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Fixed Points for Multivalued Convex Contractions on Nadler Sense Types in a Geodesic Metric Space

Amelia Bucur

Department of Mathematics and Informatics, Faculty of Sciences, Lucian Blaga University of Sibiu, Sibiu 550012, Romania; amelia.bucur@ulbsibiu.ro; Tel.: +04-074-038-1318

Received: 7 December 2018; Accepted: 18 January 2019; Published: 30 January 2019

Abstract: In 1969, based on the concept of the Hausdorff metric, Nadler Jr. introduced the notion of multivalued contractions. He demonstrated that, in a complete metric space, a multivalued contraction possesses a fixed point. Later on, Nadler's fixed point theorem was generalized by many authors in different ways. Using a method given by Angrisani, Clavelli in 1996 and Mureşan in 2002, we prove in this paper that, for a class of convex multivalued left A-contractions in the sense of Nadler and the right A-contractions with a convex metric, the fixed points set is non-empty and compact. In this paper we present the fixed point theorems for convex multivalued left A-contractions in the sense of Nadler and right A-contractions on the geodesic metric space. Our results are particular cases of some general theorems, to the multivalued left A-contractions in the sense of Nadler and right A-contractions, and particular cases of the results given by Rus (1979, 2008), Nadler (1969), Mureşan (2002, 2004), Bucur, Guran and Petruşel (2009), Petre and Bota (2013), etc., and are applicable in many fields, such as economy, management, society, biology, ecology, etc.

Keywords: fixed point; convex multivalued left A-contraction; right A-contraction; geodesic metric space; regular golbal-inf function

MSC: 47H10; 54H25

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1. Introduction

Fixed point theory plays a major role, not only in functional and nonlinear analysis, in solving problems from the theory of differential and integral equations, partial or random differential equations, differential and integral inclusions, functional equations, the approximation methods but also in management, economics, finances, computer science, and other fields [1].

The field of fixed point theory is based on the works of Poincaré, Lefschetz-Hopf, and Leray-Schauder. Their theory has been of high importance in the topological field, as well as for the degree theory.

The metric theory consists of making successive approximations in order to reveal the existence and the uniqueness of the solution. Mathematician Banach improved this theory by expanding its use to more than ordinary differential equations and integral equations. Banach's fundamental fixed point theorem was used to create the metric fixed point theory, which implies contraction mappings that are defined on a complete metric space.

In 1965, Browder, Göhde, and Kirk [2–4] developed the theory of multivalued mappings, which has applications in the following areas: Convex optimization, differential inclusions, control theory, management, finances, and economics. In addition, based on Banach's theory, in 1969 Nadler Jr. [5] demonstrated that the multivalued version of the theory has a fixed point, by using the concept of the Hausdorff metric.

In 2004, Ran and Reurings [6] demonstrated how the Banach contraction principle endowed with a partial order can be used to solve certain matrix equations. Similarly, in 2007, Nieto and Rodrigues-López [7] showed how the extension to the Banach contraction principle can be used to solve differential equations, however, Jachymski (2007) [8] used graphs instead of a partial order and obtained a more general version of the previous extensions.

Espinola and Nicolae in 2015 [9], also Nicolae and 2011 [10], and Leustean [11], used some fixed-point theorems in geodesic metric spaces.

Our results are particular cases of some general theorems, to the multivalued left A-contractions in the sense of Nadler and right A-contractions, and particular cases of the results given by Rus [12,13], Nadler [5], Mureşan [14,15], Bucur, Guran and Petruşel [16], and Petre and Bota [17], etc.

2. Literature Review

In the past years, an increasing number of papers has been published on the topic of fixed points of multivalued operators, using different methods [18].

Based on the concept of the Hausdorff metric, Nadler Jr. (1969) [5] introduced the notion of multivalued contractions and demonstrated that, in a complete metric space, a multivalued contraction possesses a fixed point.

Later on, Nadler's fixed point theorem was generalized in different ways by many authors.

For example, in 2015, using an axiomatic approach of the Pompeiu-Hausdorff metric, Coroian (2015) [19] studied the properties of the fractal operator generated by a multivalued contraction.

Aydi, Abbas, and Vetro, in their paper published in 2012 [20], also obtained a version of the Nadler fixed point theorem. They extended Nadler's fixed point theorem, obtaining results for multivalued mappings defined on complete partial metric spaces.

In 2013, Petre and Bota [17] using the concept of a generalized Pompeiu–Hausdorff functional presented some fixed and strict fixed point theorems in generalized b-metric spaces.

In 1996, Angrisani and Clavelli [21], using the class of regular-global-inf functions, presented a new method to prove fixed point theorems. We will use this method to multivalued left A-contractions in the sense of Nadler and the concept of the generalized metric space in the Perov' sense, and we prove the compactness of the fixed points set of the considered mappings.

Bucur, Guran and Petruşel (2009) [16] extended some old fixed point theorems and obtained some results on fixed points of multivalued operators on generalized metric spaces. Other results for generalized contractions in complete metric spaces were demonstrated by Kikkawa and Suzuki (2008) [22]. In the year 2011, Rezapour and Amiri [23] used Kikkawa's method and obtained new theorems on fixed points for multivalued operators defined on generalized metric spaces.

In another paper by Rezapour and Amiri [24], published in 2012, the authors obtained new theorems on fixed points for multivalued operators defined on generalized metric spaces by providing different conditions for [16] published in 2009.

Thus, there have been demonstrated fixed point theorems of multivalued operators on different types of spaces.

Some authors obtained new fixed-point results in partial metric spaces, while other authors have obtained new fixed point results in b-metric spaces.

As known, the Banach contraction principle shows that a contraction defined on a complete metric space always has a unique fixed point. In addition, this principle shows that the fixed point can be approximated by using Picard's iterates. W. A. Kirk (see reference [25]) discusses for the first time the fixed-point theory in CAT(0) spaces (Cartan-Alexandrov-Toponogov spaces), which is known to be a geodesic metric space. W. A. Kirk demonstrated that a non-expansive mapping with a compact and convex domain, subset of the CAT(0) space, always has a fixed point. Many others specialists demonstrated new fixed point theorems for various types of mappings in the CAT(0) space (for example, references [26–35]).

In this paper, we mention that the notion of convergent sequence, open subset and closed subset, Cauchy sequence, completeness, for a geodesic metric space, are analogous to those for metric spaces that are usually used.

Our results are particular cases of some general theorems, for the convex multivalued left A-contractions in the sense of Nadler, particular cases of the results given by Petruşel (1996, 2004), Rus (1979, 2008), Bucur, Guran and Petruşel (2009), and Mureşan (2002), etc.

3. Preliminaries

For (X,d) a metric space, we denote by:

$P(X)$—the set of all subsets of X, which are nonempty;

$P_c(X)$—the set of all compact subsets of X, which are nonempty.

Based on these subsets we consider the operators:

$D : P(X) \times P(X) \to [0,\infty),\ D(Z,Y) inf\{d(x,y) : x \in Z,\ y \in Y\},\ Z\ being\ part\ of\ X$—the gap functional;

$H : P(X) \times P(X) \to [0,\infty),\ H(Z,Y) max\{sup_{x\in Z} inf_{y\in Y} d(x,y),\ sup_{y\in Y} inf_{x\in Z} d(x,y)\}$—the Pompeiu-Hausdorff functional.

In addition, for $Z \in P_c(X)$, we have $diam(Z) = sup\{d(x,y) : x,y \in Z\}$ and $\alpha_K(Z) = inf\{\varepsilon > 0 : Z = U_{i\in I} Z_i,\ diam\ Z_i \leq \varepsilon\}$ the Kuratowski measure of noncompactness.

Let there be a real valued function $F : X \to R$. For any $p \in R$ we denote $L_p = \{x \in X : F(x) \leq p\}$ the p-level set and $infF = \{F(x) : x \in X\}$.

Definition 1. (see reference [14]) *Functional $F : X \to R$ is known as regular-global-inf (r.g.i.) in $x \in X$ if and only if $F(X) > infF$ implies that there is a $p > infF$ such that $D(x, L_p) > 0$. Functional F is called r.g.i.in X if is r.g.i.in any $x \in X$.*

Proposition 1. (see [14])

(i) Let $Z,Y \in P_c(X)$. For any $x \in Z$ and $q > 1$ exists $y \in Y$ with $d(x,y) \leq qH(Z,Y)$;

(ii) For all (X,d), a complete metric space, we obtain that $(P_c(X), H)$ is also a complete metric space.

Proposition 2. (see [14])

Let (X,d) is a complete metric space and $F : X \to [0,\infty)$ is a r.g.i. function in X. If $\lim\limits_{p\downarrow infF} \alpha_K(L_p) = 0$ then the set of the global minimum point of F is nonempty and compact.

Definition 2. (see [16])

We consider X a set, $X \neq \varnothing$. In addition, we consider the vector space of vectors with positive real components R^m_+, equipped with the usual component-wise partial order. The application $d : X \times X \to R^m_+$ which satisfies the usual axioms of the metric is defined as a generalized metric in the sense of Perov.

We mention that the generalized metric in Perov' sense is in fact a particular case of the K-metric.

Let (X,d) be a generalized metric space in Perov'sense. Here, if $v, r \in R^m$, $v(v_1,\ v_2,\ \ldots, v_m)$ and $r(r_1,\ r_2,\ \ldots, r_m)$, then by $v \leq r$ we mean $v_i \leq r_i$, for each $i \in \{1,2\ldots,m\}$, while $v < r$ stands for $v_i < r_i$, for each $i \in \{1,2\ldots,m\}$. In addition, $|v| = (|v_1|,\ |v_2|, \ldots, |v_m|)$. If $u, v \in R^m$, with $u(u_1,\ u_2,\ \ldots, u_m)$ and $v(v_1,\ v_2,\ \ldots, v_m)$, then $max(u,v)(max(u_1,v_1),\ max(u_2,v_2), \ldots, max(u_m,v_m))$ and, if $c \in R$ then $v_i \leq c$, for each $i \in \{1,2\ldots,m\}$.

In a generalized metric space in the sense of Perov, the concepts of convergent sequence, Cauchy sequence, completeness, and also the concepts of open and closed subsets are defined similarly to those in a metric space. If $x_0 \in X$ and $r \in R^m_+$ with $r_i > 0$ for each $i \in \{1,2\ldots,m\}$ we will denote by $B(x_0,r)\{x \in X\ : d(x_0,x) < r\}$ the open ball centred in x_0 with the radius $r(r_1,\ r_2,\ \ldots, r_m)$ and by $\overline{B}(x_0,r)\{x \in X\ : d(x_0,x) \leq r\}$ the closed ball centred in x_0 with the radius r. If $T : X \to P(X)$

is a multivalued operator, then we denote by *Fix(T)* the fixed point of T. We have $Fix(T) := \{x \in X : x \in T(x)\}$.

Notice that a generalized Pompeiu-Hausdorff functional can be introduced by the setting of a generalized metric space in the sense of Perov. Namely, if (X, d) is a generalized metric space in the sense of Perov with $d(d_1, d_2, \ldots, d_m)$ and if H_i denotes the Pompeiu-Hausdorff metric on $P_c(X)$ generated by d_i (where $i \in \{1, 2 \ldots, m\}$) then we denote by $H : P_c(X) \times P_c(X) \to R_+^m$, $H(H_1, H_2, \ldots, H_m)$ the vector-valued Pompeiu-Hausforff metric on $P_c(X)$.

Definition 3. (see [16]) *For* $A \in M_{m,m}(R_+)$*, a matrix convergent to zero and for which*

$$H(T(x),\ T(y)) \leq Ad(x, y), \textit{for all } x, y \in Y$$

is said that any A multivalued operator $T : Y \subset X \to P_c(X)$ *is a multivalued left A-contraction in the sense of Nadler*.

Definition 4. *If (X, d) is a metric space and x, y are two fixed elements from X with d(x; y) = l, a geodesic path from x to y is defined as an isometry c:[0; l]* $\to$ *c([0; 1])* $\subset$ *X for which c(0) = x, c(l) = y. The set c([0; 1]) of a geodesic path between two points x and y is defined as a geodesic segment. It is said that a metric space (X, d) is a geodesic space if between every two points x and y of X there is a geodesic segment.*

In the specialty literature, a geodesic segment between the two points *x, y* is denoted by [*x; y*]. A point z in [*x; y*] is equal by $(1 - \alpha)x \oplus \alpha\, y$ with $\alpha \in [0;\ 1]$. Thus, $[x;\ y] := \{(1 - \alpha)x \oplus \alpha\, y : \alpha \in [0;\ 1]\}$. The metric d is a convex function, and a closed ball $B[x, r] := \{y;\ d(y, x) \leq r\}$, $r > 0$ is a set metrically convex.

Notice that for $m = 1$ we get the well-known concept of contraction mapping defined by S. B. Nadler Jr. (1969). We also point out that, by the properties of the functional H, if T is a multivalued left A-contraction, then T is a multivalued left A-contraction in the sense of Nadler.

The following definition expresses a dual concept.

Definition 5. (see reference [16]) *Let* $Y \subset X$ *and* $T : Y \to P(X)$ *be a multivalued operator. Thus, T is called a multivalued right A-contraction if* $A \in M_{m,m}(R_+)$ *is a matrix convergent to zero and*

$$(H(T(x),\ T(y)))^t \leq (d(x, y))^t A, \textit{for all } x, y \in Y.$$

In a particular case, if (X, d) is a generalized metric space in the sense of Perov, then it can be a geodesic metric space.

Observation 1. (see reference [16]) *Notice that since* $\left[(d(x, y))^t A\right]^t = A^t d(x, y)$*, the right A-contraction condition on the multivalued operator T is equivalent to the left* A^t*—contraction condition given in Definition 3. In addition, a matrix A converges to zero if and only if matrix* A^t *converges to zero (due to the fact that A and* A^t *have the same eigenvalues) and* $\left[(I - A)^{-1}\right]^t = \left(I - A^t\right)^{-1}$.

Thus, a matrix A is convergent to zero if and only if $A^n \to 0$ as $n \to \infty$.

Lemma 1. (see [26,27]) *If (X, d) is a geodesic metric space, we have the following inequality*

$$d((1 - \alpha)x \oplus a\, y, z) \leq (1 - \alpha)d(x,\ z) + \alpha d(y,\ z), \textit{for all } \alpha \in [0;\ 1] \textit{ and } x,\ y,\ z \in X.$$

Definition 6. *Let (X, d) be a geodesic metric space. We say that A multivalued operator $T : Y \subset X \to P_c(X)$ is a multivalued left A-contraction in the sense of Nadler in relation to a convex metric, if $A \in M_{m,m}(R_+)$ is a matrix convergent to zero and*

$$H((1-\alpha)T(x) \oplus \alpha\, T(y),\ T(z)) \le Ad((1-\alpha)x \oplus \alpha\, y,\ z)$$

for all $x, y \in Y$.

Definition 7. *If (X, d) is a geodesic metric space, and $Y \subset X$, let $T : Y \to P(X)$ be a multivalued operator. In this case, T is called a multivalued right A-contraction in the sense of Nadler in relation to a convex metric, where $A \in M_{m,m}(R_+)$ is a matrix convergent to zero, then the following inequality takes place*

$$(H((1-\alpha)T(x) \oplus \alpha\, T(y),\ T(z)))^t \le (d((1-\alpha)x \oplus \alpha\, y,\ z))^t A, for\ all\ x, y \in Y.$$

4. Fixed Point Theorems

We now present our new results.

Theorem 1. *Let (X, d) be a complete geodesic metric space, $d(d_1, d_2, \ldots, d_m)$, and $T : X \to P_c(X)$ a convex multivalued left A-contraction in the sense of Nadler in relation to a convex metric if $A = (a_{ii}) \in M_{m,m}(R_+)$, $a_{ii} \le 1$, $i \in \{1, 2 \ldots, m\}$, is a diagonal matrix convergent to zero and*

$$H_i((1-\alpha)T(x) \oplus \alpha\, T(y),\ T(z)) \le a_{ii} d_i((1-\alpha)x \oplus \alpha\, y,\ z),$$

$i \in \{1, 2 \ldots, m\}$, for all $x, y \in X$, then $Fix(T) \neq \varnothing$ and is compact.

Proof. We define the functional

$$F : X \to [0, \infty)\,,\ D(x, T(x)) inf_{i \in \{1,2\ldots,m\}} \{d_i(x, \lambda) : \lambda \in T(x)\},$$

$$D(x,) : P(X) \to [0, \infty)$$

and we prove that:

(i) inf F = 0;

(ii) $\lim_{p \downarrow 0} \alpha_K(L_p) = 0;$

(iii) F is r.g.i. in X. □

By applying Proposition 2, the conclusion will be obtained.

(i) We take $x_0 \in X$ and $x_1 \in Tx_0$.

Considering $q_i > 1$ such that

$$\beta_i = q_i a_i < 1,\ i \in \{1, 2 \ldots, m\} \tag{1}$$

from Proposition 2, there will exist an $x_2 \in Tx_1$, such that

$$d_i(x_1, x_2) \le q_i H_i(T(x_0), T(x_1)) \le q_i a_i d_i(x_0, x_1).$$

Hence, $d_i(x_1, x_2) \le \beta_i d_i(x_0, x_1)$.

Thus, we obtain the sequence $(x_n)_{n \in N^*}$ with the properties:

$$\in T(x_{n-1}) \tag{2}$$

and

$$d_i(x_n, x_{n-1}) \leq \beta_i d_i(x_{n-1}, x_n),\ \text{for } n \in N^* \tag{3}$$

From (3), $F(x_n) \leq \beta_i^n d_i(x_0, x_1)$, which implies inf F = 0.

(ii) Considering $x_0 \in L_p$ $(D(x_0, T(x_0)) \leq p)$, because $0 < \beta_i < 1$, we find $x_1 \in T(x_0)$ such that $d_i(x_0, x_1) \leq \frac{p}{\beta_i}$ and for the sequence $(x_n)_{n \in N^*}$ we will have, from (3), for any $n \in N^*$, the following:

$$F(x_n) \leq d_i(x_n, x_{n-1}) \leq p\beta_i^n \leq p,\ i \in \{1, 2 \ldots, m\}.$$

Thus, $x_n \in L_p$ for all $n \geq 1$.

Because $d_i(x_0, x_1) \leq \frac{p}{\beta_i}$,

$$d_i(x_0, x_n) \leq\ d_i(x_0, x_1) +\ d_i(x_1, x_n) \leq \frac{p}{\beta_i} + \frac{p}{\beta_i - 1}\ for\ all\ n \geq 1,\ i \in \{1, 2 \ldots, m\},$$

$$d_i(x_n, x_{n-k}) \leq \frac{p}{\beta_i - 1}\ for\ all\ n \geq 1\ and\ k \geq 1,$$

we have

$$\alpha_K(L_p) \leq diam(x_n) \leq \frac{p}{\beta_i} + \frac{p}{\beta_i - 1},$$

which implies that $\lim_{p \downarrow 0} \alpha_K(L_p) = 0$.

(iii) We suppose that F is not r.g.i. in X. It results that there exists an $x \in X$ with the following properties:

$$F(x) > 0 \text{ and } D(x, L_p) = 0, \text{ for any } p > 0. \tag{4}$$

There will exist a sequence $(x'_n)_{n \in N^*}$ such that

$$x'_n \in L_{\frac{1}{n}} \text{ and } d_i(x, x'_n) \leq \frac{1}{n} \text{ for all } n \geq 1,\ i \in \{1, 2 \ldots, m\}. \tag{5}$$

Choosing $x'' \in T(x)$ and $x''_n \in T(x'_n)$ in anyway we have

$$d_i(x, x'') \leq\ d_i(x, x'_n) +\ d_i(x'_n, x''_n) +\ d_i(x''_n, x''),\ i \in \{1, 2 \ldots, m\}.$$

Considering this inequality $inf_{x''_n \in T(x'_n)}$, for all $n \in N^*$ we have:

$$\begin{aligned} d_i(x, x'') &\leq \frac{1}{n} + D(x'_n, T(x'_n)) + D(T(x'_n), x'') \\ &\leq \frac{1}{n} + F(x'_n) + H_i(T(x'_n), T(x)) \\ &\leq \frac{2}{n} + a_{ii}\ d_i(x'_n, x) + F(x'_n),\ i \in \{1, 2 \ldots, m\}. \end{aligned}$$

From (5), $a_{ii} \leq 1$, $i \in \{1, 2 \ldots, m\}$, and $x'_n \in L_{\frac{1}{n}}$ we have:

$$d_i(x, x'') \leq \frac{1}{n} + \frac{a_{ii}}{n} + \frac{1}{n} \leq \frac{4}{n}$$

Then, in the last inequality, considering $inf_{x'' \in Tx}$, we obtain that $F(x) = 0$. That is a contradiction with Equation (4). Then, F is r.g.i. in X.

From Observation 1 and Theorem 1 we obtain the following dual result:

Theorem 2. *Let (X,d) be a complete geodesic metric space, $d(d_1,\ d_2,\ \ldots,d_m)$, and $T:\ X \rightarrow P_c(X)$ a convex multivalued right A-contraction in relation to a convex metric if $A = (a_{ii}) \in M_{m,m}(R_+)$, $a_{ii} \leq 1$, $i \in \{1,2\ldots,m\}$, is a diagonal matrix convergent to zero and*

$$H_i(T(x),\ T(y)) \leq d_i(x,y)a_{ii}, i \in \{1,2\ldots,m\}, \text{for all } x,y \in X,$$

then the set $Fix(T)$ of the fixed point of T is nonempty and compact.

Proof. The proof is analogous to the proof from Theorem 1. □

Theorem 3. *Let (X,d) be a complete geodesic metric space, $d(d_1,\ d_2,\ \ldots,d_m)$,$x_0 \in X$ and $r = (r_1,\ r_2,\ \ldots,r_m) > 0$. Let $T:\ \overline{B}(x_0,r) \rightarrow P_c(X)$ be a convex multivalued left A-contraction in the sense of Nadler in relation to a convex metric. If $A = (a_{ii}) \in M_{m,m}(R_+)$, $a_{ii} \leq 1$, $i \in \{1,2\ldots,m\}$, is a diagonal matrix convergent to zero and*

$$H_i((1-\ \alpha)T(x)\ \oplus\ \alpha\ T(y),\ T(z)) \leq a_{ii}d_i((1-\ \alpha)x\ \oplus\ \alpha\ y,\ z), i \in \{1,2\ldots,m\},$$

for all $x,y \in X$.

Supposing that:

(i) for v, $r \in R_+^m$ the following inequality is satisfied $v(I-A)^{-1} \leq (I-A)^{-1}r$, it results that $v \leq r$;

(ii) there is an $x_1 \in Tx_0$ so that $d(x_0,x_1)(I-A)^{-1} \leq r$.

Then T has at least one fixed point.

Proof. The proof is analogous to the proof from [16].

If $x_0 \in X$ with $x_1 \in T(x_0)$, we suppose that the following inequality is satisfied $d(x_0,x_1)(I-A)^{-1} \leq r \leq (I-A)^{-1}r$. In this case, according to (i), we have $x_1 \in \overline{B}(x_0,r)$. Thus, if we apply the contraction definition, we obtain that there is an $x_2 \in T(x_1)$ for which $d(x_1,x_2) \leq Ad(x_0,x_1)$. We obtain that $d(x_1,x_2)(I-A)^{-1} \leq A\ d(x_0,x_1)(I-A)^{-1} \leq Ar$. We mention that $x_2 \in \overline{B}(x_0,r)$. From $d(x_0,x_2) \leq d(x_0,x_1) + d(x_1,x_2)$ we obtain a new inequality, $d(x_0,x_2)(I-A)^{-1} \leq\ d(x_0,x_1)(I-A)^{-1} + d(x_1,x_2)(I-A)^{-1} \leq Ir + Ar \leq (I-A)^{-1}r$, from which results that, according to (i), $d(x_0,x_2) \leq r$. Thus, by mathematical induction, we create the sequence $(x_n)_{n\in N}$ in $\overline{B}(x_0,r)$ with the following properties:

(a) $x_{n+1} \in T(x_n),\ n \in N$;

(b) $d(x_0,x_n)(I-A)^{-1} \leq (I-A)^{-1}r, \in N$, that means (by (i)) $d(x_0,x_n) \leq r$;

(c) $d(x_n,x_{n+1})(I-A)^{-1} \leq A^n r,\ n \in N$.

By (c) we get that $d\big(x_n,x_{n+p}\big)(I-A)^{-1} \leq A^n(I-A)^{-1}r,\ n,p \in N^*$.

Thus, the sequence $(x_n)_{n\in N}$ is Cauchy in the complete geodesic metric space $\big(\overline{B}(x_0,r),d\big)$. We denote by x^* its limit in $\overline{B}(x_0,r)$.

We prove that $x^* \in T(x^*)$. If $n \in N^*$, for each $x_n \in T(x_{n-1})$ there exists $u_n \in T(x^*)$ such that $d(x_n,u_n) \leq A\ d(x_{n-1},x^*)$.

On the other hand $d(x^*,u_n) \leq d(x^*,x_n) +\ d(x_n,u_n) \leq d(x^*,x_n) +\ A\ d(x_{n-1},x^*) \rightarrow 0$ as $n \rightarrow \infty$. Hence, $\lim\limits_{n\rightarrow\infty} u_n = x^*$.

Since $u_n \in T(x^*)$ for $n \in N^*$ and knowing that $T(x^*)$ is closed, it results that $x^* \in T(x^*)$. The proof is complete. □

Theorem 4. *Let (X,d) be a complete geodesic metric space, $d(d_1,\ d_2,\ \ldots,d_m)$,$x_0 \in X$ and $r = (r_1,\ r_2,\ \ldots,r_m) > 0$. Let $T:\ \overline{B}(x_0,r) \rightarrow P_c(X)$ be a convex multivalued right A-contraction in relation to a convex metric. If $A = (a_{ii}) \in M_{m,m}(R_+)$, $a_{ii} \leq 1$, $i \in \{1,2\ldots,m\}$, is a diagonal matrix convergent to zero and*

$$H_i((1-\ \alpha)T(x)\ \oplus\ \alpha\ T(y),\ T(z)) \leq a_{ii}d_i((1-\ \alpha)x\ \oplus\ \alpha\ y,\ z), i \in \{1,2\ldots,m\},$$

for all $x, y \in X$.

Supposing that:

(i) for v, $r \in R_+^m$ the following inequality is satisfied $(I-A)^{-1}v \leq r(I-A)^{-1}$, it results that;

(ii) there is an $x_1 \in T(x_0)$ so that $(I-A)^{-1}d(x_0, x_1) \leq r$.

Then T has at least one fixed point.

Proof. The proof is analogous to the proof from Theorem 3.

If $x_0 \in X$ with $x_1 \in T(x_0)$, we suppose that the following inequality is satisfied $(I-A)^{-1}d(x_0, x_1) \leq r \leq r(I-A)^{-1}$. In this case, according to (i), we have $x_1 \in \overline{B}(x_0, r)$. Thus, if we apply the contraction definition, we obtain that there is an $x_2 \in T(x_1)$ for which $d(x_1, x_2) \leq d(x_0, x_1)A$. We obtain that $(I-A)^{-1}d(x_1, x_2) \leq (I-A)^{-1}\ d(x_0, x_1)A \leq rA$. We mention that $x_2 \in \overline{B}(x_0, r)$, From $d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2)$ we obtain a new inequality, $(I-A)^{-1}d(x_0, x_2) \leq (I-A)^{-1}\ d(x_0, x_1) + (I-A)^{-1}d(x_1, x_2) \leq rI + rA \leq r(I-A)^{-1}$, from which results that, according to (i), $d(x_0, x_2) \leq r$. Thus, by mathematical induction, we create the sequence $(x_n)_{n\in N}$ in with the following properties:

(a) $x_{n+1} \in T(x_n)$, $n \in N$;

(b) $(I-A)^{-1}d(x_0, x_n) \leq r(I-A)^{-1}$, $\in N$, that means (by (i)) $d(x_0, x_n) \leq r$;

(c) $(I-A)^{-1}d(x_n, x_{n+1}) \leq rA^n$, $n \in N$.

By (c) we get that $(I-A)^{-1}d(x_n, x_{n+p}) \leq r(I-A)^{-1}A^n$, $n, p \in N^*$.

Thus, the sequence $(x_n)_{n\in N}$ is Cauchy in the complete geodesic metric space $(\overline{B}(x_0, r), d)$. We denote by x^* its limit in $\overline{B}(x_0, r)$.

We prove that $x^* \in T(x^*)$. If $n \in N^*$, for each $x_n \in T(x_{n-1})$ there exists $u_n \in T(x^*)$ such that $d(x_n, u_n) \leq d(x_{n-1}, x^*)A$.

On the other hand $d(x^*, u_n) \leq d(x^*, x_n) + d(x_n, u_n) \leq d(x^*, x_n) + d(x_{n-1}, x^*)A \to 0$ as $n \to \infty$. Hence $\lim\limits_{n\to\infty} u_n = x^*$.

Since $u_n \in T(x^*)$ for $n \in N^*$ and knowing that $T(x^*)$ is closed, it results that $x^* \in T(x^*)$. The proof is complete. □

Definition 8. (see [26,27]) *Let (X, d) be a complete geodesic metric space, $d(d_1,\ d_2,\ \ldots, d_m)$ a convex metric. A function $T:\ X \to P_c(X)$ we said to be a multivalued Lipschitz operator of X into $P_c(X)$ if and only if*

$$H_i((1-\ \alpha)T(x)\ \oplus\ \alpha\ T(y),\ T(z)) \leq ad_i((1-\ \alpha)x\ \oplus\ \alpha\ y,\ z), i \in \{1, 2\ldots, m\}, \textit{for all } x, y \in X,$$

where $a \geq 0$ is a fixed real number (H_i denotes the Pompeiu-Hausdorff metric on $P_c(X)$ generated by d_i (where $i \in \{1, 2\ldots, m\}$) and $H: P_c(X) \times P_c(X) \to R_+^m$, $H(H_1,\ H_2,\ \ldots, H_m)$ the vector-valued Pompeiu-Hausforff metric on $P_c(X)$).

If T has a Lipschitz constant $a < 1$, then T is called a multivalued contraction mapping (Nadler, 1969).

Theorem 5. *Let (X, d) be a complete geodesic metric space, $d(d_1,\ d_2,\ \ldots, d_m)$. If $\to P_c(X)$ be a convex multivalued left A-contraction in the sense of Nadler in relation to a convex metric. If $A = (a_{ii}) \in M_{m,m}(R_+)$, $a_{ii} \leq 1$, $i \in \{1, 2\ldots, m\}$, is a diagonal matrix convergent to zero and if $0 < a < 1$ be a Lipschitz constant for T:*

$$H_i((1-\ \alpha)T(x)\ \oplus\ \alpha\ T(y),\ T(z)) \leq ad_i((1-\ \alpha)x\ \oplus\ \alpha\ y,\ z) \leq\ H_i((1-\ \alpha)T(x)\ \oplus\ \alpha\ T(y),\ T(z)) + a,$$
$$i \in \{1, 2\ldots, m\}, \textit{for all } x, y \in X,$$

then T has a fixed point.

Proof. Let $a < 1$ be a Lipschitz constant for T and let $x_0 \in X$. Choose $x_1 \in T(x_0)$. Since $T(x_0),\ T(x_1) \in P_c(X)$ and $x_1 \in T(x_0)$, there is a point $x_2 \in T(x_1)$ such that:

$$H_k(T(x_0),\ T(x_1)) \le ad_k(x_1, x_2) \le H_k(T(x_0),\ T(x_1)) + a.$$

Now, since $T(x_1),\ T(x_2) \in P_c(X)$ and $x_2 \in Tx_1$, there is a point $x_3 \in T(x_2)$ such that:

$$H_k(T(x_1),\ T(x_2)) \le ad_k(x_2, x_3) \le H_k(T(x_1),\ T(x_2)) + a^2.$$

Continuing in this way we create a sequence $(x_i)_{i\in N^*}$ of points of X such that $x_{i+1} \in T(x_i)$ and

$$H_k(T(x_{i-1}),\ T(x_i)) \le ad_k(x_i, x_{i+1}) \le H_k(T(x_{i-1}),\ T(x_i)) + a^i,\ for\ all\ i \in N^*.$$

We note that

$$\begin{aligned} ad_k(x_i, x_{i+1}) \le H_k(T(x_{i-1}),\ T(x_i)) \quad & +a^i \le\ ad_k(x_i, x_{i+1}) + a^i \\ & \le a[H_k(T(x_{i-2}),\ T(x_{i-1})) + a^{i-1}]\ + a^i \\ & \le a^2 d_k(x_{i-2}, x_{i-1}) + 2a^i \le \ldots \\ & \le a^i d_k(x_0, x_1) + ia^i, \end{aligned}$$

for all $i \in N^*$. Hence

$$\begin{aligned} d_k\big(x_i, x_{i+j}\big) &\le d_k(x_i, x_{i+1}) + d_k(x_{i+1}, x_{i+2}) + \ldots + d_k\big(x_{i+j-1}, x_{i+j}\big) \\ &\le a^i d_k(x_0, x_1) + ia^i + a^{i+1} d_k(x_0, x_1) + (i+1)a^{i+1} \\ &\quad +\ldots + a^{i+j-1} d_k(x_0, x_1) + (i+j-1)a^{i+j-1} \\ &= \left(\sum_{n=i}^{i+j-1} a^n\right) d_k(x_0, x_1) + \sum_{n=1}^{i+j-1} na^n, \end{aligned}$$

for all $i, j \ge 1,\ k \in \{1, 2 \ldots, m\}$.

It follows that the sequence $(x_i)_{i\in N^*}$ is a Cauchy sequence. Since (X, d) is complete, the sequence $(x_i)_{i\in N^*}$ converges to a point $x_0 \in X$. Therefore, the sequence $(T(x_i))_{i\in N^*}$ converges to $T(x_0)$ and, since $x_i \in T(x_{i-1})$ for all i, it follows that $x_0 \in T(x_0)$. This completes the proof of the theorem. □

Theorem 6. *Let (X, d) be a complete geodesic metric space, $d(d_1,\ d_2,\ \ldots, d_m)$. If $T :\ X \to P_c(X)$ is a convex multivalued right A-contraction in relation to a convex metric. If $A = (a_{ii}) \in M_{m,m}(R_+), a_{ii} \le 1$, $i \in \{1, 2 \ldots, m\}$, is a diagonal matrix convergent to zero and if $0 < a < 1$ is a Lipschitz constant for T:*

$$H_i((1-\ \alpha)T(x)\ \oplus\ \alpha\ T(y),\ T(z)) \le ad_i((1-\ \alpha)x\ \oplus\ \alpha\ y,\ z) \le H_i((1-\ \alpha)T(x)\ \oplus\ \alpha\ T(y),\ T(z)) + a,$$
$$i \in \{1, 2 \ldots, m\},\ for\ all\ x, y \in X,$$

then T has a fixed point.

Proof. The proof is analogous to the proof from Theorem 5. □

Theorem 7. *Let (X, d) be a geodesic metric space, $d(d_1,\ d_2,\ \ldots, d_m)$, let $T_i :\ X \to P_c(X)$* be a convex multivalued left A-contraction in the sense of Nadler in relation to a convex metric, $A = (a_{ii}) \in M_{m,m}(R_+),\ a_{ii} \le 1,\ i \in \{1, 2 \ldots, m\}$, *is a diagonal matrix convergent to zero and let $T_0 :\ X \to P_c(X)$ be a convex left A-contraction in the sense of Nadler in relation to a convex metric. If the sequence $(T_i)_{i\in N^*}$ converges pointwise to T_0 and if $\big(x_{ij}\big)_{j\in N^*}$ is a convergent subsequence of $(x_i)_{i\in N^*}$, then $\big(x_{ij}\big)_{j\in N^*}$ converges to a fixed point of T_0.*

Proof. Let $x_0 = \lim_{j\to\infty} x_{ij}$ and let $\varepsilon(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) > 0$. We choose an integer M such that $H_k\big(T_{ij}(x_{ij}),\ T_0(x_0)\big) < \frac{\varepsilon_k}{2}$ and $Ad\big(x_{ij}, x_0\big) < \frac{\varepsilon}{2}$ for all $k \in \{1, 2\ldots, m\}$, $j \geq M$.

Then, if $j \geq M$,

$$\begin{aligned} H\big(T_{ij}(x_{ij}),\ T_0(x_0)\big) &\leq H\big(T_{ij}(x_{ij}),\ T_{ij}(x_0)\big) + H\big(T_{ij}(x_0),\ T_0(x_0)\big) \\ &< \ Ad\big(x_{ij}, x_0\big) + H\big(T_{ij}(x_0),\ T_0(x_0)\big) < \varepsilon. \end{aligned}$$

This proves that $\lim_{j\to\infty} T_{ij}\big(x_{ij}\big) = T_0(x_0)$. Therefore, since $x_{ij} \in T_{ij}\big(x_{ij}\big)$ for each *j = 1,2,...* it follows that $x_0 \in T_0(x_0)$. This proves the theorem. □

Theorem 8. *Let (X, d) be a geodesic metric space, $d(d_1, d_2, \ldots, d_m)$, let $T_i : X \to P_c(X)$ be a convex multivalued right A-contraction in relation to a convex metric, $A = (a_{ii}) \in M_{m,m}(R_+)$, $a_{ii} \leq 1$, $i \in \{1, 2\ldots, m\}$, is a diagonal matrix convergent to zero and let* $T_0 : X \to P_c(X)$ be a convex left A-contraction in the sense of Nadler in relation to a convex metric. If the sequence $(T_i)_{i\in N^*}$ *converges pointwise to* T_0 *and if $\big(x_{ij}\big)_{j\in N^*}$ is a convergent subsequence of $(x_i)_{i\in N^*}$, then $\big(x_{ij}\big)_{j\in N^*}$ converges to a fixed point of T_0.*

Proof. Let $x_0 = \lim_{j\to\infty} x_{ij}$ and let $\varepsilon(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) > 0$. Choose an integer M such that

$$H_k\big(T_{ij}(x_{ij}),\ T_0(x_0)\big) < \frac{\varepsilon_k}{2} \text{ and } d\big(x_{ij}, x_0\big)A < \frac{\varepsilon}{2} \text{ for all } k \in \{1, 2\ldots, m\},\ j \geq M.$$

Then, if $j \geq M$,

$$\begin{aligned} H\big(T_{ij}(x_{ij}),\ T_0(x_0)\big) &\leq H\big(T_{ij}(x_{ij}),\ T_{ij}(x_0)\big) + H\big(T_{ij}(x_0),\ T_0(x_0)\big) \\ &< \ d\big(x_{ij}, x_0\big)A + H\big(T_{ij}(x_0),\ T_0(x_0)\big) < \varepsilon. \end{aligned}$$

This proves that $\lim_{j\to\infty} T_{ij}\big(x_{ij}\big) = T_0(x_0)$. Therefore, since $x_{ij} \in T_{ij}\big(x_{ij}\big)$ for each *j = 1,2,...* it follows that $x_0 \in T_0(x_0)$. This proves the theorem. □

5. Applications

It is well-known that, often, for the study of many processes, having a certain lack of precision, which arises from economy, management, society, biology, ecology, etc., we are interested in replacing the following operator equations:

$$\begin{cases} x_1 = F_1(x_1, x_2, \ldots, x_m) \\ \qquad \cdots \\ x_m = F_m(x_1, x_2, \ldots, x_m) \end{cases},$$

(where $(X, |\ |)$ is a complete geodesic metric space and $T_i : X^m \to X$ for $i \in \{1, 2\ldots, m\}$) with the inclusion system:

$$\begin{cases} x_1 \in T_1(x_1, x_2, \ldots, x_m) \\ \qquad \cdots \\ x_m \in T_m(x_1, x_2, \ldots, x_m) \end{cases},$$

where $T_i : X^m \to P(X^m)$ for $i \in \{1, 2\ldots, m\}$ are left A-contractions in the sense of Nadler or right A-contraction with convex metric.

Theorem 9. *Let $(X, |\ |)$ be a complete geodesic metric space and let $T_i : X \to P_c(X)$ for $i \in \{1, 2\ldots, m\}$ be a convex multivalued left A-contraction in the sense of Nadler with convex metric.*

We suppose there exists $0 \leq a_{kk} \leq 1$, $k \in \{1, 2 \ldots, m\}$ *such that for each* $u(u_1, u_2, \ldots, u_m)$, $v(v_1, v_2, \ldots, v_m) \in X^m$ *and each* $y_k \in T_k(u_1, u_2, \ldots, u_m)$, $k \in \{1, 2 \ldots, m\}$ *there exists* $z_k \in T_k(v_1, v_2, \ldots, v_m)$ such that:

$$|y_k - z_k| \leq a_{kk}|u_k - v_k|, k \in \{1, 2 \ldots, m\}.$$

Then, the inclusion system:

$$\begin{cases} u_1 \in T_1(u_1, u_2, \ldots, u_m) \\ \ldots \\ u_m \in T_m(u_1, u_2, \ldots, u_m) \end{cases},$$

has at least one solution in X^m.

Proof. Consider the multivalued operator $T : X^m \to P(X^m)$ given by $T(T_1, T_2, \ldots, T_m)$. Then, the conditions from the theorem, can be represented in the following form: for each $u, v \in X^m$ and each $y \in T(u)$ there exists z$\in T(v)$ such that

$$\|y - z\| \leq A\|u - v\|.$$

Hence, Theorem 1. applies with $d(u, v) := \|u - v\|$ and implies that T has at least one fixed point $u \in T(u)$. □

Theorem 10. *Let* $(X, |\ |)$ *be a complete geodesic metric space and* $T_i : X \to P_c(X)$ *for* $i \in \{1, 2 \ldots, m\}$ *be a convex multivalued right A-contractions with convex metric.*

We suppose there exists $0 \leq a_{kk} \leq 1$, $k \in \{1, 2 \ldots, m\}$ *such that for each* $u(u_1, u_2, \ldots, u_m)$, $v(v_1, v_2, \ldots, v_m) \in X^m$ *and each* $y_k \in T_k(u_1, u_2, \ldots, u_m)$, $k \in \{1, 2 \ldots, m\}$ *there exists* $z_k \in T_k(v_1, v_2, \ldots, v_m)$ *such that:*

$$|y_k - z_k| \leq a_{kk}|u_k - v_k|, k \in \{1, 2 \ldots, m\}.$$

Then, the inclusion system:

$$\begin{cases} u_1 \in T_1(u_1, u_2, \ldots, u_m) \\ \ldots \\ u_m \in T_m(u_1, u_2, \ldots, u_m) \end{cases},$$

has at least one solution in X^m.

Proof. The proof is analogous to the proof from Theorem 9. Hence, Theorem 2. applied to $d(u, v) := \|u - v\|$ implies that T has at least one fixed point $u \in T(u)$. □

6. Conclusions

This paper presented the fixed-point theorems for convex multivalued left A-contractions in the sense of Nadler and right A-contractions on the geodesic metric space. Our results are particular cases of some general theorems, to the multivalued left A-contractions in the sense of Nadler and right A-contractions, and particular cases of the results given by Rus (1979, 2008), Nadler (1969), Mureşan (2002, 2004), Bucur, Guran and Petruşel (2009), and Petre and Bota (2013), and are applicable in many fields, such as economy, management, society, biology, and ecology.

The application of some fixed-point theorems to nonlinear domains, such as geodesic metric spaces, has its own importance.

Funding: Lucian Blaga University of Sibiu, grant number LBUS-IRG-2017-03.

Acknowledgments: Project financed from Lucian Blaga University of Sibiu research grants LBUS-IRG-2017-03.

Conflicts of Interest: The author declares no conflict of interest.

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