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Two-Step Solver for Nonlinear Equations

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Abstract: In this paper we present a two-step solver for nonlinear equations with a nondifferentiable operator. This method is based on two methods of order of convergence $1+\sqrt{2}$. We study the local and a semilocal convergence using weaker conditions in order to extend the applicability of the solver. Finally, we present the numerical example that confirms the theoretical results.

Keywords: Nondifferentiable operator; nonlinear equation; divided difference; Lipschitz condition; convergence order; local and semilocal convergence

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1. Introduction

A plethora of real-life applications from various areas, including Computational Science and Engineering, are converted via mathematical modeling to equations valued on abstract spaces such as n-dimensional Euclidean, Hilbert, Banach, and other spaces [1,2]. Then, researchers face the great challenge of finding a solution x_* in the closed form of the equation. However, this task is generally very difficult to achieve. This is why iterative methods are developed to provide a sequence approximating x_* under some initial conditions.

Newton's method, and its variations are widely used to approximate x_* [1–14]. There are problems with the implementation of these methods, since the invertibility of the linear operator involved is, in general, costly or impossible. That is why secant-type methods were also developed which are derivative-free. In these cases however, the order of convergence drops from 2 to $\frac{1+\sqrt{5}}{2}$.

Then, one considers methods that mix Newton and secant steps to increase the order of convergence. This is our first objective in this paper. Moreover, the study of iterative methods involves local convergence where knowledge about the solution x_* is used to determine upper bounds on the distances and radii of convergence. The difficulty of choosing initial points is given by local results, so they are important. In the semilocal convergence we use knowledge surrounding the initial point to find sufficient conditions for convergence. It turns out that in both cases the convergence region is small, limiting the applicability of iterative methods. That is why we use our ideas of the center-Lipschitz condition, in combination with the notion of the restricted convergence region, to present local as well as semilocal improvements leading to the extension of the applicability of iterative methods.

The novelty of the paper is that since the new Lipschitz constants are special cases of older ones, no additional cost is required for these improvements (see also the remarks and numerical examples). Our ideas can be used to improve the applicability of other iterative methods [1–14].

By E_1, E_2 we consider Banach spaces and by $\Omega \subseteq E_1$ a convex set. $F : \Omega \to E_2$ is differentiable in the Fréchet sense, $G : \Omega \to E_2$ is a continuous but its differentiability is not assumed. Then, we study equation

$$H(x) = 0, \text{ for } H(x) = F(x) + G(x). \quad (1)$$

This problem was considered by several authors. Most of them used one-step methods for finding an approximate solution of (1), for example, Newton's type method [14], difference methods [4,5] and combined methods [1–3,11].

We proposed a two-step method [6,10,12] to numerically solve (1)

$$\begin{aligned} x_{n+1} &= x_n - \left[F'\left(\frac{x_n+y_n}{2}\right) + Q(x_n,y_n)\right]^{-1}(F(x_n)+G(x_n)), \\ y_{n+1} &= x_{n+1} - \left[F'\left(\frac{x_n+y_n}{2}\right) + Q(x_n,y_n)\right]^{-1}(F(x_{n+1})+G(x_{n+1})), \quad n = 0,1,\ldots \end{aligned} \tag{2}$$

with $Q(x,y)$ a first order divided difference of the operator G at the points x and y. This method relates to methods with the order of convergence $1+\sqrt{2}$ [7,13].

If $Q : \Omega \times \Omega \to L(E_1, E_2)$, gives $Q(x,y)(x-y) = G(x) - G(y)$ for all x, y with $x \neq y$, then, we call it a divided difference.

Two-step methods have some advantages over one-step methods. First, they usually require fewer number of iterations for finding an approximate solution. Secondly, at each iteration, they solve two similar linear problems, therefore, there is a small increase in computational complexity. That is why they are often used for solving nonlinear problems [2,6,8–10,12,13].

In [6,10,12] the convergence analysis of the proposed method was provided under classical and generalized Lipschitz conditions and superquadratic convergence order was shown. Numerical results for method (2) were presented in [10,12].

2. Local Convergence

Let $S(x_*,\rho) = \{x : \|x - x_*\| < \rho\}$.

From now on by differentiable, we mean differentiable in the Fréchet sense. Moreover, F, G are assumed as previously.

Theorem 1 ([10,12]). *Assume (1) has a solution $x_* \in \Omega$, G has a first order divided difference Q in Ω, and there exist $[T(x;y)]^{-1} = \left[F'\left(\frac{x+y}{2}\right) + Q(x,y)\right]^{-1}$ for each $x \neq y$ and $\|[T(x;y)]^{-1}\| \leq B$. Moreover, assume for each x, y, u, $v \in \Omega$, $x \neq y$*

$$\|F'(x) - F'(y)\| \leq 2p_1\|x-y\|, \tag{3}$$

$$\|F''(x) - F''(y)\| \leq p_2\|x-y\|^{\alpha}, \ \alpha \in (0,1], \tag{4}$$

$$\|Q(x,y) - Q(u,v)\| \leq q_1(\|x-u\| + \|y-v\|). \tag{5}$$

Assume $S(x_*,r_*) \subset \Omega$, where r_* is the minimal positive zero of

$$\begin{gathered} q(r) = 1, \\ 3B(p_1+q_1)rq(r) = 1, \\ q(r) = B\left[(p_1+q_1)r + \frac{p_2}{4(\alpha+1)(\alpha+2)}r^{1+\alpha}\right]. \end{gathered}$$

Then, the sequences $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ for x_0, $y_0 \in S(x_*,r_*)$ remain in $S(x_*,r_*)$ with $\lim_{n\to\infty} x_n = x_*$, and

$$\|x_{n+1} - x_*\| \leq B\left[(p_1+q_1)\|y_n - x_*\| + \frac{p_2}{4(\alpha+1)(\alpha+2)}\|x_n - x_*\|^{1+\alpha}\right]\|x_n - x_*\|, \tag{6}$$

$$\|y_{n+1} - x_*\| \leq B(p_1+q_1)\Big[\|y_n - x_*\| + \|x_n - x_*\| + \|x_{n+1} - x_*\|\Big]\|x_{n+1} - x_*\|. \tag{7}$$

The condition $\|[T(x;y)]^{-1}\| \leq B$ used in [10,12] is very strong in general. That is why in what follows, we provide a weaker alternative. Indeed, assume that there exists $a > 0$ and $b > 0$ such that

$$\begin{aligned} \|F'(x_*) - F'(x)\| &\leq a\|x_* - x\|, & (8)\\ \|Q(x,y) - G'(x_*)\| &\leq b(\|x - x_*\| + \|y - x_*\|) \text{ for each } x, y \in \Omega. & (9) \end{aligned}$$

Set $c = (a+2b)\|T_*^{-1}\|$, $\Omega_0 = \Omega \cap S(x_*, \frac{1}{c})$ and $T_* = F'(x_*) + G'(x_*)$. It follows, for each $x, y \in S(x_*, r)$, $r \in [0, \frac{1}{c}]$ we get in turn by (8) and (9) provided that T_*^{-1} exists

$$\begin{aligned} \|T_*^{-1}\|\|T(x;y) - T_*\| &\leq \|T_*^{-1}\|\Big[\|F'(\frac{x+y}{2}) - F'(x_*)\| + \|Q(x,y) - G'(x_*)\|\Big] \\ &\leq \|T_*^{-1}\|\Big[\frac{a}{2}(\|x - x_*\| + \|y - x_*\|) + b(\|x - x_*\| + \|y - x_*\|)\Big] \\ &\leq \|T_*^{-1}\|(\frac{a}{2} + b)[\|x - x_*\| + \|y - x_*\|] \\ &< \|T_*^{-1}\|\Big[(\frac{a}{2} + b) + (\frac{a}{2} + b)\Big]\frac{1}{c} = 1. \qquad (10) \end{aligned}$$

Then, (10) and the Banach lemma on invertible operators [2] assure $T(x;y)^{-1}$ exists with

$$\|T(x;y)^{-1}\| \leq \bar{B} = \bar{B}(r) = \frac{\|T_*^{-1}\|}{1 - cr}. \qquad (11)$$

Then, Theorem 1 holds but with $\bar{B}$, $\bar{p}_1$, $\bar{q}_1$, $\bar{p}_2$, $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_*$ replacing B, p_1, q_1, p_2, r_1, r_2, r_*, respectively.

Next, we provide a weaker alternative to the Theorem 1.

Theorem 2. *Assume $x_* \in \Omega$, exists with $F(x_*) + G(x_*) = 0$, $T_*^{-1} \in L(E_2, E_1)$ and together with conditions (8) and (9) following items hold for each $x, y, u, v \in \Omega_0$*

$$\begin{aligned} \|F'(y) - F'(x)\| &\leq 2\bar{p}_1\|y - x\|, \\ \|F''(y) - F''(x)\| &\leq \bar{p}_2\|y - x\|^{\alpha},\ \alpha \in (0,1], \\ \|Q(x,y) - Q(u,v)\| &\leq \bar{q}_1(\|x - u\| + \|y - v\|). \end{aligned}$$

Let $\bar{r}_1$, $\bar{r}_2$ be the minimal positive zeros of equations

$$\bar{q}(r) = 1,$$
$$3\bar{B}(\bar{p}_1 + \bar{q}_1)r\bar{q}(r) = 1,$$

respectively, where

$$\bar{q}(r) = \bar{B}\Big[(\bar{p}_1 + \bar{q}_1)r + \frac{\bar{p}_2}{4(\alpha+1)(\alpha+2)}r^{1+\alpha}\Big]$$

and set $\bar{r}_* = \min\{\bar{r}_1, \bar{r}_2\}$. Moreover, assume that $S(x_*, \bar{r}_*) \subset \Omega$.

Then, the sequences $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ for x_0, $y_0 \in S(x_*, \bar{r}_*)$ remain in $S(x_*, \bar{r}_*)$, $\lim_{n\to\infty} x_n = x_*$, and

$$\begin{aligned} \|x_{n+1} - x_*\| &\leq \bar{B}\Big[(\bar{p}_1 + \bar{q}_1)\|y_n - x_*\| + \frac{\bar{p}_2}{4(\alpha+1)(\alpha+2)}\|x_n - x_*\|^{1+\alpha}\Big]\|x_n - x_*\|, & (12)\\ \|y_{n+1} - x_*\| &\leq \bar{B}(\bar{p}_1 + \bar{q}_1)\Big[\|y_n - x_*\| + \|x_n - x_*\| + \|x_{n+1} - x_*\|\Big]\|x_{n+1} - x_*\|. & (13) \end{aligned}$$

Proof. It follows from the proof of Theorem 1, (10), (11) and the preceding replacements. □

Corollary 1. *Assume hypotheses of Theorem 2 hold. Then, the order of convergence of method (2) is $1 + \sqrt{1+\alpha}$.*

Proof. Let

$$a_n = \|x_n - x_*\|,\ b_n = \|y_n - x_*\|,\ \bar{C}_1 = \bar{B}(\bar{p}_1 + \bar{q}_1),\ \bar{C}_2 = \frac{\bar{B}\bar{p}_2}{4(\alpha+1)(\alpha+2)}.$$

By (12) and (13), we get

$$\begin{aligned} a_{n+1} &\leq \bar{C}_1 a_n b_n + \bar{C}_2 a_n^{2+\alpha}, \\ b_{n+1} &\leq \bar{C}_1(a_{n+1} + a_n + b_n)a_{n+1} \leq \bar{C}_1(2a_n + b_n)a_{n+1} \\ &\leq \bar{C}_1(2a_n + \bar{C}_1(2a_0 + b_0)a_n)a_{n+1} = \bar{C}_1(2 + \bar{C}_1(2a_0 + b_0))a_n a_{n+1}, \end{aligned}$$

Then, for large n and $a_{n-1} < 1$, from previous inequalities, we obtain

$$\begin{aligned} a_{n+1} &\leq \bar{C}_1 a_n b_n + \bar{C}_2 a_n^2 a_{n-1}^{\alpha} \\ &\leq \bar{C}_1^2(2 + \bar{C}_1(2a_0 + b_0))a_n^2 a_{n-1} + \bar{C}_2 a_n^2 a_{n-1}^{\alpha} \\ &\leq [\bar{C}_1^2(2 + \bar{C}_1(2a_0 + b_0)) + \bar{C}_2]a_n^2 a_{n-1}^{\alpha}. \end{aligned} \tag{14}$$

From (14) we relate (2) to $t^2 - 2t - \alpha = 0$, leading to the solution $t^* = 1 + \sqrt{1+\alpha}$. □

Remark 1. *To relate Theorem 1 and Corollary 2 in [12] to our Theorem 2 and Corollary 1 respectively, let us notice that under (3)–(5) B_1 can replace B in these results, where $B_1 = B_1(r) = \frac{\|T_*^{-1}\|}{1 - c_1 r}$, $c_1 = 2(p_1 + q_1)\|T_*^{-1}\|$.*

Then, we have

$$\begin{aligned} \bar{p}_1 &\leq p_1, \\ \bar{p}_2 &\leq p_2, \\ \bar{q}_1 &\leq q_1, \\ c &\leq c_1, \\ \bar{B}(t) &\leq B_1(t) \ \textit{for each} \ t \in [0, \frac{1}{c_1}), \\ \bar{C}_1 &\leq C_1, \\ \bar{C}_2 &\leq C_2 \end{aligned}$$

and

$$\Omega_0 \subseteq \Omega$$

since $r_* \leq \bar{r}_*$, which justify the advantages claimed in the Introduction of this study.

3. Semilocal Convergence

Theorem 3 ([12]). *We assume that $S(x_0, r_0) \subset \Omega$, the linear operator $T_0 = F'\left(\frac{x_0 + y_0}{2}\right) + Q(x_0, y_0)$, where $x_0,\ y_0 \in \Omega$, is invertible and the Lipschitz conditions are fulfilled*

$$\|T_0^{-1}(F'(y) - F'(x))\| \leq 2p_0\|y - x\|, \tag{15}$$

$$\|T_0^{-1}(Q(x,y) - Q(u,v))\| \leq q_0(\|x - u\| + \|y - v\|). \tag{16}$$

Let's λ, μ $(\mu > \lambda)$, r_0 be non-negative numbers such that

$$\|x_0 - x_{-1}\| \leq \lambda, \quad \|T_0^{-1}(F(x_0) + G(x_0))\| \leq \mu, \tag{17}$$

$$r_0 \geq \mu/(1-\gamma), \quad (p_0 + q_0)(2r_0 - \lambda) < 1,$$

$$\gamma = \frac{(p_0+q_0)(r_0-\lambda)+0.5p_0r_0}{1-(p_0+q_0)(2r_0-\lambda)}, \quad 0 \le \gamma < 1.$$

Then, for each $n = 0, 1, 2, \ldots$

$$\|x_n - x_{n+1}\| \le t_n - t_{n+1}, \quad \|y_n - x_{n+1}\| \le s_n - t_{n+1},$$
$$\|x_n - x_*\| \le t_n - t^*, \quad \|y_n - x_*\| \le s_n - t^*,$$

where

$$\begin{aligned} t_0 &= r_0, \quad s_0 = r_0 - \lambda, \quad t_1 = r_0 - \mu, \\ t_{n+1} - t_{n+2} &= \frac{(p_0+q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1-(p_0+q_0)[(t_0 - t_{n+1}) + (s_0 - s_{n+1})]}(t_n - t_{n+1}), \quad (18) \\ t_{n+1} - s_{n+1} &= \frac{(p_0+q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1-(p_0+q_0)[(t_0 - t_n) + (s_0 - s_n)]}(t_n - t_{n+1}), \quad (19) \end{aligned}$$

$\{t_n\}_{n\ge 0}$, $\{s_n\}_{n\ge 0}$ *are non-negative, decreasing sequences that converge to some* t^* *such that* $r_0 - \mu/(1-\gamma) \le t^* < t_0$*; sequences* $\{x_n\}_{n\ge 0}$, $\{y_n\}_{n\ge 0} \subseteq S(x_0, t^*)$ *and converge to a solution* x_* *of equation (1).*

Next, we present the analogous improvements in the semilocal convergence case. Assume that for all $x, y, u, v \in \Omega$

$$\|T_0^{-1}(F'(z) - F'(x))\| \le 2\bar{p}_0\|z - x\|, \; z = \frac{x_0 + y_0}{2} \quad (20)$$

and

$$\|T_0^{-1}(Q(x,y) - Q(x_0,y_0))\| \le \bar{q}_0(\|x - x_0\| + \|y - y_0\|). \quad (21)$$

Set $\Omega_0 = \Omega \cap S(x_0, \bar{r}_0)$, where $\bar{r}_0 = \dfrac{1+\lambda(\bar{p}_0+\bar{q}_0)}{2(\bar{p}_0+\bar{q}_0)}$. Define parameter $\bar{\gamma}$ and sequences $\{\bar{t}_n\}$, $\{\bar{s}_n\}$ for each $n = 0, 1, 2, \ldots$ by $\bar{\gamma} = \dfrac{(p_0^0+q_0^0)(\bar{r}_0-\lambda)+0.5p_0^0\bar{r}_0}{1-(\bar{p}_0+\bar{q}_0)(2\bar{r}_0-\lambda)}$,

$$\bar{t}_0 = \bar{r}_0, \; \bar{s}_0 = \bar{r}_0 - \lambda, \; \bar{t}_1 = \bar{r}_0 - \mu,$$

$$\begin{aligned} \bar{t}_{n+1} - \bar{t}_{n+2} &= \frac{(p_0^0+q_0^0)(\bar{s}_n - \bar{t}_{n+1}) + 0.5p_0^0(\bar{t}_n - \bar{t}_{n+1})}{1-(\bar{p}_0+\bar{q}_0)[(\bar{t}_0 - \bar{t}_{n+1}) + (\bar{s}_0 - \bar{s}_{n+1})]}(\bar{t}_n - \bar{t}_{n+1}), \quad (22) \\ \bar{t}_{n+1} - \bar{s}_{n+1} &= \frac{(p_0^0+q_0^0)(\bar{s}_n - \bar{t}_{n+1}) + 0.5p_0^0(\bar{t}_n - \bar{t}_{n+1})}{1-(\bar{p}_0+\bar{q}_0)[(\bar{t}_0 - \bar{t}_n) + (\bar{s}_0 - \bar{s}_n)]}(\bar{t}_n - \bar{t}_{n+1}). \quad (23) \end{aligned}$$

As in the local convergence case, we assume instead of (15) and (16) the restricted Lipschitz-type conditions for each $x, y, u, v \in \Omega_0$

$$\begin{aligned} \|T_0^{-1}(F'(x) - F'(y))\| &\le 2p_0^0\|x - y\|, \quad (24) \\ \|T_0^{-1}(Q(x,y) - Q(u,v))\| &\le q_0^0(\|x - u\| + \|y - v\|). \quad (25) \end{aligned}$$

Then, instead of the estimate in [12] using (15) and (16):

$$\begin{aligned}\|T_0^{-1}[T_0 - T_{n+1}]\| &\leq \left\|T_0^{-1}\left[F'\left(\frac{x_0+y_0}{2}\right) - F'\left(\frac{x_{n+1}+y_{n+1}}{2}\right)\right]\right\| + \|T_0^{-1}[Q(x_0,y_0) - Q(x_{n+1},y_{n+1})]\| \\ &\leq 2p_0\left(\frac{\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|}{2}\right) + q_0(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \\ &= (p_0+q_0)(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \leq (p_0+q_0)(t_0 - t_{n+1} + s_0 - s_{n+1}) \\ &\leq (p_0+q_0)(t_0+s_0) = (p_0+q_0)(2r_0 - \lambda) < 1, \end{aligned} \tag{26}$$

we obtain more precise results using (20) and (21)

$$\begin{aligned}\|T_0^{-1}[T_0 - T_{n+1}]\| &\leq \leq 2\bar{p}_0\left(\frac{\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|}{2}\right) + \bar{q}_0(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \\ &\leq (\bar{p}_0+\bar{q}_0)(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \\ &\leq (\bar{p}_0+\bar{q}_0)(\bar{t}_0 - \bar{t}_{n+1} + \bar{s}_0 - \bar{s}_{n+1}) \\ &\leq (\bar{p}_0+\bar{q}_0)(\bar{t}_0+\bar{s}_0) = (\bar{p}_0+\bar{q}_0)(2\bar{r}_0 - \lambda) < 1, \end{aligned}$$

since

$$\begin{aligned}\Omega_0 &\subseteq \Omega, \\ \bar{p}_0 &\leq p_0, \\ \bar{q}_0 &\leq q_0, \\ p_0^0 &\leq p_0, \\ q_0^0 &\leq q_0, \\ \bar{\gamma} &\leq \gamma, \\ \text{and } \bar{r}_0 &\geq r_0. \end{aligned} \tag{27}$$

Then, by replacing p_0, q_0, r_0, γ, t_n, s_n, (26) with p_0^0, q_0^0 (at the numerator in (18) and (19)), or $\bar{p}_0$, $\bar{q}_0$ (at the denominator in (18) and (19)), and with $\bar{r}_0$, $\bar{\gamma}$, $\bar{t}_n$, $\bar{s}_n$, (27) respectively, we arrive at the following improvement of Theorem 3.

Theorem 4. *Assume together with (17), (20), (21), (24), (25) that* $\bar{r}_0 \geq \mu(1-\bar{\gamma})$, $(\bar{p}_0+\bar{q}_0)(2\bar{r}_0 - \lambda) < 1$ *and* $\bar{\gamma} \in [0,1]$. *Then, for each* $n = 0,1,2,\ldots$

$$\|x_n - x_{n+1}\| \leq \bar{t}_n - \bar{t}_{n+1}, \quad \|y_n - x_{n+1}\| \leq \bar{s}_n - \bar{t}_{n+1}, \tag{28}$$

$$\|x_n - x_*\| \leq \bar{t}_n - t^*, \quad \|y_n - x_*\| \leq \bar{s}_n - t^*, \tag{29}$$

with sequences $\{\bar{t}_n\}_{n\geq 0}$, $\{\bar{s}_n\}_{n\geq 0}$ *given in (22) and (23) decreasing, non-negative sequences that converge to some* t^* *such that* $r_0 - \mu/(1-\bar{\gamma}) \leq t^* < \bar{t}_0$. *Moreover, sequences* $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0} \subseteq S(x_0, \bar{t}^*)$ *for each* $n = 0,1,2,\ldots$, *and* $\lim_{n\to\infty} x_n = x_*$.

Remark 2. *It follows (27) that by hypotheses of Theorem 3, Theorem 4, and by a simple inductive argument that the following items hold*

$$\begin{aligned} t_n &\leq \bar{t}_n, \\ s_n &\leq \bar{s}_n, \\ o \leq \bar{t}_n - \bar{t}_{n+1} &\leq t_n - t_{n+1}, \\ o \leq \bar{s}_n - \bar{t}_{n+1} &\leq s_n - t_{n+1}, \\ \text{and } t^* &\leq \bar{t}^*. \end{aligned}$$

Hence, the new results extend the applicability of the method (2).

Remark 3. *If we choose $F(x) = 0$, $p_1 = 0$, $p_2 = 0$. Then, the estimates (6) and (7) reduce to similar ones in [7] for the case $\alpha = 1$.*

Remark 4. *Section 3 contains existence results. The uniqueness results are omitted, since they can be found in [2,6] but with center-Lipschitz constants replacing the larger Lipschitz constants.*

4. Numerical Experiments

Let $E_1 = E_2 = R^3$ and $\Omega = S(x_*, 1)$. Define functions F and G for $v = (v_1, v_2, v_3)^T$ on Ω by

$$\begin{aligned} F(v) &= \left(e^{v_1} - 1,\ \frac{e-1}{2}v_2^2 + v_2,\ v_3\right)^T, \\ G(v) &= \left(|v_1|,\ |v_2|,\ |v_3|\right)^T, \end{aligned}$$

and set $H(v) = F(v) + G(v)$. Moreover, define a divided difference $Q(\cdot, \cdot)$ by

$$Q(v, \bar{v}) = diag\left(\frac{|\bar{v}_1| - |v_1|}{\bar{v}_1 - v_1}, \frac{|\bar{v}_2| - |v_2|}{\bar{v}_2 - v_2}, \frac{|\bar{v}_3| - |v_3|}{\bar{v}_3 - v_3}\right)$$

if $v_i \neq \bar{v}_i$, $i = 1, 2, 3$. Otherwise, set $Q(v, \bar{v}) = diag(1, 1, 1)$. Then, $T_* = 2diag(1, 1, 1)$, so $\|T_*^{-1}\| = 0.5$. Notice that $x_* = (0, 0, 0)^T$ solves equation $H(v) = 0$. Furthermore, we have $\Omega_0 = S(x_*, \frac{2}{e+1})$, so

$$p_1 = \frac{e}{2}, p_2 = e, q_1 = 1, B = B(t) = \frac{1}{2(1 - c_1 t)},$$
$$b = 1, \alpha = 1, a = e - 1, \bar{p}_1 = \bar{p}_2 = \frac{1}{2}e^{\frac{2}{e+1}}, \bar{q}_1 = 1$$

and Ω_0 is a strict subset of Ω. As well, the new parameters and functions are also more strict than the old ones in [12]. Hence, the aforementioned advantages hold. In particular, $r_* \approx 0.2265878$ and $\bar{r}_* \approx 0.2880938$.

Let's give results obtained by the method (2) for approximate solving the considered system of nonlinear equations. We chose initial approximations as $x_0 = (0.1; 0.1; 0.1)d$ (d is a real number) and $y_0 = x_0 + 0.0001$. The iterative process was stopped under the condition $\|x_{n+1} - x_n\| \leq 10^{-10}$ and $\|H(x_{n+1})\| \leq 10^{-10}$. We used the Euclidean norm. The obtained results are shown in Table 1.

Table 1. Value of $\|x_n - x_{n-1}\|$ for each iteration.

n	$d = 1$	$d = 10$	$d = 50$
1	0.1694750	1.4579613	5.9053855
2	0.0047049	0.3433874	1.9962504
3	0.0000005	0.0112749	1.3190118
4	4.284×10^{-16}	0.0000037	1.0454772
5		2.031×10^{-14}	0.4157737
6			0.0260385
7			0.0000271
8			1.389×10^{-12}

5. Conclusions

The convergence region of iterative methods is, in general, small under Lipschitz-type conditions, leading to a limited choice of initial points. Therefore, extending the choice of initial points without imposing additional, more restrictive, conditions than before is extremely important in computational sciences. This difficult task has been achieved by defining a convergence region where the iterates lie, that is more restricted than before, ensuring the Lipschitz constants are at least as small as in previous works. Hence, we achieve: more initial points, fewer iterations to achieve a predetermined error accuracy, and a better knowledge of where the solution lies. These are obtained without additional cost because the new Lipschitz constants are special cases of the old ones. This technique can be applied to other iterative methods.

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