$ with $|\mathbb{F}_q| = \ell + 1$. Then the probability that we pick a point in the affine subspace $a_1 \neq 0$ in the projective ray state space $\mathbb{P}^{m-1}(\mathbb{F}_{1^\ell})$ or $\mathbb{P}^{m-1}(\mathbb{F}_q)$ of $\mathcal{H}_A$, is

$$\frac{\left(\ell+1\right)^{m-1}}{\left((\ell+1)^m - 1\right)/\ell} = \frac{\ell}{\ell + (1 - 1/(\ell+1)^{m-1})} =: P_{a_1}. \tag{25}$$

We have that

$$\lim_{m\mapsto\infty} P_{a_1} = \frac{\ell}{\ell+1}, \quad \text{while} \quad \lim_{\ell\mapsto\infty} P_{a_1} = 1. \tag{26}$$

If we would take MQT as a model for quantum theory, the value ℓ would be very large in concrete situations, so the limit $\lim_{\ell\mapsto\infty} P_{a_1}$ is highly relevant in the modal setting.

7.1.2. Over $\mathbb{C}$, and Other Fields/Division Rings

Now let $\mathcal{H}_A$ be a vector space over an infinite field or division ring k. To fix ideas, one can put $k = \mathbb{C}$. We cannot define a uniform distribution on $\mathcal{H}_A$ (which we identify with the diagonal subspace $\{|\phi\rangle_A \otimes |\phi\rangle_B\}$ of $\mathcal{H}_A \otimes \mathcal{H}_B$), but on the other hand, it is well known that the Lebesgue measure of a hyperplane in a projective space $\mathbb{P}^n(k)$ is zero. We interpret this fact as the idea that the probability of choosing a point outside a given hyperplane in $\mathbb{P}^{m-1}(k)$ tends to 1.

7.1.3. Interpretation

We interpret the probability considerations in this subsection as follows:

> The almost unitary operator $U : \mathcal{H}_A \otimes \mathcal{H}_B \mapsto \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are copies of the same Hilbert space (over any choice of division ring or field) virtually deletes all projective wave states.

7.2. Cloning for Almost Unitary Operators

Note that by considering a nonsimple $|\phi\rangle_A$ (and a simple $|e\rangle_B$ as above), one already sees that cloning is not possible for almost unitary operators as well.

8. Conclusions

We started this paper with formalizing and completing the "quantum $\mathbb{F}_{un}$ theory" of Chang, Lewis, Minic and Takeuchi [3], adding unitary (time evolution) operators and Hermitian (observable) operators to the story. To introduce more elbow room, we then extended that theory to extensions of $\mathbb{F}_1$, relying on the general quantum theory founded in our paper [9]. After a careful analysis of the possible involutions (which each define a standard "inner product"), and hence the unitary and Hermitian operators, we obtained the no-cloning result for quantum theories over arbitrary extensions of $\mathbb{F}_1$.

Then, we introduced "almost unitary operators" as singular analogues of unitary operators, and showed that in every quantum theory, there exist irreversible almost unitary operators which delete any copy of a wave state in projective wave state space outside an affine subspace, thus deleting arbitrary wave states with a probability tending to 1.

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References

1. Chang, L.N.; Lewis, Z.; Minic, D.; Takeuchi, T. Galois field quantum mechanics. *Modern Phys. Lett. B* **2013**, *27*, 1350064. [CrossRef]
2. Chang, L.N.; Lewis, Z.; Minic, D.; Takeuchi, T. Spin and rotations in Galois field quantum mechanics. *J. Phys. A* **2013**, *46*, 065304. [CrossRef]
3. Chang, L.N.; Lewis, Z.; Minic, D.; Takeuchi, T. Quantum $\mathbb{F}_{un}$: The $q = 1$ limit of Galois field quantum mechanics, projective geometry, and the field with one element. *J. Phys. A* **2014**, *47*, 405304. [CrossRef]
4. Lev, F.M. Introduction to a quantum theory over a Galois field. *Symmetry* **2010**, *2*, 1810–1845. [CrossRef]
5. Schumacher, B.; Westmoreland, M.D. Modal quantum theory. *Found. Phys.* **2012**, *42*, 918–925. [CrossRef]
6. Thas, K. *Absolute Arithmetic and $\mathbb{F}_1$-Geometry*; European Mathematical Society EMS: Zürich, Switzerland, 2016
7. Hirschfeld, J.W.P. *Projective Geometries over Finite Fields*, 2nd ed.; Oxford Mathematical Monographs; Oxford University Press: New York, NY, USA, 1998.
8. Shult, E.E. *Points and Lines. Characterizing the Classical Geometries*; Universitext; Springer: Heidelberg/Berlin, Germany, 2011.
9. Thas, K. General quantum theory. *arXiv* **2018**, arXiv:1712.04669.

10. Veblen, O.; Young, J.W. A set of assumptions for projective geometry. *Am. J. Math.* **1908**, *30*, 347–380. [CrossRef]
11. Wootters, W.K.; Zurek, W.H. A single quantum cannot be cloned. *Nature* **1982**, *299*, 802–803. [CrossRef]
12. Dieks, D. Communication by EPR devices. *Phys. Lett. A* **1982**, *92*, 271–272. [CrossRef]
13. Pati, A.K.; Braunstein, S.L. Impossibility of deleting an unknown quantum state. *Nature* **2000**, *404*, 164–165. [CrossRef] [PubMed]
14. Aleksandrova, A.; Borish, V.; Wootters, W.K. Real-vector-space quantum theory with a universal quantum bit. *Phys. Rev. A* **2013**, *87*, 052106. [CrossRef]
15. Wootters, W.K. Entanglement sharing in real-vector-space quantum theory. *Found. Phys.* **2012**, *42*, 19–28. [CrossRef]
16. Wootters, W.K. *Fundamental Theories of Physics*; Springer: Dordrecht, The Netherlands, 2016; Volume 181; pp. 21–43.
17. Thas, K. Projective spaces over $\mathbb{F}_{1^\ell}$. *J. Combin. Des.* **2019**, *27*, 55–74. [CrossRef]
18. Thas, K. The combinatorial-motivic nature of $\mathbb{F}_1$-schemes. In *Absolute Arithmetic and* $\mathbb{F}_1$*-Geometry*; European Mathematical Society EMS: Zürich, Switzerland, 2016; pp. 83–159.
19. Wootters, W.K.; Zurek, W.H. The no-cloning theorem. *Phys. Today* **2009**, *2*, 76–77. [CrossRef]