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Hybrid Algorithms for Variational Inequalities Involving a Strict Pseudocontraction

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Abstract: In a real Hilbert space, we investigate the Tseng's extragradient algorithms with hybrid adaptive step-sizes for treating a Lipschitzian pseudomonotone variational inequality problem and a strict pseudocontraction fixed-point problem, which are symmetry. By imposing some appropriate weak assumptions on parameters, we obtain a norm solution of the problems, which solves a certain hierarchical variational inequality.

Keywords: extragradient method; variational inequality; fixed point; pseudocontractions; sequentially weak continuity

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1. Introduction

In a real Hilbert space H, one employs $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to stand for its inner product and norm. Let P_C be the projection operator from the space H onto a nonempty convex and closed set C, where $C \subset H$. Let us denote by $\mathrm{Fix}(S)$ the set of all fixed points of an operator $S : C \to H$. The notations $\to$, $\mathbf{R}$, and $\rightharpoonup$ will be used to stand for the strong convergence, the set of real numbers, and the weak convergence, respectively. A self-operator $S : C \to C$ is named ς-strictly pseudocontractive if $\exists \varsigma \in [0, 1)$ such that

$$\|Su - Sv\|^2 \leq \varsigma \|(I - S)u - (I - S)v\|^2 + \|u - v\|^2 \quad \forall u, v \in C.$$

In particular, whenever $\varsigma = 0$, S is called nonexpansive. This means that the class of nonexpansive mappings is a proper subclass of the one of strict pseudocontractions. Recall that an operator $S : C \to H$ is called

(i) Lipschitz with module L if $\exists L > 0$ such that

$$\|Su - Sv\| \leq L\|u - v\| \ \forall u, v \in C;$$

(ii) monotone if $\langle u - v, Su - Sv \rangle \geq 0, \forall u, v \in C$;

(iii) pseudomonotone if

$$\langle v - u, Su \rangle \geq 0 \Rightarrow \langle v - u, Sv \rangle \geq 0 \ \forall u, v \in C;$$

(iv) strongly monotone with module β if $\exists \beta > 0$ s.t.

$$\langle u - v, Su - Sv \rangle \geq \beta \|u - v\|^2 \ \forall u, v \in C;$$

(v) sequentially weakly continuous if $u_n \rightharpoonup u \Rightarrow Su_n \rightharpoonup Su \ \ \forall \{u_n\} \subset C$.

It is not hard to see that the pseudomonotone operators may not be monotone. In addition, recall that the operator $S : C \to C$ is ς-strictly pseudocontractive with constant $\varsigma \in [0,1)$ iff the following inequality holds: $2\langle Su - Sv, u - v\rangle \le 2\|u - v\|^2 - (1-\varsigma)\|(I-S)u - (I-S)v\|^2 \ \forall u, v \in C$. It is obvious that if S is a ς-strict pseudocontraction, then S satisfies Lipschitz condition $\|Su - Sv\| \le \frac{1+\varsigma}{1-\varsigma}\|u - v\| \ \forall u, v \in C$. For each point $u \in H$, we know that there exists a unique nearest point in C, denoted by $P_C u$, such that $\|u - P_C u\| \le \|u - v\| \ \ \forall v \in C$. The operator P_C is called the metric projection of H onto C.

Consider an operator $A : H \to H$. The classical monotone variational inequality problem (VIP) consists of finding $u^* \in C$ s.t. $\langle v - u^*, Au^*\rangle \ge 0 \ \ \forall v \in C$. The solution set of such a VIP is denoted by VI(C, A). Korpelevich [1] first designed an extragradient method with two projections

$$\begin{cases} v_n = P_C(u_n - \ell A u_n), \\ u_{n+1} = P_C(u_n - \ell A v_n), \end{cases}$$

with $\ell \in (0, \frac{1}{L})$, which has been one of the most popular methods for dealing with the VIP up until now. If VI$(C, A) \neq \emptyset$, it was shown in [1] that $\{x_n\}$ weakly converges to a vector in VI(C, A). The gradient (reduced) type iterative schemes are under the spotlight of investigators of applied mathematicians and engineers in the communities of nonlinear and optimization. Based on this approach, a number of authors have conducted various investigations on efficient iterative algorithms; for examples, see [2–11].

Let both the operators A and B be inverse-strongly monotone from C to H and the self-mapping $S : C \to C$ be ς-strictly pseudocontractive. In 2010, via the extragradient approach, Yao et al. [12] designed an efficient, fast algorithm for obtaining a feasibility point in a common solution set:

$$\begin{cases} w_n = P_C(u_n - \mu B u_n), \\ v_n = (1-\beta_n) P_C(w_n - \lambda A w_n) + \beta_n f(u_n), \\ u_{n+1} = \gamma_n P_C(w_n - \lambda A w_n) + \delta_n S v_n + \sigma_n u_n, \quad \forall n \ge 0, \end{cases}$$

where $f : C \to C$ is a δ-contractive map with $\delta \in [0, \frac{1}{2})$, and $\{\beta_n\}, \{\sigma_n\}, \{\gamma_n\}, \{\delta_n\}$ are four sequences in $[0,1]$ s.t. $\sigma_n + \gamma_n + \delta_n = 1$, $(\gamma_n + \delta_n)\varsigma \le \gamma_n < (1-2\delta)\delta_n$, $\sum_{n=0}^{\infty} \beta_n = \infty$, $\liminf_{n\to\infty} \delta_n > 0$, $\liminf_{n\to\infty} \sigma_n > 0$, and $\lim_{n\to\infty}(\frac{\gamma_{n+1}}{1-\sigma_{n+1}} - \frac{\gamma_n}{1-\sigma_n}) = \lim_{n\to\infty} \beta_n = 0$. They claimed the strong convergence of the sequence in H.

In the extragradient approach, one has to compute two projection operators onto C. It is clear that the projection operator onto the convex set C is closely related to a minimum distance problem. In the case where C is a general convex set, the computation of two projections might be prohibitively time-consuming. Via Korpelevich's extragradient approach, Censor et al. [13] suggested a subgradient algorithm, in which the second projection operator onto the subset C is changed onto a half-space. Recently, numerous methods of reduced-gradient-type are focused and extensively investigated in both infinite and infinite dimensional spaces; see, for example [14–23]. Based on inertial effects, Thong and Hieu [24] proposed an inertial subgradient method, and also proved the weak convergence of their algorithms. In addition, the authors [25] investigated subgradient-based fast algorithms with inertial effects.

Inspired by the above research works in [12,24–27], we are concerned with hybrid-adaptive step-sizes Tseng's extragradient algorithms, that are more advantageous and more subtle than the above iterative algorithms because they involve solving the VIP with Lipschitzian, pseudomonotone operators, and the common fixed-point problem of a finite family of strict pseudocontractions in Hilbert spaces. By imposing some appropriate weak assumptions on parameters, one obtains a norm solution of the problems, which solves a certain hierarchical variational inequality. The outline of this article is organized below. In Section 2, a toolbox containing definitions and preliminary results is provided. In Section 3, we propose and investigate the iterative algorithms and their convergence criteria. In Section 4, theorems of norm solutions are employed as illustrating examples to support the convergence criteria.

2. Preliminaries

Lemma 1 ([28]). *Let $S : C \to C$ be a ς-strict pseudocontraction. If $\{u_n\}$ is a sequence in C such that $(Id - S)u_n \to 0$, where Id is the identity operator on H, and $u_n \rightharpoonup u \in C$, then $u = Su$. Further, S is $\frac{1+\zeta}{1-\zeta}$ Lipschitz continuous.*

Lemma 2 ([29]). *Let $S : C \to C$ be a ς-strict pseudocontraction, and let γ and β be real numbers in $[0, +\infty)$. Then, $\|\gamma(u - v) + \beta(Sv - Su)\| \leq (\beta + \gamma)\|v - u\|$ $\forall v, u \in C$ provided that $\gamma \geq \varsigma(\beta + \gamma)$.*

Lemma 3 ([30]). *Let f be a pseudomonotone mapping from C into H which is continuous on finite-dimensional subspaces. Then, $x \in C$ is a solution of $\langle u - x, f(x)\rangle \geq 0$, $\forall u \in C$ iff $\langle u - x, f(u)\rangle \geq 0$, $\forall u \in C$.*

Lemma 4 ([31]). *Let $\{a_n\}$ be a sequence in $[0, +\infty)$ satisfying the condition $a_{n+1} \leq a_n + s_n b_n - s_n a_n$ $\forall n \geq 1$, where $\{s_n\} \in (0, 1)$ and $\{b_n\} \in (-\infty, \infty)$ s.t. (a) $\sum_{n=1}^{\infty} |s_n b_n| < \infty$, $\limsup_{n\to\infty} b_n \leq 0$ (b) $\sum_{n=1}^{\infty} s_n = \infty$. Then, $a_n \to 0$ as $n \to \infty$.*

3. Results

From now on, one can always assume that our feasibility set $\Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ is consistent.

Put $n := n + 1$ and return to Step 1 (Algorithm 1), where $\{\epsilon_n\} \subset (0, 1]$ and $\{\beta_n\}, \{\sigma_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ are such that $\sigma_n + \gamma_n + \delta_n = 1$; $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = \lim_{n\to\infty} \beta_n = 0$; $\liminf_{n\to\infty} \delta_n > 0$; $\liminf_{n\to\infty} \sigma_n > 0$; $\liminf_{n\to\infty}((1 - 2\delta)\delta_n - \gamma_n) > 0$; $\limsup_{n\to\infty} \sigma_n < 1$; $\sum_{n=1}^{\infty} \beta_n = \infty$; $(\gamma_n + \delta_n)\zeta \leq \gamma_n < (1 - 2\delta)\delta_n$; the pseudomonotone self-operator A is Lipschitz continuous with module L and sequentially weakly continuous on H; T is a ζ-strictly pseudocontractive self-operator on H; and f : is a δ-contraction operator, where $\delta \in [0, \frac{1}{2})$, from H to C.

Algorithm 1: Initial Step: Fix two initials x_0 and x_1 in H and set $\alpha > 0$, $\tau_1 > 0$, $\mu \in (0, 1)$.

Iteration Steps: calculate iterative sequence x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose α_n s.t. $0 \leq \alpha_n \leq \bar{\alpha}_n$, where

$$\bar{\alpha}_n = \begin{cases} \min\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } x_n \neq x_{n-1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{1}$$

Step 2. Let $w_n = x_n + \alpha_n(x_n - x_{n-1})$ and calculate $y_n = P_C(w_n - \tau_n A w_n)$.

Step 3. Calculate $x_{n+1} = \sigma_n x_n + \gamma_n(y_n - \tau_n(Ay_n - Aw_n)) + \delta_n T z_n$, where $z_n = \beta_n f(x_n) + (1 - \beta_n)(y_n - \tau_n(Ay_n - Aw_n))$. Update

$$\tau_{n+1} = \begin{cases} \min\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \tau_n\} & \text{if } Aw_n - Ay_n \neq 0, \\ \tau_n & \text{otherwise.} \end{cases} \tag{2}$$

Remark 1. *We show $\lim_{n\to\infty}(\alpha_n\|x_n - x_{n-1}\|)/\beta_n = 0$. It follows from (1) that $\epsilon_n \geq \alpha_n\|x_n - x_{n-1}\|$ $\forall n \geq 1$. Since $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$, one sees that $0 = \lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} \geq \limsup_{n\to\infty} \frac{\alpha_n\|x_n - x_{n-1}\|}{\beta_n}$.*

Lemma 5. *Let $\{\tau_n\}$ be generated by (2). Then, $\{\tau_n\}$ is a nonincreasing sequence with $\tau_n \geq \tilde{\tau} := \min\{\tau_1, \frac{\mu}{L}\}$ $\forall n \geq 1$ and $\lim_{n\to\infty} \tau_n \geq \tilde{\tau} := \min\{\tau_1, \frac{\mu}{L}\}$.*

Proof. By borrowing (2), one concludes that $\tau_n \geq \tau_{n+1}\ \forall n \geq 1$. One also has

$$\|Aw_n - Ay_n\| \leq L\|w_n - y_n\| \ \Rightarrow\ \tau_{n+1} \geq \min\{\tau_n, \frac{\mu}{L}\}.$$

Note that $\tau_1 \geq \tilde{\tau} := \min\{\tau_1, \frac{\mu}{L}\}$. So, $\tau_n \geq \tilde{\tau} := \min\{\tau_1, \frac{\mu}{L}\}\ \forall n \geq 1$. □

Lemma 6. *Let $\{y_n\}$, $\{w_n\}$, and $\{z_n\}$ be three iterative vector sequences defined by Algorithm 1. We have*

$$\begin{aligned}\|p - z_n\|^2 &\leq \beta_n\delta\|p - x_n\|^2 + (1-\beta_n)\|p - w_n\|^2 - (1-\beta_n)(1 - \mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|w_n - y_n\|^2 \\ &\quad + 2\beta_n\langle (f - I)p, z_n - p\rangle,\ \forall p \in \Omega,\end{aligned} \tag{3}$$

where $u_n := y_n - \tau_n(Ay_n - Aw_n)$.

Proof. Fixing $p \in \Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ arbitrarily, one asserts $\langle Ap, y_n - p\rangle \geq 0$ and $Tp = p$. This yields

$$\begin{aligned}&\|p - u_n\|^2 = \|p - y_n\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle \\ &= \|p - w_n\|^2 + \|w_n - y_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\ &\quad - 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle \\ &= \|p - w_n\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\ &\quad - 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle.\end{aligned}$$

Thanks to $y_n = P_C(w_n - \tau_n Aw_n)$, we have

$$\langle y_n - w_n, y_n - p\rangle \leq -\tau_n\langle Aw_n, y_n - p\rangle.$$

This ensures that

$$\begin{aligned}\|p - u_n\|^2 &\leq \|p - w_n\|^2 - \|w_n - y_n\|^2 - 2\tau_n\langle Aw_n, y_n - p\rangle + \tau_n^2\|Ay_n - Aw_n\|^2 \\ &\quad - 2\tau_n\langle y_n - p, Ay_n - Aw_n\rangle \\ &= \|p - w_n\|^2 - \|w_n - y_n\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 - 2\tau_n\langle Ay_n, y_n - p\rangle.\end{aligned}$$

By using the fact that $\langle Ap, y_n - p\rangle \geq 0$, one obtains that $\langle Ay_n, y_n - p\rangle \geq 0$. Hence,

$$\|p - u_n\|^2 \leq \|p - w_n\|^2 + \tau_n^2\|Ay_n - Aw_n\|^2 - \|w_n - y_n\|^2. \tag{4}$$

Moreover, from (2), it follows that

$$\tau_{n+1}\|Aw_n - Ay_n\| \leq \mu\|w_n - y_n\| \quad \forall n \geq 1. \tag{5}$$

Combining (4) and (5), we obtain

$$\|p - u_n\|^2 \leq \|p - w_n\|^2 - (1 - \mu^2\frac{\tau_n^2}{\tau_{n+1}^2})\|w_n - y_n\|^2. \tag{6}$$

On the other hand,

$$\begin{aligned}z_n - p &= (1-\beta_n)(y_n - \tau_n(Ay_n - Aw_n)) - p + \beta_n f(x_n) \\ &= (1-\beta_n)(u_n - p) + \beta_n(f - I)p + \beta_n(f(x_n) - f(p)).\end{aligned}$$

Using the convexity of the norm function, we get

$$\begin{aligned}\|z_n - p\|^2 &\le 2\beta_n\langle fp - p, z_n - p\rangle + \|(1-\beta_n)(u_n - p) - \beta_n(f(p) - f(x_n))\|^2 \\ &\le 2\beta_n\langle (fp - p), z_n - p\rangle + [(1-\beta_n)\|p - u_n\| + \beta_n\delta\|p - x_n\|]^2 \\ &\le 2\beta_n\langle (fp - p), z_n - p\rangle + (1-\beta_n)\|p - u_n\|^2 + \beta_n\delta\|p - x_n\|^2 \\ &\le (1-\beta_n)[\|p - w_n\|^2 - (1 - \mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|w_n - y_n\|^2] + \beta_n\delta\|p - x_n\|^2 \\ &\quad + 2\beta_n\langle (fp - p), z_n - p\rangle \\ &= (1-\beta_n)\|p - w_n\|^2 - (1-\beta_n)(1 - \mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|w_n - y_n\|^2 + \beta_n\delta\|p - x_n\|^2 \\ &\quad + 2\beta_n\langle (fp - p), z_n - p\rangle.\end{aligned}$$

This completes the proof. □

Lemma 7. *Let $\{z_n\}$, $\{y_n\}$, and $\{x_n\}$ be three iterative sequences, which are bounded, defined by Algorithm 1. Suppose that there exists a subsequence $\{w_{n_k}\}$ of the weakly convergent sequence $\{w_n\}$ such that $w_{n_k} \rightharpoonup z \in H$. If $\|x_n - x_{n+1}\| \to 0$, $\|w_n - y_n\| \to 0$, $\|w_n - z_n\| \to 0$, then $z \in \Omega$.*

Proof. Algorithm 1 shows $\|x_n - w_n\| = \alpha_n\|x_{n-1} - x_n\|$. Utilizing Remark 1, we have $\lim_{n\to\infty}\|w_n - x_n\| = 0$. This, together with the assumption $w_n - z_n \to 0$, yields that

$$\|z_n - x_n\| \le \|z_n - w_n\| + \|w_n - x_n\| \to 0 \quad (n \to \infty).$$

Since $\{x_n\}$ is bounded and $\alpha_n(x_n - x_{n-1}) \to 0$, one asserts that $\{w_n\}$ is bounded. Note that (4) yields

$$\|u_n - p\|^2 \le \|w_n - p\|^2 + \tau_1^2 L^2\|y_n - w_n\|^2.$$

Hence, $\{u_n\}$ is bounded, where $u_n := y_n - \tau_n(Ay_n - Aw_n)$. By Algorithm 1, we also get

$$z_n - x_n = \beta_n f(x_n) + u_n - x_n - \beta_n u_n.$$

So, it follows from the boundedness of $\{x_n\}$ and $\{u_n\}$ that

$$\|u_n - x_n\| = \|z_n - x_n - \beta_n f(x_n) + \beta_n u_n\| \le \|z_n - x_n\| + \beta_n(\|f(x_n)\| + \|u_n\|),$$

which indicates $u_n - x_n$ tends to 0 as n tends to the infinity. Using Algorithm 1 again, we get

$$\begin{aligned}x_{n+1} - z_n &= \sigma_n(x_n - z_n) + \gamma_n(u_n - z_n) + \delta_n(Tz_n - z_n) \\ &= \sigma_n(x_n - z_n) + \gamma_n(u_n - x_n + x_n - z_n) + \delta_n(Tz_n - z_n) \\ &= (1-\delta_n)(x_n - z_n) + \gamma_n(u_n - x_n) + \delta_n(Tz_n - z_n),\end{aligned}$$

which immediately leads to

$$\begin{aligned}\delta_n\|z_n - Tz_n\| &= \|x_{n+1} - \gamma_n(u_n - x_n) - z_n - (1-\delta_n)(x_n - z_n)\| \\ &= \|x_{n+1} - (1-\delta_n)(x_n - z_n) - x_n + x_n - z_n - \gamma_n(u_n - x_n)\| \\ &= \|x_{n+1} - x_n + \delta_n(x_n - z_n) - \gamma_n(u_n - x_n)\| \\ &\le \|x_{n+1} - x_n\| + \|x_n - z_n\| + \|u_n - x_n\|.\end{aligned}$$

Since $x_n - x_{n+1}$, $z_n - x_n$, and $u_n - x_n$ tend to 0 as n tends to the infinity and $\liminf_{n\to\infty}\delta_n > 0$, we obtain

$$\lim_{n\to\infty}\|z_n - Tz_n\| = 0,$$

which, together with Lemma 1, ensures that

$$\begin{aligned} \|x_n - Tx_n\| &\leq \|x_n - z_n\| + \|z_n - Tz_n\| + \|Tz_n - Tx_n\| \\ &\leq \|x_n - z_n\| + \|z_n - Tz_n\| + \tfrac{1+\zeta}{1-\zeta}\|z_n - x_n\| \\ &\leq \tfrac{2}{1-\zeta}\|x_n - z_n\| + \|z_n - Tz_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{7}$$

From the restriction on the operator A, we have

$$\tau_n\langle Aw_n, x - w_n\rangle \geq \langle w_n - y_n, x - y_n\rangle - \tau_n\langle Aw_n, w_n - y_n\rangle \quad \forall x \in C. \tag{8}$$

Using the boundedness of $\{w_{n_k}\}$ and Lipschitzian property of A, we get the boundedness of $\{Aw_{n_k}\}$. Note that $\tau_n \geq \tilde{\tau} := \min\{\tau_1, \frac{\mu}{L}\}$ and the boundedness of $\{y_{n_k}\}$. Inequality (8) deduces $\liminf_{k\to\infty}\langle w_{n_k} - x, Aw_{n_k}\rangle \leq 0\ \forall x \in C$. Borrowing the facts that $\lim_{n\to\infty}\|w_n - y_n\| = 0$ and A is Lipschitz continuous with moudle L, one concludes that $\lim_{n\to\infty}\|Ay_n - Aw_n\| = 0$, which combines with (8) and sends us to the situation $\liminf_{k\to\infty}\langle y_{n_k} - x, Ay_{n_k}\rangle \leq 0\ \forall x \in C$.

One now focuses on $z \in \mathrm{Fix}(T)$. Thanks to the weak convergence $w_{n_k} \rightharpoonup z$, as $k \to \infty$, one reaches $x_{n_k} \rightharpoonup z$. Without loss of generality, we may assume $l = n_k \mathrm{mod} N$ for all k. Since by the assumption $x_n - x_{n+1} \to 0$ we have $x_{n_k+j} \rightharpoonup z$ for all $j \geq 1$, we deduce from (7) that

$$\|x_{n_k+j} - T_{l+j}x_{n_k+j}\| = \|T_{n_k+j}x_{n_k+j} - x_{n_k+j}\| \to 0$$

as $k \to \infty$. An application of Lemma 1 is to yield $z \in \mathrm{Fix}(T_{l+j})$ for all j. This amounts to

$$z \in \mathrm{Fix}(T). \tag{9}$$

Let $\{\varepsilon_k\}$ be a decreasing real sequence in $(0, 1)$ converging to 0 and let

$$\varepsilon_k + \langle x - y_{n_j}, Ay_{n_j}\rangle \geq 0 \tag{10}$$

for all $j \geq m_k$, where m_k is the smallest integer satisfying the above inequality. Note that sequence $\{m_k\}$ is increasing and $Ay_{m_k} \neq 0$. It follows that $\langle Ay_{m_k}, h_{m_k}\rangle = 1$, where $h_{m_k} := \frac{Ay_{m_k}}{\|Ay_{m_k}\|^2}$. This sends us to $\langle y_{m_k} - x - \varepsilon_k h_{m_k}, Ay_{m_k}\rangle \leq 0$, which guarantees

$$\langle y_{m_k} - x - \varepsilon_k h_{m_k}, A(\varepsilon_k h_{m_k} + x)\rangle \leq 0.$$

This sends us to

$$\langle Ax, y_{m_k} - x\rangle \leq \langle y_{m_k} - x - \varepsilon_k h_{m_k}, Ax - A(\varepsilon_k h_{m_k} + x)\rangle - \varepsilon_k\langle Ax, h_{m_k}\rangle. \tag{11}$$

On the other hand, one has $\lim_{n\to\infty}\|w_n - y_n\| = 0$ and $w_{n_k} \rightharpoonup z$ as $k \to \infty$. This infers $w_{n_k} \rightharpoonup z$, which lies in C, as k goes to the infinity. So, $Ay_{n_k} \rightharpoonup Az$ as k goes to the infinity. This shows that z is not a solution. In the sense of norms, one obtains $\liminf_{k\to\infty}\|Ay_{n_k}\| \geq \|Az\|$. This further concludes that

$$\begin{aligned} 0 &= \limsup_{k\to\infty}\varepsilon_k / \liminf_{k\to\infty}\|Ay_{n_k}\| \\ &\geq \limsup_{k\to\infty}\varepsilon_k / \|Ay_{m_k}\| \\ &\geq \limsup_{k\to\infty}\|h_{m_k}\varepsilon_k\| \\ &\geq 0, \end{aligned}$$

which reaches that $h_{m_k}\varepsilon_k \to 0$ as $k \to \infty$.

Finally, one focuses on the desired point z. (11), the boundedness of sequences $\{h_{m_k}\}$ and $\{y_{m_k}\}$, and the fact that $h_{m_k}\epsilon_k \to 0$ as $k \to \infty$, yield that $\langle z - x, Ax\rangle = \liminf_{k\to\infty}\langle y_{m_k} - x, Ax\rangle \leq 0$ for all x in C. Lemma 3 asserts that the desired point z is a solution to the VIP, e.g., $z \in \mathrm{VI}(C, A)$. Therefore, we have from (9) that $z \in \Omega := \mathrm{VI}(C, A) \cap \mathrm{Fix}(T)$. The proof is complete. □

Theorem 1. *Let $\{x_n\}$ be a vector sequence constructed by Algorithm 1 and let $A(H)$ be bounded. Suppose that x^* is in Ω, which uniquely solves $\langle x^* - fx^*, x^* - x\rangle \leq 0, \forall x \in \Omega$. Then,*

$$x_n \to x^* \in \Omega \quad \Leftrightarrow \quad \begin{cases} x_n - x_{n+1} \to 0, \\ \sup_{n\geq 1}\|(I-f)x_n\| < \infty. \end{cases}$$

Proof. Noticing condition (iv) on $\{\sigma_n\}$, one may assume that $\{\sigma_n\} \subset [a, b]$, which is a subset of $(0, 1)$. Using the Banach Fixed Point Theory, one deduces that a unique point x^* in H s.t. $x^* = P_\Omega f(x^*)$. Hence, there is a solution $x^* \in \Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ to the HVI problem

$$\langle x^* - x, x^* - fx^*\rangle \leq 0 \tag{12}$$

for any point x in Ω. If $\lim_{n\to\infty}\|x_n - x^*\| = 0$, then

$$\sup_{n\geq 1}(\|x^* - x_n\| + \|f(x^*) - f(x_n)\| + \|f(x^*) - x^*\|) \geq \sup_{n\geq 1}\|x_n - f(x_n)\|$$

and

$$\|x_{n+1} - x^*\| + \|x^* - x_n\| \geq \|x_{n+1} - x_n\| \geq 0.$$

So, $x_{n+1} - x_n \to 0$ as $n \to \infty$. In order to prove the sufficiency of the theorem, one supposes $\|x_n - x_{n+1}\| \to 0$ and $\sup_{n\geq 1}\|(I-f)x_n\| < \infty$. Then, we divide the proof of the sufficiency into several steps. □

Step 1. One proves the boundedness of $\{x_n\}$. In fact, taking an arbitrary $p \in \Omega$, one has $p = Tp$ and (6), that is,

$$\|p - u_n\|^2 + (1 - \mu^2\frac{\tau_n^2}{\tau_{n+1}^2})\|w_n - y_n\|^2 \leq \|p - w_n\|^2. \tag{13}$$

Since $\lim_{n\to\infty}(1 - \mu^2\frac{\tau_n^2}{\tau_{n+1}^2}) = 1 - \mu^2 > 0$, there exists an integer $n_0 \geq 1$ with

$$1 - \mu^2\frac{\tau_n^2}{\tau_{n+1}^2} > 0 \; \forall n \geq n_0.$$

Using (13), we have

$$\|p - u_n\| \leq \|p - w_n\|, \quad \forall n \geq n_0. \tag{14}$$

So,

$$\|p - w_n\| \leq \alpha_n\|x_n - x_{n-1}\| + \|p - x_n\| = \beta_n \cdot \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| + \|p - x_n\|. \tag{15}$$

From Remark 1, we have $\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \to 0$ $(n \to \infty)$. This ensures that $\exists M_1 > 0$ s.t.

$$\frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\| \leq M_1 \quad \forall n \geq 1. \tag{16}$$

Combining (14), (15), and (16), we have

$$\|u_n - p\| \leq \|w_n - p\| \leq \|x_n - p\| + \beta_n M_1 \quad \forall n \geq n_0. \tag{17}$$

Note that $A(H)$ is bounded, $y_n = P_C(w_n - \tau_n A w_n)$, $f(H) \subset C$, and

$$u_n = y_n - \tau_n(Ay_n - Aw_n).$$

Hence, we know that $\{Ay_n\}$ and $\{Aw_n\}$ are both bounded. From $\sup_{n\geq 1} \|(I-f)x_n\| < \infty$ and $\alpha_n\|x_n - x_{n-1}\| \to 0$, we conclude that

$$\begin{aligned} \|u_n - f(x_n)\| &= \|y_n - \tau_n(Ay_n - Aw_n) - P_C f(x_n)\| \\ &\leq \|P_C(w_n - \tau_n Aw_n) - P_C f(x_n)\| + \tau_n\|Ay_n - Aw_n\| \\ &\leq \alpha_n\|x_n - x_{n-1}\| + \|x_n - f(x_n)\| + \tau_1(\|Aw_n\| + \|Ay_n - Aw_n\|) \leq M_0, \end{aligned}$$

where

$$\sup_{n\geq 1}\{\alpha_n\|x_n - x_{n-1}\| + \|x_n - f(x_n)\| + \tau_1(\|Aw_n\| + \|Ay_n - Aw_n\|)\} \leq M_0$$

for some $M_0 > 0$. By using (17), one concludes

$$\begin{aligned} \|p - z_n\| &\leq (1-\beta_n)\|p - u_n\| + \beta_n\|fp - p\| + \beta_n\delta\|p - x_n\| \\ &\leq (1-\beta_n)(\|p - x_n\| + \beta_n M_1) + \beta_n\|fp - p\| + \beta_n\delta\|p - x_n\| \\ &\leq (1 - \beta_n(1-\delta))\|p - x_n\| + \beta_n(\|fp - p\| + M_1), \end{aligned}$$

which, together with Lemma 2 and $(\gamma_n + \delta_n)\zeta \leq \gamma_n$, yields

$$\begin{aligned} \|p - x_{n+1}\| &\leq (1-\sigma_n)\|\tfrac{1}{1-\sigma_n}[\gamma_n(z_n - p) + \delta_n(Tz_n - p)]\| + \gamma_n\|u_n - z_n\| + \sigma_n\|p - x_n\| \\ &\leq (1-\sigma_n)\|p - z_n\| + \gamma_n\beta_n\|u_n - f(x_n)\| + \sigma_n\|p - x_n\| \\ &\leq (1-\sigma_n)[(1-\beta_n(1-\delta))\|p - x_n\| + \beta_n(M_0 + M_1 + \|(f-I)p\|)] + \sigma_n\|p - x_n\| \\ &= [1 - \beta_n(1-\sigma_n)(1-\delta)]\|p - x_n\| + \beta_n(1-\sigma_n)(1-\delta)\tfrac{M_1 + M_0 + \|(I-f)p\|}{1-\delta} \\ &\leq \max\{\tfrac{M_0 + M_1 + \|(f-I)p\|}{1-\delta}, \|p - x_n\|\}. \end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \leq \max\{\frac{M_0 + M_1 + \|(I-f)p\|}{1-\delta}, \|x_{n_0} - p\|\}.$$

This indicates that all the vector sequence $\{x_n\}, \{u_n\}, \{w_n\}, \{y_n\}$, and $\{z_n\}$ are bounded sequences.

Step 2. We claim

$$(1-\beta_n)(1-\sigma_n)(1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2})\|y_n - w_n\|^2 \leq \|p - x_n\|^2 + \alpha_n M_4 - \|p - x_{n+1}\|^2, \quad \forall n \geq n_0,$$

for some $M_4 > 0$. Indeed, using Lemma 2, Lemma 6, and the convexity of $\|\cdot\|^2$, we have from $(\gamma_n + \delta_n)\zeta \leq \gamma_n$ that $\forall n \geq n_0$,

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 = \|\sigma_n(x_n-p)+\gamma_n(z_n-p)+\delta_n(Tz_n-p)+\gamma_n(u_n-z_n)\|^2\\
&\le \|\sigma_n(x_n-p)+\gamma_n(z_n-p)+\delta_n(Tz_n-p)\|^2+2\gamma_n\beta_n\langle u_n-f(x_n),x_{n+1}-p\rangle\\
&\le \sigma_n\|p-x_n\|^2+(1-\sigma_n)\|\tfrac{1}{1-\sigma_n}[\gamma_n(z_n-p)+\delta_n(Tz_n-p)]\|^2\\
&\quad +2(1-\sigma_n)\beta_n\|u_n-f(x_n)\|\|x_{n+1}-p\|\\
&\le \sigma_n\|p-x_n\|^2+(1-\sigma_n)\|z_n-p\|^2+2(1-\sigma_n)\beta_n\|u_n-f(x_n)\|\|p-x_{n+1}\|\\
&\le \sigma_n\|p-x_n\|^2+(1-\sigma_n)\{\beta_n\delta\|x_n-p\|^2+(1-\beta_n)\|p-w_n\|^2\\
&\quad -(1-\beta_n)(1-\mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|w_n-y_n\|^2+2\beta_n\langle(f-I)p,z_n-p\rangle\}\\
&\quad +2(1-\sigma_n)\beta_n\|u_n-f(x_n)\|\|x_{n+1}-p\|\\
&\le \sigma_n\|p-x_n\|^2+(1-\sigma_n)\{\beta_n\delta\|p-x_n\|^2+(1-\beta_n)\|p-w_n\|^2\\
&\quad -(1-\beta_n)(1-\mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|w_n-y_n\|^2+\beta_nM_2\},
\end{aligned}
\tag{18}
$$

where $\sup_{n\ge1}2(\|(f-I)p\|\|z_n-p\|+\|u_n-f(x_n)\|\|x_{n+1}-p\|)\le M_2$ for some $M_2>0$. In addition, from (17) we get

$$
\begin{aligned}
\|p-w_n\|^2 &= \beta_n(2M_1\|p-x_n\|+\beta_nM_1^2)+\|p-x_n\|^2\\
&\le \beta_nM_3+\|p-x_n\|^2,
\end{aligned}
\tag{19}
$$

where $\sup_{n\ge1}(\beta_nM_1^2+2M_1\|p-x_n\|)\le M_3$ for some $M_3>0$. Substituting (19) for (18), we obtain that for all $n\ge n_0$,

$$
\begin{aligned}
\|p-x_{n+1}\|^2 &\le \sigma_n\|p-x_n\|^2+(1-\sigma_n)\{\delta\beta_n\|p-x_n\|^2+(1-\beta_n)[\|p-x_n\|^2+\beta_nM_3]\\
&\quad -(1-\beta_n)(1-\mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|y_n-w_n\|^2+\beta_nM_2\}\\
&= [1-(1-\sigma_n)(1-\delta)\beta_n]\|p-x_n\|^2+(1-\sigma_n)\beta_n(1-\beta_n)M_3\\
&\quad -(1-\beta_n)(1-\sigma_n)(1-\mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|y_n-w_n\|^2+(1-\sigma_n)\beta_nM_2\\
&\le \|p-x_n\|^2-(1-\beta_n)(1-\sigma_n)(1-\mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|y_n-w_n\|^2+\beta_nM_4,
\end{aligned}
\tag{20}
$$

where $M_4:=M_2+M_3$. This immediately implies that for all $n\ge n_0$,

$$
(1-\beta_n)(1-\sigma_n)(1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2})\|y_n-x_n\|^2\le\|p-x_n\|^2+\beta_nM_4-\|p-x_{n+1}\|^2. \tag{21}
$$

Step 3. One proves

$$
\begin{aligned}
&\|p-x_{n+1}\|^2\\
&\le [1-\tfrac{(1-2\delta)\delta_n-\gamma_n}{1-\beta_n\gamma_n}\beta_n]\|x_n-p\|^2+\tfrac{[(1-2\delta)\delta_n-\gamma_n]\beta_n}{1-\beta_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_{n+1}\|\\
&\quad +\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_n\|+\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(p)-p,x_n-p\rangle\\
&\quad +\tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}\cdot\tfrac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|3M\},\quad \forall n\ge n_0,
\end{aligned}
$$

where M is some appropriate constant.

$$
\begin{aligned}
3M\alpha_n\|x_n-x_{n-1}\|+\|p-x_n\|^2 &\ge \|p-x_n\|^2+\alpha_n(\alpha_n\|x_n-x_{n-1}\|+2\|p-x_n\|)\|x_n-x_{n-1}\|\\
&= (\|p-x_n\|+\alpha_n\|x_n-x_{n-1}\|)^2\\
&\ge \|p-w_n\|^2,
\end{aligned}
\tag{22}
$$

where $M\ge\sup_{n\ge1}\{\alpha_n\|x_n-x_{n-1}\|,\|p-x_n\|\}$. From the convexity of $\|\cdot\|^2$, one arrives at

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\sigma_n(x_n-p)+\gamma_n(z_n-p)+\delta_n(Tz_n-p)+\gamma_n(u_n-z_n)\|^2\\
&\le \|\sigma_n(x_n-p)+\gamma_n(z_n-p)+\delta_n(Tz_n-p)\|^2+2\gamma_n\beta_n\langle u_n-f(x_n),x_{n+1}-p\rangle\\
&\le \sigma_n\|x_n-p\|^2+(1-\sigma_n)\|\tfrac{1}{1-\sigma_n}[\gamma_n(z_n-p)+\delta_n(Tz_n-p)]\|^2\\
&\quad +2\gamma_n\beta_n\langle u_n-p,x_{n+1}-p\rangle+2\gamma_n\beta_n\langle p-f(x_n),x_{n+1}-p\rangle,
\end{aligned}
$$

which yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\sigma_n)\|p-z_n\|^2+\sigma_n\|p-x_n\|^2+2\gamma_n\beta_n\|p-u_n\|\|p-x_{n+1}\| \\ &\quad +2\gamma_n\beta_n\langle p-f(x_n),x_{n+1}-p\rangle \\ &\le \sigma_n\|p-x_n\|^2+(1-\sigma_n)[(1-\beta_n)\|p-u_n\|^2+2\beta_n\langle f(x_n)-p,z_n-p\rangle] \\ &\quad +\gamma_n\beta_n(\|p-u_n\|^2+\|p-x_{n+1}\|^2)+2\gamma_n\beta_n\langle p-f(x_n),x_{n+1}-p\rangle.\end{aligned}$$

From (17) and (22) we know that $\|u_n-p\|^2\le\|x_n-p\|^2+\alpha_n\|x_n-x_{n-1}\|3M\ \forall n\ge n_0$. Hence, we have, $\forall n\ge n_0$, that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \sigma_n\|p-x_n\|^2+(1-\sigma_n)(1-\beta_n)(\|p-x_n\|^2+\alpha_n\|x_n-x_{n-1}\|3M) \\ &\quad +2\beta_n(1-\sigma_n)\langle f(x_n)-p,z_n-p\rangle+\gamma_n\beta_n(\|p-x_n\|^2+\|p-x_{n+1}\|^2 \\ &\quad +\alpha_n\|x_n-x_{n-1}\|3M)+2\gamma_n\beta_n\langle p-f(x_n),x_{n+1}-p\rangle \\ &\le [1-\beta_n(1-\sigma_n)]\|p-x_n\|^2+2\beta_n\delta_n\langle f(x_n)-p,z_n-p\rangle+\gamma_n\beta_n(\|p-x_n\|^2 \\ &\quad +\|p-x_{n+1}\|^2)+(1-\sigma_n)\alpha_n\|x_n-x_{n-1}\|3M+2\gamma_n\beta_n\langle f(x_n)-p,z_n-x_{n+1}\rangle \\ &\le [1-\beta_n(1-\sigma_n)]\|p-x_n\|^2+2\gamma_n\beta_n\|p-f(x_n)\|\|z_n-x_{n+1}\| \\ &\quad +2\beta_n\delta_n\langle f(x_n)-p,x_n-p\rangle+2\beta_n\delta_n\langle f(x_n)-p,z_n-x_n\rangle \\ &\quad +\gamma_n\beta_n(\|p-x_n\|^2+\|p-x_{n+1}\|^2)+(1-\sigma_n)\alpha_n\|x_n-x_{n-1}\|3M \\ &\le [1-\beta_n(1-\sigma_n)]\|p-x_n\|^2+2\gamma_n\beta_n\|p-f(x_n)\|\|z_n-x_{n+1}\| \\ &\quad +2\beta_n\delta_n\delta\|p-x_n\|^2+2\beta_n\delta_n\langle f(p)-p,x_n-p\rangle+2\beta_n\delta_n\|p-f(x_n)\|\|z_n-x_n\| \\ &\quad +\gamma_n\beta_n(\|p-x_n\|^2+\|p-x_{n+1}\|^2)+(1-\sigma_n)\alpha_n\|x_n-x_{n-1}\|3M,\end{aligned}$$

which immediately yields

$$\begin{aligned}&\|x_{n+1}-p\|^2 \\ &\le [1-\tfrac{(1-2\delta)\delta_n-\gamma_n}{1-\beta_n\gamma_n}\beta_n]\|x_n-p\|^2+\tfrac{[(1-2\delta)\delta_n-\gamma_n]\beta_n}{1-\beta_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_{n+1}\| \\ &\quad +\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-p\|\|z_n-x_n\|+\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(p)-p,x_n-p\rangle \\ &\quad +\tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}\cdot\tfrac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|3M\}.\end{aligned} \tag{23}$$

Step 4. We claim strong convergence of vector sequence $\{x_n\}$ to the unique solution of HVI (12), $x^*\in\Omega$. One lets $p=x^*$, and use (23) to obtain

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \\ &\le [1-\tfrac{(1-2\delta)\delta_n-\gamma_n}{1-\beta_n\gamma_n}\beta_n]\|x_n-x^*\|^2+\tfrac{[(1-2\delta)\delta_n-\gamma_n]\beta_n}{1-\beta_n\gamma_n}\cdot\{\tfrac{2\gamma_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-x^*\|\|z_n-x_{n+1}\| \\ &\quad +\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\|f(x_n)-x^*\|\|z_n-x_n\|+\tfrac{2\delta_n}{(1-2\delta)\delta_n-\gamma_n}\langle f(x^*)-x^*,x_n-x^*\rangle \\ &\quad +\tfrac{\gamma_n+\delta_n}{(1-2\delta)\delta_n-\gamma_n}\cdot\tfrac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|3M\}.\end{aligned} \tag{24}$$

From (21), $x_n-x_{n+1}\to 0,\ \beta_n\to 0,\ 1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\to 1-\mu^2$, and $\{\sigma_n\}\subset[a,b]\subset(0,1)$, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}(1-\beta_n)(1-b)(1-\mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|w_n-y_n\|^2 \\ &\le \limsup_{n\to\infty}(1-\beta_n)(1-\sigma_n)(1-\mu^2\tfrac{\tau_n^2}{\tau_{n+1}^2})\|w_n-y_n\|^2 \\ &\le \limsup_{n\to\infty}[\|p-x_n\|^2-\|p-x_{n+1}\|^2+\beta_nM_4] \\ &\le \limsup_{n\to\infty}[\|p-x_n\|^2-\|p-x_{n+1}\|^2]+\limsup_{n\to\infty}\beta_nM_4 \\ &\le \limsup_{n\to\infty}(\|p-x_n\|+\|p-x_{n+1}\|)\|x_n-x_{n+1}\|=0.\end{aligned}$$

This immediately implies that $w_n - y_n \to 0$. From the Lipschitzian property of A, we have $\|u_n - y_n\| = \tau_n \|Ay_n - Aw_n\| \le \tau_1 L \|y_n - w_n\|$. Consequently,

$$\lim_{n\to\infty} \|u_n - y_n\| = \lim_{n\to\infty} \|w_n - y_n\| = 0. \tag{25}$$

Thus, we get

$$\|x_n - y_n\| \le \|x_n - w_n\| + \|w_n - y_n\| \to 0 \quad (n \to \infty).$$

Since $z_n = \beta_n f(x_n) + (1-\beta_n)u_n$ with $u_n := y_n - \tau_n(Ay_n - Aw_n)$, from (25) and the boundedness of $\{x_n\}, \{u_n\}$, we get

$$\begin{aligned} \|z_n - y_n\| &= \|\beta_n f(x_n) - \beta_n u_n + u_n - y_n\| \\ &\le \beta_n(\|f(x_n)\| + \|u_n\|) + \|u_n - y_n\| \to 0 \quad (n \to \infty), \end{aligned} \tag{26}$$

and hence,

$$\|z_n - x_n\| \le \|z_n - y_n\| + \|y_n - x_n\| \to 0 \quad (n \to \infty). \tag{27}$$

Obviously, combining (25) and (26) guarantees that $\|w_n - z_n\| \le \|y_n - w_n\| + \|z_n - y_n\|$, which indicates that $\lim_{n\to\infty} \|w_n - z_n\| = 0$. Let vector sequence $\{x_{n_k}\}$ be a subsequence of original sequence $\{x_n\}$. From its boundedness, one asserts that

$$\lim_{k\to\infty} \langle (f-I)x^*, x^* - x_{n_k}\rangle = \limsup_{n\to\infty} \langle (f-I)x^*, x^* - x_n\rangle. \tag{28}$$

Without loss of generality, one lets $x_{n_k} \rightharpoonup \tilde{x}$. (28) implies

$$\limsup_{n\to\infty} \langle (f-I)x^*, x_n - x^*\rangle = \lim_{k\to\infty} \langle (f-I)x^*, x_{n_k} - x^*\rangle = \langle (f-I)x^*, \tilde{x} - x^*\rangle. \tag{29}$$

On the other hand, one has $\lim_{k\to\infty} \|w_{n_k} - x_{n_k} = 0$. This indicates $w_{n_k} \rightharpoonup \tilde{x}$. Lemma 7 guarantees that $\tilde{x}$ is in Ω. Therefore, (12) and (29) amount to $\limsup_{n\to\infty} \langle x_n - x^*, (f-I)x^*\rangle = \langle \tilde{x} - x^*, (f-I)x^*\rangle \le 0$. Note that $\liminf_{n\to\infty} \frac{(1-2\delta)\delta_n - \gamma_n}{1-\beta_n\gamma_n} > 0$. It follows that $\sum_{n=1}^{\infty} \frac{(1-2\delta)\delta_n - \gamma_n}{1-\beta_n\gamma_n}\beta_n = \infty$. It is clear that

$$\begin{aligned} \limsup_{n\to\infty} \{ &\tfrac{2\gamma_n}{(1-2\delta)\delta_n - \gamma_n} \|f(x_n) - x^*\| \|z_n - x_{n+1}\| + \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n} \|f(x_n) - x^*\| \|z_n - x_n\| \\ &+ \tfrac{2\delta_n}{(1-2\delta)\delta_n - \gamma_n} \langle f(x^*) - x^*, x_n - x^*\rangle + \tfrac{\gamma_n + \delta_n}{(1-2\delta)\delta_n - \gamma_n} \cdot \tfrac{\alpha_n}{\beta_n} \|x_n - x_{n-1}\| 3M\} \le 0. \end{aligned}$$

By utilizing Lemma 4, one concludes $x_n \to x^*$ easily. The proof is complete.

4. Applications

In this section, our main results are applied to solve the VIP and CFPP in an illustrating example. The initial point $x_0 = x_1$ is randomly chosen in $\mathbf{R}$. Take $f(x) = \frac{\sin x}{4}$, $\alpha = \tau_1 = \mu = \frac{1}{2}$, $\epsilon_n = \frac{1}{n^2}$, $\beta_n = \frac{1}{n+1}$, $\sigma_n = \frac{1}{3}$, $\gamma_n = \frac{1}{6}$, and $\delta_n = \frac{1}{2}$.

We first provide an example of Lipschitzian, pseudomonotone operator A satisfying the boundedness of $A(H)$ and strictly pseudocontractive operator T_1 with $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{VI}(C, A) \ne \emptyset$. Let $C = [-1, 1]$ and $H = \mathbf{R}$ with the inner product $\langle a, b\rangle = ab$ and induced norm $\|\cdot\| = |\cdot|$. Then, f is a δ-contractive map with $\delta = \frac{1}{4} \in [0, \frac{1}{2})$ and $f(H) \subset C$ because $\|f(x) - f(y)\| = \frac{1}{4}\|\sin x - \sin y\| \le \frac{1}{4}\|x - y\|$ for all $x, y \in H$.

Let $A : H \to H$ and $T_1 : H \to H$ be defined as $Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|}$ and $T_1 x := \frac{5}{8}x - \frac{1}{4}\sin x$ for all $x \in H$. Now, we first show that A is Lipschitzian, pseudomonotone operator with $L = 2$, such that $A(H)$ is bounded. Indeed, for all $x, y \in H$, we have

$$\begin{aligned}\|Ax - Ay\| &\le \left|\frac{1}{1+\|x\|} - \frac{1}{1+\|y\|}\right| + \left|\frac{1}{1+\|\sin x\|} - \frac{1}{1+\|\sin y\|}\right| \\ &\le \frac{\|x-y\|}{(1+\|x\|)(1+\|y\|)} + \frac{\|\sin x - \sin y\|}{(1+\|\sin x\|)(1+\|\sin y\|)} \\ &\le 2\|x-y\|.\end{aligned}$$

This implies that A is Lipschitzian operator with $L = 2$. Next, we verify that A is pseudomonotone. For any given $x, y \in H$, it is clear that the relation holds:

$$\langle Ax, y - x\rangle = \left(\frac{1}{1+|\sin x|} - \frac{1}{1+|x|}\right)(y-x) \ge 0 \Rightarrow \langle Ay, y - x\rangle = \left(\frac{1}{1+|\sin y|} - \frac{1}{1+|y|}\right)(y-x) \ge 0.$$

Furthermore, it is easy to see that T_1 is strictly pseudocontractive with constant $\zeta_1 = \frac{1}{4}$. Indeed, we observe that for all $x, y \in H$,

$$\|Tx - Ty\| \le \frac{5}{8}\|x-y\| + \frac{1}{4}\|\sin x - \sin y\| \le \|x-y\| + \frac{1}{4}\|(I-T)x - (I-T)y\|.$$

It is clear that $(\gamma_n + \delta_n)\zeta_1 = (\frac{1}{6} + \frac{1}{2}) \cdot \frac{1}{4} \le \frac{1}{6} = \gamma_n < (1 - 2\delta)\delta_n = (1 - 2 \cdot \frac{1}{4})\frac{1}{2} = \frac{1}{4}$ for all $n \ge 1$. In addition, it is clear that $\mathrm{Fix}(T_1) = \{0\}$ and $A0 = 0$ because the derivative $d(T_1 u)/du = \frac{5}{8} - \frac{1}{4}\cos u > 0$. Therefore, $\Omega = \mathrm{Fix}(T_1) \cap \mathrm{VI}(C, A) = \{0\} \ne \emptyset$. In this case, Algorithm 1 can be rewritten below:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \tau_n A w_n), \\ z_n = \frac{1}{n+1} f(x_n) + \frac{n}{n+1}(y_n - \tau_n(Ay_n - Aw_n)), \\ x_{n+1} = \frac{1}{3}x_n + \frac{1}{6}(y_n - \tau_n(Ay_n - Aw_n)) + \frac{1}{2}T_1 z_n \quad \forall n \ge 1, \end{cases}$$

where, for each $n \ge 1$, $\bar{\alpha}_n$ $(= \alpha_n)$ and τ_n are chosen as in Algorithm 1. Then, by Theorem 1, we know that $x_n \to 0 \in \Omega$ iff $x_n - x_{n+1} \to 0$ $(n \to \infty)$ and $\sup_{n \ge 1} |(I - f)x_n| < \infty$.

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References

1. Korpelevich, G.M. The extragradient method for finding saddle points and other problems. *Matecon* **1976**, *12*, 747–756.
2. Qin, X.; An, N.T. Smoothing algorithms for computing the projection onto a Minkowski sum of convex sets. *Comput. Optim. Appl.* **2019**, *74*, 821–850. [CrossRef]
3. Dehaish, B.A.B. Weak and strong convergence of algorithms for the sum of two accretive operators with applications. *J. Nonlinear Convex Anal.* **2015**, *16*, 1321–1336.
4. Ceng, L.C.; Shang, M. Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings. *Optimization* **2019**. [CrossRef]
5. Gibali, A.; Shehu, Y. An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces. *Optimization* **2019**, *68*, 13–32. [CrossRef]
6. Qin, X.; Yao, J.C. A viscosity iterative method for a split feasibility problem. *J. Nonlinear Convex Anal.* **2019**, *20*, 1497–1506.
7. Qin, X.; Yao, J.C. Projection splitting algorithms for nonself operators. *J. Nonlinear Convex Anal.* **2017**, *18*, 925–935.
8. An, N.T.; Nam, N.M.; Qin, X. Solving k-center problems involving sets based on optimization techniques. *J. Glob. Optim.* **2019**. [CrossRef]
9. Ansari, Q.H.; Babu, F. Regularization of proximal point algorithms in Hadamard manifolds. *J. Fixed Point Theory Appl.* **2019**, *21*, 25. [CrossRef]

10. Zhao, X.; Ng, K.F.; Li, C.; Yao, J.C. Linear regularity and linear convergence of projection-based methods for solving convex feasibility problems. *Appl. Math. Optim.* **2018**, *78*, 613–641. [CrossRef]
11. Qin, X.; Petrusel, A. CQ iterative algorithms for fixed points of nonexpansive mappings and split feasibility problems in Hilbert spaces. *J. Nonlinear Convex Anal.* **2018**, *19*, 157–165.
12. Yao, Y.; Liou, Y.C.; Kang, S.M. Approach to common elements of variational inequality problems and fixed point problems via a relaxed extragradient method. *Comput. Math. Appl.* **2010**, *59*, 3472–3480. [CrossRef]
13. Censor, Y.; Gibali, A.; Reich, S. The subgradient extragradient method for solving variational inequalities in Hilbert space. *J. Optim. Theory Appl.* **2011**, *148*, 318–335. [CrossRef] [PubMed]
14. Qin, X.; Yao, J.C. Weak convergence of a Mann-like algorithm for nonexpansive and accretive operators. *J. Inequal. Appl.* **2016**, *2016*, Article ID 232. [CrossRef]
15. Chang, S.S. Common zero point for a finite family of inclusion problems of accretive mappings in Banach spaces. *Optimization* **2018**, *67*, 1183–1196. [CrossRef]
16. Chang, S.S.; Wen, C.F.; Yao, J.C. Zero point problem of accretive operators in Banach spaces. *Bull. Malaysian Math. Sci. Soc.* **2019**, *42*, 105–118. [CrossRef]
17. Chidume, C.E.; Romanus, O.M.; Nnyaba, U.V. An iterative algorithm for solving split equilibrium problems and split equality variational inclusions for a class of nonexpansive-type maps. *Optimization* **2018**, *67*, 1949–1962. [CrossRef]
18. Ye, M. An improved projection method for solving generalized variational inequality problems. *Optimization* **2018**, *67*, 1523–1533. [CrossRef]
19. Takahashi, W.; Wen, C.F. The shrinking projection method for a finite family of demimetric mappings with variational inequality problems in a Hilbert space. *Fixed Point Theory* **2018**, *19*, 407–419. [CrossRef]
20. Bin Dehaish, B.A. A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces. *J. Inequal. Appl.* **2015**, *2015*, 51. [CrossRef]
21. Ceng, L.C.; Petrusel, A.; Yao, J.C.; Yao, Y. Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. *Fixed Point Theory* **2018**, *19*, 487–502. [CrossRef]
22. Ceng, L.C.; Petrusel, A.; Yao, J.C.; Yao, Y. Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. *Fixed Point Theory* **2019**, *20*, 113–134. [CrossRef]
23. Takahahsi, W.; Yao, J.C. The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces. *J. Nonlinear Convex Anal.* **2019**, *20*, 173–195.
24. Thong, D.V.; Hieu, D.V. Modified subgradient extragradient method for variational inequality problems. *Numer. Algorithms* **2018**, *79*, 597–610. [CrossRef]
25. Thong, D.V.; Hieu, D.V. Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. *Numer. Algorithms* **2019**, *80*, 1283–1307. [CrossRef]
26. Zhang, L.; Zhao, H.; Lv, Y. A modified inertial projection and contraction algorithms for quasi-variational inequalities. *Appl. Set-Valued Anal. Optim.* **2019**, *1*, 63–76.
27. Kraikaew, R.; Saejung, S. Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. *J. Optim. Theory Appl.* **2014**, *163*, 399–412. [CrossRef]
28. Marino, G.; Xu, H.K. Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. *J. Math. Anal. Appl.* **2007**, *329*, 336–346. [CrossRef]
29. Ceng, L.C.; Kong, Z.R.; Wen, C.F. On general systems of variational inequalities. *Comput. Math. Appl.* **2013**, *66*, 1514–1532. [CrossRef]
30. Cottle, R.W.; Yao, J.C. Pseudo-monotone complementarity problems in Hilbert space. *J. Optim. Theory Appl.* **1992**, *75*, 281–295. [CrossRef]
31. Xue, Z.; Zhou, H.; Cho, Y.J. Iterative solutions of nonlinear equations for m-accretive operators in Banach spaces. *J. Nonlinear Convex Anal.* **2000**, *1*, 313–320.