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Numerical Method for Dirichlet Problem with Degeneration of the Solution on the Entire Boundary

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Abstract: The finite element method (FEM) with a special graded mesh is constructed for the Dirichlet boundary value problem with degeneration of the solution on the entire boundary of the two-dimensional domain. A comparative numerical analysis is performed for the proposed method and the classical finite element method for a set of model problems in symmetric domain. Experimental confirmation of theoretical estimates of accuracy is obtained and conclusions are made.

Keywords: boundary value problems with degeneration of the solution on entire boundary of the domain; the method of finite elements; special graded mesh

1. Introduction

As is known, classical solutions for boundary value problems for elliptic equations with discontinuous coefficients do not exist. Therefore, the notion of a generalized (weak) solution was introduced. Based on this definition and on the Galerkin method, numerous numerical methods were developed for finding approximate solutions of such problems. However, these methods for boundary value problems with singularity lose accuracy, which depends on the smoothness of the solution of the original differential problem (see, for example, [1,2]). The singularity of the solution of the boundary value problem can be caused by the presence of re-entrant corners on the domain boundary, by the degeneration of the coefficients and right-hand sides of equation and boundary conditions, or by the internal properties of the solution (see, for example, [3–9]). For boundary problems with a singularity, we proposed to determine an R_ν -generalized solution. The existence, uniqueness and differential properties of this kind of solution in the weighted Sobolev spaces were studied in [10–15]. Based the R_ν -generalized solution, a weighted finite element method was developed for boundary value problems for elliptic equations in two-dimensional domain with a singularity in a finite set of boundary points [16–20]. A weighted FEM was constructed and studied for the Lamé system in a domain with re-entrant corners [21,22]. To find an approximate solution of Maxwell's equations in an L-shaped domain, a weighted edge-based finite element method was proposed in [23,24]. In [25,26] a weight analogue of the condition of Ladyzhenskaya-Babuška-Brezzi was proved, a numerical method was developed for the Stokes and Oseen problems in domains with corner singularity. The main feature of all the developed methods is the convergence of the approximate solution to the exact one with the rate $O(h)$ in the norms of the Sobolev and Monk weighted spaces, regardless of the reasons causing the solution singularity and its value.

In this paper we consider the Dirichlet problem for an elliptic equation with degeneration of the solution on the entire boundary of a two-dimensional domain. In [27] a finite element method was constructed for this problem and the convergence of this method was established. The paper [28] singles out the weighted subspace of functions for which the approximate solutions converge to

an exact solution with a speed $O(h)$ on a mesh with a special compression of nodes close to the boundary (see [29]). The compression parameters depend on the constructed subspace. Our method of constructing mesh with a special compression of nodes differs from the methods proposed by other authors (see, for example, [30–32]).

Here we test model problems with singularities in a symmetric domain. We carry out a comparative numerical analysis of finite element methods on quasi-uniform meshes and meshes with a special compression of nodes close to the boundary. We obtain experimental confirmation of theoretical estimates and demonstrate the advantage of the proposed method over the classical finite element method. By analogy with [22], we found that it is impossible to use FEM with a strong thickening of mesh, and introduction of an R-generalized solution is required. The existence and uniqueness of the R-generalized solution for this problem were proved in [33].

2. Problem Formulation

Let Ω be a bounded convex two-dimensional domain with twice differentiable boundary $\partial\Omega$, and let $\bar{\Omega}$ be the closure of Ω , i.e., $\bar{\Omega} = \Omega \cup \partial\Omega$; $x = (x_1, x_2)$ and $dx = dx_1 dx_2$.

We assume that a positive function $\rho(x)$ belongs to the space $C^{(2)}(\Omega)$ and coincides in the boundary strip of width $d > 0$ with the distance from x ($x \in \Omega$) to the boundary $\partial\Omega$.

We introduce the weighted Sobolev space $W_{2,\eta}^s(\Omega)$ with the norm

$$\|v\|_{W_{2,\eta}^s(\Omega)} = \|v\|_{L_2(\Omega)} + \sum_{\substack{m_1, m_2=0 \\ |m|=s}} \left\| \rho^{-\eta} \frac{\partial^{|m|} v}{\partial x_1^{m_1} \partial x_2^{m_2}} \right\|_{L_2(\Omega)},$$

where η is a real number satisfying the inequalities $\frac{1}{2} - s < \eta < \frac{1}{2}$; $s = 1, 2$; $m = (m_1, m_2)$, $|m| = m_1 + m_2$, m_1, m_2 are integer nonnegative numbers.

Let

$$\dot{W}_{2,\eta}^s(\Omega) = \{v: v \in W_{2,\eta}^s(\Omega), \quad v|_{\partial\Omega} = 0\}.$$

We denote by $L_{2,-1-\eta}(\Omega)$ the space of functions f with the norm

$$\|f\|_{L_{2,-1-\eta}(\Omega)} = \left(\int_{\Omega} |\rho^{1+\eta} f|^2 dx \right)^{1/2}.$$

We consider the first boundary value problem for a second order elliptic equation

$$\begin{aligned} - \sum_{k=1}^2 \frac{\partial}{\partial x_k} \left(a_{kk}(x) \frac{\partial u}{\partial x_k} \right) + a(x)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1)$$

We suppose that the input data of Equation (1) satisfy the conditions:

(a)

$$f \in L_{2,-1-\alpha}(\Omega), \quad (2)$$

(b) $a_{kk}(x)$ ($k = 1, 2$) are differentiable functions on Ω , such that the inequalities

$$|a_{kk}(x)| \leq C_1 \rho^{-2\alpha}(x), \quad (3)$$

$$\left| \frac{\partial a_{kk}(x)}{\partial x_1} \right|, \left| \frac{\partial a_{kk}(x)}{\partial x_2} \right| \leq C_2 \rho^{-2\alpha-1}(x), \quad (4)$$

$$\sum_{k=1}^2 a_{kk}(x) \xi_k^2 \geq C_3 \rho^{-2\alpha}(x) \sum_{k=1}^2 \xi_k^2, \quad x \in \Omega, \quad C_3 > 0, \quad (5)$$

- hold,
(c) the function $a(x)$ satisfies the inequalities

$$0 < a(x) \leq C_4 \rho^{-2\alpha-2}(x), \quad x \in \Omega. \quad (6)$$

Here C_i , ($i = 1, \dots, 4$) are constants independent of x , ξ_1 and ξ_2 are any real parameters, $\alpha \in (-\frac{1}{2}, \frac{1}{2})$.

Remark 1. If Conditions (2)–(6) are fulfilled for the input data, Equation (1) is called a Dirichlet boundary value problem for an elliptic equation with degeneration of the solution on the entire boundary of a two-dimensional domain. Such problems are encountered in gas dynamics, electromagnetism and other subject areas of mathematical physics. The differential properties of solutions of problems with degeneracy on the entire boundary were studied, for the first time, in [7–9].

We introduce the bilinear and linear forms

$$E(u, w) = \int_{\Omega} \left(\sum_{k=1}^2 a_{kk}(x) \frac{\partial u}{\partial x_k} \frac{\partial w}{\partial x_k} + a(x) u w \right) dx, \quad (f, w) = \int_{\Omega} f w dx.$$

A function u in $\dot{W}_{2,\alpha}^1(\Omega)$ is called a generalized solution of the first boundary value Equation (1) if for any w in $\dot{W}_{2,\alpha}^1(\Omega)$ the identity

$$E(u, w) = (f, w)$$

holds.

We note that if Conditions (2)–(6) are satisfied, then there exists a unique generalized solution of the Equation (1) in the space $\dot{W}_{2,\alpha}^1(\Omega)$ (see Theorem 1 from [8]). In addition $u \in \dot{W}_{2,\alpha-1}^2(\Omega)$ (see Theorem 1 from [9]). Moreover, if the function $f \in L_{2,-1-\alpha+\lambda}(\Omega) \subset L_{2,-1-\alpha}(\Omega)$ ($-\frac{1}{2} < \alpha < \alpha + \lambda < \frac{1}{2}$) and the parameter λ is sufficiently small, then the generalized solution u belongs to the space $\dot{W}_{2,\alpha+\lambda-1}^2(\Omega)$ which is a subspace of $\dot{W}_{2,\alpha-1}^2(\Omega)$ (see [28]).

Remark 2. Knowing that the solution belongs to the space $\dot{W}_{2,\alpha+\lambda-1}^2(\Omega)$ allows us to construct a finite element method for finding a generalized solution for the Dirichlet problem with the degeneration of the solution on the entire boundary of the domain with a convergence speed $O(h)$ in the norm $W_{2,\alpha}^1(\Omega)$.

3. The Scheme of the Finite Element Method

We construct a scheme of the finite element method for finding an approximate generalized solution of the first boundary value Equation (1). We perform a triangulation of the domain Ω (see, for example, Figure 1).

We draw the curves Γ_j , $j = 0, \dots, n$, at distance $b(\frac{j}{n})^r$, $j = 0, \dots, n$, to the boundary $\partial\Omega$. Here r is the exponent of compression and $r > 1$; $0 < b < \frac{\delta_{\Omega}}{2} \leq d$, δ_{Ω} is the diameter of the circle inscribed in Ω . In this case the line Γ_n divides the domain Ω into two subdomains Ω_1 and Ω_2 . The subdomain Ω_1 is the outer domain on the boundary strip of width b , Ω_2 is the inner domain. On each curve Γ_j , $j = 0, \dots, n$, ($\Gamma_0 = \partial\Omega$, $\Gamma_n = \partial\Omega_1$) we fix M_j equidistant points, which we call the nodes. Here $M_j = \left\lceil l_j / \left(b \left((\frac{j}{n})^r - (\frac{j-1}{n})^r \right) \right) \right\rceil + 1$, $j = 1, \dots, n$, l_j is the length of the curve Γ_j ($[x]$ denotes the integer part of x) and $M_0 = 2M_1$. All nodes on the curve Γ_j , $j = 0, \dots, n$, are connected by the broken line. Then, we connect each node on the curve Γ_{j-1} , $j = 1, \dots, n$, with closest nodes on the curve Γ_j . As a

result, the subdomain Ω_1 is divided into triangles with the compression of nodes to the boundary $\partial\Omega$. The union of all triangles with vertices on Γ_{j-1} and Γ_j is a layer Q_j^h . (In Figure 1 the subdomain Ω_1 is divided into the layers $Q_1^h, \dots, Q_4^h$, $\Gamma_4 = \partial\Omega_1$). The parameter h denotes the greatest in length of the sides of the triangles in Q_n^h .

Figure 1. Triangulation of domain Ω .

The subdomain Ω_2 is divided quasi-uniformly into a finite number of the triangles. The sides of these triangles can not be greater than h . Moreover, the vertices of the triangles on the boundary $\partial\Omega_1$ belong to the set of vertices of the triangles in $\partial\Omega_2$.

The algorithm and code description of this triangulation are given in [29].

Let Ω^h be the union of closed triangles $\{K\} = \{K_1, \dots, K_{N_h}\}$, and K_j , $j = 1, \dots, N_h$, is the finite element. The vertices G_i , $i = 1, \dots, N$, of these triangles are the nodes of the triangulation. We denote by N' the number of the internal nodes. To each node G_i , $i = 1, \dots, N'$, we assign the function $\varphi_i(x)$ which is equal to 1 at the point G_i and zero at all other nodes, and $\varphi_i(x)$ is linear on each triangle K . We denote by V^h the linear span $\{\varphi_i\}_{i=1}^{N'}$. Next, we associate the following discrete problem with the constructed finite-dimensional space $V^h \subset \dot{W}_{2,\alpha}^1(\Omega)$: find the function $u_h \in V^h$ satisfying the equality

$$E(u_h, w_h) = (f, w_h)$$

for any function $w_h \in V^h$.

An approximate (finite element) generalized solution will be found in the form

$$u_h(x) = \sum_{i=1}^{N'} a_i \varphi_i(x),$$

where $a_i = u_h(G_i)$. We assume that $u_h(x) = 0$, $x \in \Omega \setminus \Omega^h$.

The coefficients a_i are defined from system of equations

$$E(u_h, \varphi_i) = (f, \varphi_i), \quad i = 1, \dots, N'$$

or

$$\hat{A}a = F,$$

where

$$a = (a_1, \dots, a_{N'})^T, \quad F = (F_1, \dots, F_{N'})^T, \quad \hat{A} = (A_{ij}),$$

$$A_{ij} = E(\varphi_i, \varphi_j), \quad F_i = (f, \varphi_i), \quad i, j = 1, \dots, N'.$$

It is obvious that the approximate generalized solution of Problems (1)–(6) by the finite element method exists and is unique.

For the performed triangulation of the domain Ω with the exponent of the compression of the mesh $r = \frac{1}{\lambda}$ and for functions in the space $\hat{W}_{2,\alpha+\lambda-1}^2(\Omega)$ we have convergence estimates:

$$\|u - u_h\|_{W_{2,\alpha}^1(\Omega)} \leq C_5 h \|f \rho^{1+\alpha-\lambda}\|_{L_2(\Omega)},$$

$$\|u - u_h\|_{L_2(\Omega)} \leq C_6 h^2 \|f \rho^{1+\alpha-\lambda}\|_{L_2(\Omega)}.$$

Here, the positive constants C_5, C_6 are independent of u, u_h, f and h .

4. Numerical Experiments

In this section we present the results of numerical experiments for two model problems.

Let Ω be a circle of unit radius with center at the point $(2, 2)$. We consider the boundary value Equation (1) in the domain Ω . The right-hand side and coefficients of Equation (1) are given as

$$f(x) = 4(1 + \mu - t) \left((\mu - 2\alpha - t) \tilde{\rho}^{\mu-2\alpha-t-1}(x) (1 - \tilde{\rho}(x)) - \tilde{\rho}^{\mu-2\alpha-t}(x) \right) + \tilde{\rho}^{1+\mu-t}(x),$$

$$a_{11}(x) = a_{22}(x) = \tilde{\rho}^{-2\alpha}(x), \quad a(x) = 1, \quad \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right), \quad \mu \in \left(\alpha, \frac{1}{2}\right), \quad t < \frac{1}{2},$$

where $\tilde{\rho}(x) = 1 - (x_1 - 2)^2 - (x_2 - 2)^2$ and $\tilde{\rho}(x)$ be a function that is infinitely differentiable and satisfies the following conditions:

$$C_7 \rho(x) \leq \tilde{\rho}(x) \leq C_8 \rho(x) \text{ for each point } x \text{ of the domain } \Omega; 0 < C_7 \leq C_8 < \infty.$$

The exact solution of this problem is $u(x) = \tilde{\rho}^{1-\mu-t}(x)$.

For finding an approximate solution of model problems we used mesh with the compression of nodes to the boundary (R_c), quasi-uniform mesh (R_q) and the finite element method scheme from paragraph three. For the mesh R_c we set the number of layers n and the exponent of compression of the mesh $r = \frac{1}{\tau}$, $\tau = \mu - \alpha$.

We investigate the convergence rate of the approximate solution u_h to the exact one in the norms of the spaces $L_2(\Omega)$ and $W_{2,\alpha}^1(\Omega)$ on the mesh R_c and R_q . The absolute value of the error $e = |u - u_h|$ in the mesh nodes G_i on the mesh R_c and R_q is analyzed.

Model Problem 1. We set the parameters $\alpha = 0.01$, $\mu = 0.49$, $t = 0.499$, at which the solution, the coefficients and the right-hand side of the equation in Equation (1) have the form

$$u(x) = \tilde{\rho}^{0.991}(x),$$

$$a_{11}(x) = a_{22}(x) = \tilde{\rho}^{-0.02}(x), \quad a(x) = 1,$$

$$f(x) = 0.114956 \cdot \tilde{\rho}^{-1.029}(x) + 3.849044 \cdot \tilde{\rho}^{-0.029}(x) + \tilde{\rho}^{0.991}(x),$$

the exponent of compression of the mesh $r = 2.08(3)$.

In Table 1 we give the number of nodes and their percentage to the total number of mesh nodes N , in which the absolute value of the error $e = |u - u_h|$ is not less than the specified value of the limit error. In this Table the patterns of the absolute error distribution at the nodes of the R_q and R_c meshes are also showed. We present data on the R_c mesh for N nodes and on the R_q mesh for N and $\frac{N}{2}$ nodes.

In Table 2 we present the norms of the difference between an exact and an approximate solution $\|e\|_{L_2(\Omega)} = \|u - u_h\|_{L_2(\Omega)}$ and $\|e\|_{W_{2,\alpha}^1(\Omega)} = \|u - u_h\|_{W_{2,\alpha}^1(\Omega)}$ for R_q and R_c and find the ratios of the norms β when the mesh parameter h is reduced by a factor two. The value of the parameter h in the domain Ω_2 for R_c varies by changing number of layers n .

Table 1. Absolute value of the error e for Model Problem 1.

Quasi-Uniform Mesh (R_q)		Absolute Error Distribution	Specified Limited Error	Percent	Number of Nodes
Number of nodes N	10,849,474		$e \geq 3 \times 10^{-6}$	0.00%	69
			$1 \times 10^{-6} \leq e < 3 \times 10^{-6}$	89.37%	9,696,362
			$7 \times 10^{-7} \leq e < 1 \times 10^{-6}$	10.61%	1,151,232
h	0.00055		$3 \times 10^{-7} \leq e < 7 \times 10^{-7}$	0.01%	1196
			$1 \times 10^{-7} \leq e < 3 \times 10^{-7}$	0.00%	372
			$0 \leq e < 1 \times 10^{-7}$	0.00%	243
Refined Mesh (R_c)		Absolute Error Distribution	Specified Limited Error	Percent	Number of Nodes
N	10,755,478		$e \geq 3 \times 10^{-6}$	0.00%	0
Number of nodes in domain Ω_2	10,661,162		$1 \times 10^{-6} \leq e < 3 \times 10^{-6}$	0.00%	0
			$7 \times 10^{-7} \leq e < 1 \times 10^{-6}$	0.00%	4
			$3 \times 10^{-7} \leq e < 7 \times 10^{-7}$	44.46%	4,781,879
h	0.000556		$1 \times 10^{-7} \leq e < 3 \times 10^{-7}$	55.44%	5,962,761
n	3		$0 \leq e < 1 \times 10^{-7}$	0.10%	10,834
b	1/1024				
Refined Mesh (R_c)		Absolute Error Distribution	Specified Limited Error	Percent	Number of Nodes
N	4,974,486		$e \geq 3 \times 10^{-6}$	0.00%	0
Number of nodes in domain Ω_2	4,241,164		$1 \times 10^{-6} \leq e < 3 \times 10^{-6}$	0.00%	0
			$7 \times 10^{-7} \leq e < 1 \times 10^{-6}$	0.00%	0
			$3 \times 10^{-7} \leq e < 7 \times 10^{-7}$	2.75%	136,821
h	0.00087		$1 \times 10^{-7} \leq e < 3 \times 10^{-7}$	83.18%	4,137,684
n	18		$0 \leq e < 1 \times 10^{-7}$	14.07%	699,981
b	1/128				

Table 2. The errors $\|e\|_{L_2(\Omega)}$ and $\|e\|_{W_{2,\alpha}^1(\Omega)}$ for meshes R_q and R_c for Model Problem 1.

Quasi-Uniform Mesh (R_q)					Refined Mesh (R_c), $b = 1/128$				
h	$\ e\ _{L_2(\Omega)}$	β	$\ e\ _{W_{2,\alpha}^1(\Omega)}$	β	h	$\ e\ _{L_2(\Omega)}$	β	$\ e\ _{W_{2,\alpha}^1(\Omega)}$	β
0.0022	5.40×10^{-6}	1.68	3.83×10^{-3}	1.85	0.0035	3.00×10^{-6}	4.21	5.11×10^{-3}	2.06
0.0011	3.21×10^{-6}		2.22×10^{-3}		0.00169	7.12×10^{-7}		2.48×10^{-3}	
0.00055	1.75×10^{-6}	1.83	1.36×10^{-3}	1.72	0.00083	1.73×10^{-7}	4.11	1.23×10^{-3}	2.03

The distribution of the absolute values of the error e in the mesh nodes with a decrease in the h parameter by a factor of two on the meshes R_q and R_c is given in Table 3.

Table 3. The distribution of the error e on the grids R_q and R_c as h changes for Model Problem 1.

Specified Limited Error		Quasi-Uniform Mesh (R_q)
$e \geq 3 \times 10^{-6}$

$1 \times 10^{-6} \leq e < 3 \times 10^{-6}$
$7 \times 10^{-7} \leq e < 1 \times 10^{-6}$
$3 \times 10^{-7} \leq e < 7 \times 10^{-7}$
$1 \times 10^{-7} \leq e < 3 \times 10^{-7}$
$0 \leq e < 1 \times 10^{-7}$			
N		338,449	1,355,498	5,427,739
h		0.0031	0.0015	0.00078
Specified Limited Error		Refined Mesh (R_c), $b = 1/128$		
		$n = 5$	$n = 10$	$n = 20$
$e \geq 3 \times 10^{-6}$

$1 \times 10^{-6} \leq e < 3 \times 10^{-6}$
$7 \times 10^{-7} \leq e < 1 \times 10^{-6}$
$3 \times 10^{-7} \leq e < 7 \times 10^{-7}$
$1 \times 10^{-7} \leq e < 3 \times 10^{-7}$
$0 \leq e < 1 \times 10^{-7}$			
N		425,760	1,569,052	6,129,755
Number of nodes in domain Ω_2		386,628	1,375,684	5,200,079
h in domain Ω_2		0.0029	0.0015	0.00079

Model Problem 2. We set the parameters $\alpha = -0.49$, $\mu = 0.01$, $t = 0.49$, at which the solution, the coefficients and the right-hand side of Equation (1) have the form

$$u(x) = \tilde{\rho}^{0.52}(x),$$

$$a_{11}(x) = a_{22}(x) = \tilde{\rho}^{0.98}(x), \quad a(x) = 1,$$

$$f(x) = -1.04 \cdot \tilde{\rho}^{-0.5}(x) + 3.12 \cdot \tilde{\rho}^{0.5}(x) + \tilde{\rho}^{0.52}(x),$$

the exponent of compression of the mesh $r = 2$.

A numerical analysis of this problem was carried out by analogy with Model Problem 1. The results of the research are presented in Tables 4–6.

Table 4. Absolute value of the error e for Model Problem 2.

Quasi-Uniform Mesh (R_q)		Absolute Error Distribution	Specified Limited Error	Percent	Number of Nodes	
N	2,713,152		$e \geq 5 \times 10^{-3}$	50.40%	1,367,499	
			$2 \times 10^{-3} \leq e < 5 \times 10^{-3}$	49.60%	1,345,653	
			$1 \times 10^{-3} \leq e < 2 \times 10^{-3}$	0.00%	0	
h	0.00055		$7 \times 10^{-4} \leq e < 1 \times 10^{-3}$	0.00%	0	
			$4 \times 10^{-4} \leq e < 7 \times 10^{-4}$	0.00%	0	
Refined Mesh (R_c)		Absolute Error Distribution	Specified Limited Error	Percent	Number of Nodes	
N	3,254,432		$e \geq 5 \times 10^{-3}$	0.00%	0	
Number of nodes in domain Ω_2	2,484,744		$2 \times 10^{-3} \leq e < 5 \times 10^{-3}$	0.00%	0	
h in domain Ω_2	0.0011		$1 \times 10^{-3} \leq e < 2 \times 10^{-3}$	26.32%	856,521	
n	27		$7 \times 10^{-4} \leq e < 1 \times 10^{-3}$	31.94%	1,039,571	
b	1/64		$4 \times 10^{-4} \leq e < 7 \times 10^{-4}$	41.74%	1,358,340	

Table 5. The error $\|e\|_{W_{2,\alpha}^1(\Omega)}$ for meshes R_q and R_c for Model Problem 2.

Quasi-Uniform Mesh (R_q)				Refined Mesh (R_c), $b = 1/64$		
h	$\ e\ _{W_{2,\alpha}^1(\Omega)}$	β	n	h in domain Ω_2	$\ e\ _{W_{2,\alpha}^1(\Omega)}$	β
0.0022	0.065816	1.43	3	0.0068	0.04562	2.08
0.0011	0.046016		8	0.0032	0.021939	2.00
0.00055	0.032220	1.43	18	0.0016	0.010989	

Table 6. The distribution of the error e on the grids R_q and R_c as h changes for Model Problem 2.

Specified Limited Error	Quasi-Uniform Mesh (R_q)		
$e \geq 5 \times 10^{-3}$			
$2 \times 10^{-3} \leq e < 5 \times 10^{-3}$			
$1 \times 10^{-3} \leq e < 2 \times 10^{-3}$			
$7 \times 10^{-4} \leq e < 1 \times 10^{-3}$			
$4 \times 10^{-4} \leq e < 7 \times 10^{-4}$			
N	168,670	677,704	2,713,152
h	0.0044	0.0022	0.0011

Table 6. Cont.

Specified Limited Error	Refined Mesh (R_c)		
	$n = 6$	$n = 12$	$n = 24$
$e \geq 5 \times 10^{-3}$

$2 \times 10^{-3} \leq e < 5 \times 10^{-3}$
$1 \times 10^{-3} \leq e < 2 \times 10^{-3}$
$7 \times 10^{-4} \leq e < 1 \times 10^{-3}$
$4 \times 10^{-4} \leq e < 7 \times 10^{-4}$			
N	166,751	643,498	2,567,442
Number of nodes in domain Ω_2	139,635	514,952	1,972,944
h in domain Ω_2	0.0050	0.0025	0.00127

In Figure 2a,b we present graphs of the error $\|e\|_{W_{2,\alpha}^1(\Omega)} = \|u - u_h\|_{W_{2,\alpha}^1(\Omega)}$ as a function of the parameter h on the grids R_q and R_c in a logarithmic scale. In the first case the parameter h decreases due to an increase in the number of layers n at a fixed value $b = 1/64$ (Figure 2a). In the second case h changes due to a decrease in the width of the border strip b at a fixed number n (Figure 2b).

Figure 2. The graph of the error $\|e\|_{W_{2,\alpha}^1(\Omega)}$ on the grids R_q and R_c as h changes in a logarithmic scale for Model Problem 2. For R_c (a) $b = \text{const} = 1/64$, n is a variable; (b) n is a constant, b is a variable.

5. Conclusions

We can conclude according to the results of numerical experiments:

- An approximate generalized solution of Equation (1) on grids with an appropriate compression of nodes (close) to the boundary of the domain converges to an exact solution with a speed $O(h^2)$ in the norm of the space $L_2(\Omega)$ and $O(h)$ in the norm of the space $W_{2,\alpha}^1(\Omega)$ (see Tables 2 and 5);

- the absolute value of the error is an order of magnitude smaller on the mesh with the compression of nodes to the boundary of the domain and with an optimal choice of parameters b and n than on a quasi-uniform mesh (see Tables 1, 3, 4 and 6);
- to reduce the absolute value of the error it is more expedient to increase the number of layers n than to reduce the width of the boundary ring domain; in this case the absolute value of the error decreases faster;
- for meshes of large dimensionality it is advisable to use the weighted finite element method.

In the next papers we plan to develop the proposed finite element method for boundary value problems with inhomogeneous boundary conditions for self-adjoint differential equations of second and higher orders with weaker conditions on the input data of the problem, in particular

$$a(x) \geq -c, \quad x \in \Omega, \quad c \text{ is finite constant.}$$

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