

A Note on Distributions in the Second Chaos

Pauliina Ilmonen ¹ and Lauri Viitasaari ^{2,*} 

¹ Department of Mathematics and Systems Analysis, Aalto University School of Science, 00076 Aalto, Finland; pauliina.ilmonen@aalto.fi

² Department of Information and Service Management, Aalto University School of Business, 00076 Aalto, Finland

* Correspondence: lauri.viitasaari@iki.fi

Received: 29 October 2019; Accepted: 3 December 2019; Published: 6 December 2019



Abstract: In this article we study basic properties of random variables X , and their associated distributions, in the second chaos, meaning that X has a representation $X = \sum_{k \geq 1} \lambda_k (\xi_k^2 - 1)$, where $\xi_k \sim N(0, 1)$ are independent. We compute the Lévy-Khintchine representations which we then use to study the smoothness of each density function. In particular, we prove the existence of a smooth density with asymptotically vanishing derivatives whenever $\lambda_k \neq 0$ infinitely often. Our work generalises some known results presented in the literature.

Keywords: second chaos; infinitely divisible distribution; smooth density; symmetrised distributions

MSC: 60E07; 60E10

1. Introduction

In this article we study random variables (and their corresponding distributions) in the second chaos, meaning that they are of the form

$$X = \sum_{k=1}^{\infty} \lambda_k (\xi_k^2 - 1), \quad (1)$$

where ξ_k represents independent standard normal random variables, and $\lambda = (\lambda_k)_{k \geq 1}$ is a general sequence of coefficients $\lambda \in l^2(\mathbb{Z}_+)$. Such random variables include, among others, all centred χ^2 -random variables with arbitrary degrees of freedom. In this case the sum in (1) is finite. In addition, the class includes the so-called *Rosenblatt distribution* as a particular example of the case where $\lambda_k \neq 0$ for all k . These special cases arise naturally in different applications. Indeed, χ^2 -distributions are well-known and widely applied in statistics. Similarly, the Rosenblatt distribution arises naturally in non-central limit theorems for long memory stationary sequences. Consequently, it appears also as the asymptotic distribution of certain statistical estimators; see, e.g., [1–3]. Due to its special importance, the Rosenblatt distribution has received a lot of attention in the literature, and is still under active research. To simply name a few articles on the topic, we must mention [3–6]. For an overview on the Rosenblatt distribution we refer the reader to [7], and for details on long-memory processes we refer the reader to the monograph [8]. For details on self-similar processes, including the Rosenblatt process (and distribution), see monographs [9,10].

There exists a large literature on the convergence and on the absolute continuity of the densities of random variables in a fixed chaos (or for random variables that can be given as a finite sum of variables in different chaoses). The absolute continuity and the convergence in total variation in Wiener chaos has been studied, e.g., in [11–13] to simply name a few. For an overview on the topic, see [14] and the references therein. The convergence in the second chaos has been studied in detail in [15].

On a related study, we also emphasise [16], which is close to our work. In [16] the authors studied the Rosenblatt distribution and proved, by using the special form of the sequence $(\lambda_k)_{k \geq 1}$, that the Rosenblatt distribution is infinitely divisible and has an infinitely differentiable density with respect to the Lebesgue measure (for details on infinitely divisible distributions; see [17,18]). See also [19], for a generalisation to the multidimensional setting.

In the present article, we consider random variables of the form (1) with general coefficients $(\lambda_k)_{k \geq 1} \in l^2(\mathbb{Z}_+)$. Throughout, we consider general coefficient sequences, while the analysis of some particular cases, such as the χ^2 -distribution or the Rosenblatt distribution, applies the special structure of the sequence $(\lambda_k)_{k \geq 1}$. We complement the work [16] by proving that each such random variable is infinitely divisible, and we compute the characteristic function in terms of Lévy-Khintchine representation. As a consequence, we obtain that each such distribution admits a density with respect to a Lebesgue measure, a result that is not surprising and could alternatively be proven by the means of Malliavin calculus. To our knowledge, no proof of this result, apart from certain specific cases, such as the case of the Rosenblatt distribution [16], has been presented in the literature. In this article, we provide a self-contained proof that does not require any knowledge on Malliavin calculus or Gaussian analysis, as our proof is based purely on known facts about infinitely divisible distributions. Moreover, we study the smoothness of the density and show that the density is infinitely differentiable with vanishing derivatives whenever $\lambda_k \neq 0$ for infinitely many indices k . Finally, we consider the symmetrised version of the distribution, and compute the associated Lévy-Khintchine representation. In this case, we also observe the existence of an infinitely-many-times differentiable density function with asymptotically vanishing derivatives whenever $\lambda_k \neq 0$ for infinitely many k s. Thus, as random variables of the form (1) arise naturally in different applications of statistics and in limit theorems, the results of this paper could be useful by providing more information on such random variables.

The rest of the paper is divided into two sections. In Section 2 we introduce and discuss our main results, while all the proofs are postponed to Section 3.

2. Distributions in the Second Chaos

We study random variables of the form

$$X = \sum_{k=1}^{\infty} \lambda_k (\xi_k^2 - 1), \quad (2)$$

where $\xi_k \sim N(0, 1)$ are independent. It is known [4,20] that random variables living in the second Wiener chaos (see, e.g., [21,22] for details on Wiener chaoses) admit such representation, with the sequence $(\lambda_k)_{k \geq 1}$ being the singular value sequence of a certain integral operator. Thus, we say that such random variables belong to the second chaos. For the coefficient sequence we assume that $\lambda = (\lambda_k)_{k \geq 1} \in l^2(\mathbb{Z}_+)$; i.e.,

$$\sum_{k=1}^{\infty} \lambda_k^2 < \infty. \quad (3)$$

This implies that the series in (2) converges in L^2 , and, by [18] (Proposition 27.17), almost surely. Throughout the article, we use the asymptotic notation $f \sim g$ as $x \rightarrow a \in [-\infty, \infty]$, meaning that $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \text{a constant}$. We denote by c and C generic unimportant constants that may vary from time to time. We also use the standard notation C^∞ for the space of functions that are infinitely many times differentiable.

Example 1. Let $\lambda_k = 1$ for $k = 1, \dots, n$ and $\lambda_k = 0$ for $k \geq n + 1$. Then, X is just a centred, chi-squared distribution with n degrees of freedom.

Example 2. Let $H \in (\frac{1}{2}, 1)$ be fixed and let $\lambda_k := \lambda_k^H$ be the eigenvalue sequence of the integral operator $A : L^2(|y|^{-H}dy) \mapsto L^2(|y|^{-H}dy)$ given by

$$(Af)(x) = \int_{\mathbb{R}} \frac{\sum_{j=1}^n (e^{it_j(x-y)} - 1)}{i(x-y)} |y|^{-H} f(y) dy.$$

Now the distribution of X is the Rosenblatt distribution, and the free parameter H is called the Hurst index. It can be proven that in this case, we have $\lambda_k > 0$ for all k s. Moreover, by [16] (Theorem 3.2) we have the asymptotic formula $\lambda_k \sim k^{-H}$.

Since $(\zeta_k)_{k \geq 1}$ are independent, we may order the sequence $(\lambda_k)_{k \geq 1}$ freely. Throughout, we assume that the sequence is ordered to be decreasing in the absolute value; i.e.,

$$|\lambda_1| \geq |\lambda_2| \geq \dots$$

Recall that a random variable X is infinitely divisible, if for every $n \geq 1$, there exists a random variable Y such that

$$Y_1 + \dots + Y_n,$$

where Y_i are independent copies of Y , distributed as X . In this case, the characteristic function $\phi_X(\theta) = \mathbb{E}e^{i\theta X}$ has a representation

$$\phi_X(\theta) = \exp \left(ai\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x \mathbf{1}_{|x| < 1} \right) \Pi(dx) \right),$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$, and $\Pi(dx)$ are sigma-finite measures satisfying

$$\Pi(\{0\}) = 0, \quad \int_{\mathbb{R}} (1 \wedge x^2) \Pi(dx) < \infty.$$

The measure $\Pi(dx)$ is called the Lévy-measure of X , and the triplet (a, σ^2, Π) is called the Lévy-Khintchine triplet. For details, we refer to monographs [17,18].

The following result shows that each X of the form (2) is an infinitely divisible random variable.

Theorem 1. Let X be of the form (2). Then X is infinitely divisible with a characteristic function

$$\phi_X(\theta) = \exp \left(\int_{-\infty}^{\infty} \left(e^{iu\theta} - iu\theta - 1 \right) \frac{v(u)}{2|u|} du \right), \quad (4)$$

where

$$v(u) = \sum_{k=1}^{\infty} e^{-\frac{|u|}{2|\lambda_k|}} [\mathbf{1}_{u, \lambda_k < 0} + \mathbf{1}_{u, \lambda_k > 0}]. \quad (5)$$

Our second main result concerns the existence and smoothness of the densities.

Theorem 2. Let X be of the form (2). Then the density function f_X of X exists. Moreover, if $\lambda_k \neq 0$ for infinitely many indices k , then $f_X \in C^\infty$ with all the derivatives vanishing at infinity.

Remark 1. Our proof is based on general existence and smoothness results for the density of infinitely divisible random variables. Consequently, we could also consider the case where the sum in (2) is finite and obtain the existence of densities that are smooth up to a certain order. However, the infinite sum in (2) is more interesting

for us, while the finite sum is relatively simple to study case by case. Indeed, in the finite case, each random variable $\lambda_k(\xi_k^2 - 1)$ with $\lambda_k > 0$ admits a density

$$f(y) = \frac{1}{2\lambda_k} \left(\frac{y}{\lambda_k} + 1 \right)^{-\frac{1}{2}} e^{-\frac{y+\lambda_k}{2\lambda_k}}.$$

The case $\lambda_k < 0$ can be treated by symmetry. Thus, the exact density of a linear combination of such random variables can be computed directly by using the convolution theorem; see Lemma 1. In particular, with $\lambda_k = 1$ for $k = 1, 2, \dots, n$, we obtain the well-known expression for the density of the χ^2 -distribution.

Consider next the symmetrised version of X given by

$$Y = X - \tilde{X},$$

where \tilde{X} is an independent copy of X . Such object can arise, e.g., as the limiting distribution of the difference of two long range, dependent stationary sequences.

By straightforward manipulations, we note that Y can be represented as

$$Y = \sum_{k=1}^{\infty} |\lambda_k|(\xi_k^2 - 1) - \sum_{k=1}^{\infty} |\lambda_k|(\tilde{\xi}_k^2 - 1), \quad (6)$$

where $\tilde{\xi}_k$ are i.i.d. standard normals, independent of ξ_k . Consequently, we obtain the following result, similar to Theorems 1 and 2.

Theorem 3. Let Y be of the form (6). Then, Y is infinitely divisible with a characteristic function

$$\phi_Y(\theta) = \exp \left(\int_0^{\infty} (\cos(u\theta) - 1) \frac{v(u)}{u} du \right), \quad (7)$$

where

$$v(u) = \sum_{k=1}^{\infty} e^{-\frac{u}{|\lambda_k|}}. \quad (8)$$

Moreover, Y has a density f_Y , and the density $f_Y \in C^{\infty}$ with all the derivatives vanishing at infinity provided that $|\lambda_k| > 0$ for all $k = 1, 2, \dots$

3. Proofs

We begin with some auxiliary lemmas. The first one is rather standard and can be found in basic textbooks.

Lemma 1. Suppose that the random variables X and Y are independent and have densities f_X and f_Y , respectively. Then, the random variable $Z = X + Y$ has a density f_Z given by the convolution

$$f_Z(x) = (f_X * f_Y)(x) = \int_{-\infty}^{\infty} f_X(x - y) f_Y(y) dy. \quad (9)$$

In particular, if f_X is infinitely differentiable with bounded derivatives, then f_Z is infinitely differentiable with bounded derivatives.

Lemma 2. Let $(\lambda_k)_{k \geq 1} \in l^2(\mathbb{Z}_+)$ and let $v(u)$ be given by (5). Then,

$$\int_{-\infty}^{\infty} |u| v(u) du < \infty. \quad (10)$$

Proof. From (5) we get

$$v(u) \leq 2 \sum_{k=1}^{\infty} e^{-\frac{|u|}{2|\lambda_k|}}.$$

By change of variable $u = |\lambda_k|v$, we get that

$$\int_0^{\infty} u e^{-\frac{u}{2|\lambda_k|}} du = \lambda_k^2 \int_0^{\infty} v e^{-\frac{v}{2}} dv.$$

Thus we obtain, thanks to symmetry, that

$$\begin{aligned} & \int_{-\infty}^{\infty} |u| v(u) du \\ & \leq 4 \sum_{k=1}^{\infty} \int_0^{\infty} u e^{-\frac{u}{2|\lambda_k|}} du \\ & = 4 \int_0^{\infty} u e^{-\frac{u}{2}} du \sum_{k=1}^{\infty} |\lambda_k|^2 \\ & < \infty. \end{aligned}$$

This completes the proof. \square

Lemma 3. Let $(\lambda_k)_{k \geq 1} \in l^2(\mathbb{Z}_+)$ and let $v(u)$ be given by (5). Then, $v(u)$ is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$. Moreover, $v(u)$ is continuous on $\mathbb{R} \setminus \{0\}$.

Proof. By symmetry, it suffices to consider only the interval $(0, \infty)$. Now the fact that $v(u)$ is monotonic is evident from (5). In order to prove continuity, we define a sequence of continuous function by

$$v_n(u) = \sum_{k=1}^n e^{-\frac{u}{2\lambda_k}} \mathbf{1}_{\lambda_k > 0}.$$

Now $v_n(u)$ is an increasing sequence of continuous functions converging to $v(u)$, and thus, Dini's theorem implies that the convergence is uniform. This further implies that $\mu(u)$ is continuous on $(0, \infty)$, which completes the proof. \square

We are now in position to prove our first main result.

Proof of Theorem 1. Let X_n be the approximation of X given by

$$X_n = \sum_{k=1}^n \lambda_k (\xi_k^2 - 1).$$

Since each $\lambda_k (\xi_k^2 - 1)$ is an infinitely divisible random variable, it follows that X_n is infinitely divisible as a finite sum of independent infinitely divisible random variables. Moreover, since $X_n \rightarrow X$ in L^2 , it also converges weakly. By [18] (Lemma 7.8), any weak limit μ of infinitely divisible probability measures μ_n remains infinitely divisible. Consequently, X is infinitely divisible. Proving (4) yet remains undone. For this, we begin by recalling the explicit formula for the characteristic function of the χ^2 -distribution and its Lévy-Khintchine representation (see [23] and Example 1.3.22, [17]) leading to

$$\phi_{\chi^2(1)}(\theta\lambda) = \frac{1}{(1 - 2i\theta\lambda)^{1/2}} = \exp \left(\int_0^{\infty} (e^{iv\theta\lambda} - 1) \frac{e^{-\frac{v}{2}}}{2v} dv \right).$$

By independence, we get

$$\begin{aligned}\phi_{X_n}(\theta) &= \prod_{k=1}^n \frac{e^{-i\theta\lambda_k}}{(1 - 2i\theta\lambda_k)^{1/2}} \\ &= \prod_{k=1}^n \exp\left(-i\theta\lambda_k + \int_0^\infty (e^{iv\theta\lambda_k} - 1) \frac{e^{-\frac{v}{2}}}{2v} dv\right) \\ &= \prod_{k=1}^n \exp\left(\int_0^\infty (e^{iv\theta\lambda_k} - iv\theta\lambda_k - 1) \frac{e^{-\frac{v}{2}}}{2v} dv\right) \\ &= \prod_{k=1}^n \exp\left(\int_{-\infty}^\infty (e^{iu\theta} - iu\theta - 1) \frac{e^{-\frac{|u|}{2|\lambda_k|}}}{2|u|} [\mathbf{1}_{u,\lambda_k < 0} + \mathbf{1}_{u,\lambda_k > 0}] du\right),\end{aligned}$$

where we have used the fact

$$\int_0^\infty e^{-\frac{v}{2}} dv = 2$$

and the change of variable $u = \lambda_k v$. Thus, by using $\prod_{k=1}^n e^{x_k} = e^{\sum_{k=1}^n x_k}$, we obtain

$$\phi_{X_n}(\theta) = \exp\left(\int_{-\infty}^\infty (e^{iu\theta} - iu\theta - 1) \frac{v_n(u)}{2|u|} du\right),$$

where

$$v_n(u) = \sum_{k=1}^n e^{-\frac{|u|}{2|\lambda_k|}} [\mathbf{1}_{u,\lambda_k < 0} + \mathbf{1}_{u,\lambda_k > 0}].$$

If $\lambda_k \neq 0$ for only finitely many indices we have arrived to (4). For the general case, it suffices to argue why we can pass to the limit. For this we use the inequality, valid for all real numbers u and θ ,

$$|e^{iu\theta} - iu\theta - 1| \leq \frac{u^2\theta^2}{2}.$$

Using this together with Lemma 2, we obtain next:

$$\int_{-\infty}^\infty |e^{iu\theta} - iu\theta - 1| \frac{v_n(u)}{2|u|} du \leq \frac{\theta^2}{4} \int_{-\infty}^\infty |u| v_n(u) du \leq \frac{\theta^2}{4} \int_{-\infty}^\infty |u| v(u) du < \infty.$$

Thus, we may apply the dominated convergence theorem which shows that (4) is valid. This completes the proof. \square

Proof of Theorem 2. By Theorem 1, we have

$$\phi_X(\theta) = \exp\left(\int_{-\infty}^\infty (e^{iu\theta} - iu\theta - 1) \frac{v(u)}{2|u|} du\right),$$

where $v(u)$ is given by (5). Let us verify that ([18], Theorem 28.4) is applicable. The positivity of $v(u)$ is evident from (5), and by Lemma 3 we have that $v(u)$ is continuous on $\mathbb{R} \setminus \{0\}$, increasing on $(-\infty, 0)$, and decreasing on $(0, \infty)$. Moreover, we have

$$v(0+) + v(0-) = \infty.$$

Thus Theorem 28.4 of [18] is applicable and the claim follows. This completes the proof. \square

Proof of Theorem 3. Using the representation (6) and independence, we get

$$\phi_Y(\theta) = \phi_Z(\theta)\phi_Z(-\theta),$$

where

$$Z = \sum_{k=1}^{\infty} |\lambda_k| (\xi_k^2 - 1).$$

Let also

$$Z_n = \sum_{k=1}^n |\lambda_k| (\xi_k^2 - 1)$$

be the approximation of Z and Y_n be the approximation of Y given by

$$Y_n = Z_n - \tilde{Z}_n,$$

where \tilde{Z}_n is an independent copy of Z_n . By Theorem 1 we get

$$\phi_{Z_n}(\theta) = \exp \left(\int_0^{\infty} (e^{iu\theta} - iu\theta - 1) \frac{v_n(u)}{2u} du \right),$$

where

$$v_n(u) = \sum_{k=1}^n \exp \left(-\frac{u}{2|\lambda_k|} \right).$$

By recalling the identity

$$e^{iu\theta} - e^{-iu\theta} = 2i \sin(u\theta)$$

we directly get that

$$\phi_{Y_n}(\theta) = \exp \left(\int_0^{\infty} (\cos(u\theta) - 1) \frac{v_n(u)}{u} du \right).$$

From this we observe (7) immediately, provided that $\lambda_k \neq 0$ only for finitely many k . For the general case, we apply the inequality

$$|\cos(u\theta) - 1| \leq C\theta^2 u^2.$$

This gives us

$$\int_0^{\infty} |\cos(u\theta) - 1| \frac{v_n(u)}{u} du \leq C \int_0^{\infty} uv(u) du,$$

which is finite by Lemma 2. Thus we may apply the dominated convergence theorem in order to pass to the limit, which proves (7). Finally, we can conclude the proof by noting that, by Theorem 2, the random variable Z has a density f_Z that is infinitely many times differentiable with all the derivatives vanishing at infinity. Hence, thanks to Lemma 1, $Y = Z - \tilde{Z}$ also has a density with the same properties. This finishes the proof. \square

Remark 2. We remark that in general the approximation argument used in the above proof cannot be avoided. Indeed, the underlying reason is that one cannot compute

$$i \int_0^{\infty} \sin(u\theta) \frac{v(u)}{u} du - i \int_0^{\infty} \sin(u\theta) \frac{v(u)}{u} du = 0$$

directly, since

$$\int_0^{\infty} \sin(u\theta) \frac{v(u)}{u} du < \infty$$

only if $\int_0^1 v(u) du < \infty$. This is not the case for all $(\lambda_k)_{k \geq 1} \in l^2(\mathbb{Z}_+)$.

Author Contributions: P.I. and L.V. contributed equally to the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

- Dehling, H.; Taqqu, M. The empirical process of some long-range dependent sequences with an application to U-statistics. *Ann. Stat.* **1989**, *17*, 1767–1783. [[CrossRef](#)]
- Dehling, H.; Rooch, A.; Taqqu, M. Non-parametric change-point tests for long-range dependent data. *Scand. J. Stat.* **2013**, *40*, 153–173. [[CrossRef](#)]
- Tudor, C.A. Analysis of the Rosenblatt process. *ESAIM Probab. Stat.* **2008**, *12*, 230–257. [[CrossRef](#)]
- Dobrushin, R.L.; Major, P. Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete* **1979**, *50*, 27–52. [[CrossRef](#)]
- Maejima, M.; Tudor, C.A. Selfsimilar processes with stationary increments in the second Wiener chaos. *Probab. Math. Stat.* **2012**, *32*, 167–186.
- Taqqu, M.S. Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Probab. Theory Rel. Fields* **1975**, *31*, 287–302.
- Taqqu, M.S. The Rosenblatt process. In *Selected Works of Murray Rosenblatt*; Davis, R., Lii, K.S., Politis, D., Eds.; Springer: New York, NY, USA, 2011; pp. 159–179.
- Beran, J.; Feng, Y.; Ghosh, S.; Kulik, R. *Long-Memory Processes: Probabilistic Properties and Statistical Methods*; Springer: Berlin/Heidelberg, Germany, 2013.
- Tudor, C.A. *Analysis of Variations for Self-similar Processes: A Stochastic Calculus Approach*; Springer International Publishing: Manhattan, KS, USA, 2013.
- Embrechts, P.; Maejima, M. *Selfsimilar Processes*; Princeton Series in Applied Mathematics; Princeton University Press: Princeton, NJ, USA, 2002.
- Kusuoka, S. On the absolute continuity of the law of a system of multiple Wiener integral. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **1983**, *30*, 191–197.
- Nourdin, I.; Nualart, D.; Poly, G. Absolute continuity and convergence of densities for random vectors on Wiener chaos. *Electron. J. Probab.* **2013**, *18*, 22. [[CrossRef](#)]
- Nourdin, I.; Poly, G. Convergence in total variation on Wiener chaos. *Stoch. Proc. Appl.* **2013**, *123*, 651–674. [[CrossRef](#)]
- Nualart, D. Normal approximation on a finite Wiener chaos. In *Stochastic Analysis and Applications 2014. Springer Proceedings in Mathematics & Statistics*; Crisan, D., Hambly, B., Zariphopoulou, T., Eds.; Springer International Publishing: Manhattan, KS, USA, 2014; Volume 100.
- Nourdin, I.; Poly, G. Convergence in law in the second Wiener/Wigner chaos. *Electron. Commun. Probab.* **2012**, *17*, 36.
- Veillette, M.S.; Taqqu, M.S. Properties and numerical evaluation of the Rosenblatt distribution. *Bernoulli* **2013**, *19*, 982–1005. [[CrossRef](#)]
- Applebaum, D. *Lévy Processes and Stochastic Calculus*; Cambridge Studies in Advanced Mathematics; Cambridge University Press: Cambridge, UK, 2004.
- Sato, K. *Lévy Processes and Infinitely Divisible Distributions*; Cambridge Studies in Advanced Mathematics; Cambridge University Press: Cambridge, UK, 1999.
- Leonenko, N.N.; Ruiz-Medina, M.D.; Taqqu, M.S. Rosenblatt distribution subordinated to Gaussian random fields with long-range dependence. *Stoch. Anal. Appl.* **2017**, *35*, 144–177. [[CrossRef](#)]
- Nourdin, I.; Peccati, G. *Normal Approximations Using Malliavin Calculus: From Stein's Method to Universality*; Cambridge Tracts in Mathematics; Cambridge University Press: Cambridge, UK, 2012.
- Janson, S. *Gaussian Hilbert Spaces*; Cambridge Tracts in Mathematics; Cambridge University Press: Cambridge, UK, 1997; Volume 129.
- Nualart, D. *The Malliavin Calculus and Related Topics*, 2nd ed.; Probability and Its Applications; Springer: Berlin, Germany, 2006.
- Simon, M.K. *Probability Distributions Involving Gaussian Random Variables*; Springer: New York, NY, USA, 2002.

