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Residual Symmetries and Bäcklund Transformations of Strongly Coupled Boussinesq–Burgers System

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Abstract: In this article, we construct a new strongly coupled Boussinesq–Burgers system taking values in a commutative subalgebra Z_2. A residual symmetry of the strongly coupled Boussinesq–Burgers system is achieved by a given truncated Painlevé expansion. The residue symmetry with respect to the singularity manifold is a nonlocal symmetry. Then, we introduce a suitable enlarged system to localize the nonlocal residual symmetry. In addition, a Bäcklund transformation is obtained with the help of Lie's first theorem. Further, the linear superposition of multiple residual symmetries is localized to a Lie point symmetry, and a N-th Bäcklund transformation is also obtained.

Keywords: (2+1)-dimensional strongly coupled Boussinesq–Burgers system; residual symmetry; Bäcklund transformation; Painlevé analysis

1. Introduction

Integrable equations have wide applications in the field of nonlinear science, such as plasma physics [1–4], hydrodynamics [5–7], nonlinear optics [8,9], and so on [10–15]. The Painlevé analysis is an effective way to study the integrability of nonlinear systems [16,17]. In [18–20], the authors proposed a residual symmetry in the process of the residue of the truncated Painlevé expansion for the bosonized super symmetric KdV equation which is a nonlocal symmetry.

The Boussinesq–Burgers (B–B) system arises in the study of shallow water waves for two-layered fluid flow and describes propagation of shallow water waves in lake and ocean beaches [21–27]. In [28,29], the authors were concerned with the application of the nonlocal residual symmetry analysis to the Boussinesq–Burgers (B–B) system, which has the form as follows:

$$p_t - \frac{1}{2}(\beta - 1)p_{xx} - 2pp_x - \frac{1}{2}r_x = 0, \tag{1a}$$

$$r_t + \frac{1}{2}(\beta - 1)r_{xx} + \beta(\frac{\beta}{2} - 1)p_{xxx} - 2(pr)_x = 0, \tag{1b}$$

where $v = v(x,t)$ is the height deviating from the equilibrium position of water, $u = u(x,t)$ is the field of horizontal velocity, β is a constant representing different dispersive power.

In [30,31], a hierarchy called the Frobenius-valued Kakomtsev–Petviashvili hierarchy which takes values in a maximal commutative subalgebra of $gl(m,\mathbb{C})$ was constructed. In [32], the authors considered the Hirota quadratic equation of the commutative version of extended multi-component Toda hierarchy, which should be useful to Frobenius manifold theory [33]. In addition, we studied the Z_n-Painlevé IV equation and Frobenius Painlevé equations [34]. In this paper, we consider a new strongly coupled B–B system which is defined by us through taking values in a commutative subalgebra $Z_2 = \mathbb{C}[\Gamma]/(\Gamma^2)$. The form of the strongly coupled B–B system is as follows:

$$\begin{aligned} p_t - \frac{1}{2}(\beta-1)p_{xx} - 2pp_x - 2qq_x - \frac{1}{2}r_x = 0, \\ q_t - \frac{1}{2}(\beta-1)q_{xx} - 2pq_x - 2qp_x - \frac{1}{2}s_x = 0, \end{aligned} \tag{2a}$$

$$\begin{aligned} r_t + \frac{1}{2}(\beta-1)r_{xx} + \beta(\frac{\beta}{2}-1)p_{xxx} - 2(pr+qs)_x = 0, \\ s_t + \frac{1}{2}(\beta-1)s_{xx} + \beta(\frac{\beta}{2}-1)q_{xxx} - 2(ps+qr)_x = 0. \end{aligned} \tag{2b}$$

Then, the residual symmetry of the strongly coupled Boussinesq–Burgers system is achieved. In order to further explore the residual symmetry, we use Lie's first theorem: Every local analytical group determines on its tangent vector space at the identity the structure of a unique Lie algebra with the Lie bracket given by the formula

$$[x,y] = \lim_{t\to 0} t^{-2}(\frac{tx \circ ty}{ty \circ tx}),$$

where the local group operation $\circ$ is transported into the tangent space, and where g/h is written for $g \circ h^{-1}$.

The aim of this paper is to promote a Boussinesq–Burgers system to a Frobenius integrable system, which is called a strongly coupled Boussinesq–Burgers system. We apply the nonlocal residual symmetry analysis to the strongly coupled Boussinesq–Burgers system.

2. Residual Symmetries of (2+1)-Dimensional Strongly Coupled Burgers System

We first introduce the truncated Painlevé expansion:

$$\begin{aligned} p &= \sum_{i=0}^{\alpha_0}(p_i(\psi+\phi)^{i-\alpha_0} + p_i(\psi-\phi)^{i-\alpha_0} - q_i(\psi-\phi)^{i-\alpha_0} + q_i(\psi+\phi)^{i-\alpha_0}), \\ q &= \sum_{i=0}^{\alpha_1}(q_i(\psi+\phi)^{i-\alpha_1} + q_i(\psi-\phi)^{i-\alpha_1} - p_i(\psi-\phi)^{i-\alpha_1} + p_i(\psi+\phi)^{i-\alpha_1}), \\ r &= \sum_{i=0}^{\alpha_2}(r_i(\psi+\phi)^{i-\alpha_2} + r_i(\psi-\phi)^{i-\alpha_2} - s_i(\psi-\phi)^{i-\alpha_2} + s_i(\psi+\phi)^{i-\alpha_2}), \\ s &= \sum_{i=0}^{\alpha_3}(s_i(\psi+\phi)^{i-\alpha_3} + s_i(\psi-\phi)^{i-\alpha_3} - r_i(\psi-\phi)^{i-\alpha_3} + r_i(\psi+\phi)^{i-\alpha_3}), \end{aligned} \tag{3}$$

where p_α, q_α are a set of arbitrary solutions of the equation, and $p_{\alpha-1}, p_{\alpha-2}, \cdots, p_0, q_{\alpha-1}, q_{\alpha-2}, \cdots, q_0$ to be expressed by derivatives of ψ and ϕ. By balancing the dispersion and nonlinear terms according to the leader order analysis to the system Equation (2), the truncated Painlevé expansion has the following form:

$$\begin{aligned} p &= \frac{p_0\psi - q_0\phi}{\psi^2-\phi^2} + p_1, \\ q &= \frac{q_0\psi - p_0\phi}{\psi^2-\phi^2} + q_1, \end{aligned} \tag{4a}$$

$$\begin{aligned} r &= \frac{r_0(\psi^2+\phi^2) - 2s_0\psi\phi}{(\psi^2-\phi^2)^2} + \frac{r_1\psi - s_1\phi}{\psi^2-\phi^2} + r_2, \\ s &= \frac{s_0(\psi^2+\phi^2) - 2r_0\psi\phi}{(\psi^2-\phi^2)^2} + \frac{s_1\psi - r_1\phi}{\psi^2-\phi^2} + s_2. \end{aligned} \tag{4b}$$

Then, plugging Equation (4) into Equation (2), and vanishing all the coefficients of each power of $(\psi+\phi)^{-1}+(\psi-\phi)^{-1}$ and $(\psi-\phi)^{-1}-(\psi+\phi)^{-1}$, we obtain

$$p_0 = -\frac{1}{2}\psi_x, \quad q_0 = -\frac{1}{2}\phi_x, \tag{5a}$$

$$p_1 = \frac{\psi_x(\psi_{xx}+2\psi_t)-\phi_x(\phi_{xx}+2\phi_t)}{4(\psi_x^2-\phi_x^2)}, \quad q_1 = \frac{\psi_x(\phi_{xx}+2\phi_t)-\phi_x(\psi_{xx}+2\psi_t)}{4(\psi_x^2-\phi_x^2)}, \tag{5b}$$

$$r_0 = -\frac{\beta}{2}(\psi_x^2+\phi_x^2), \quad s_0 = -\beta\psi_x\phi_x, \tag{5c}$$

$$r_1 = \frac{\beta}{2}\psi_{xx}, \quad s_1 = \frac{\beta}{2}\phi_{xx}, \tag{5d}$$

$$\begin{aligned} r_2 =& \frac{\beta}{4}\frac{(\psi_{xx}^2+\phi_{xx}^2)(\psi_x^2+\phi_x^2)-4\psi_x\phi_x\psi_{xx}\phi_{xx}}{(\psi_x^2-\phi_x^2)^2} - \frac{\beta}{4}\frac{\psi_{xxx}\psi_x-\phi_{xxx}\phi_x}{\psi_x^2-\phi_x^2} \\ &+ \frac{\beta}{2}\frac{(\psi_{xx}\psi_t+\phi_{xx}\phi_t)(\psi_x^2+\phi_x^2)-2\psi_x\phi_x(\psi_{xx}\phi_t+\phi_{xx}\psi_t)}{(\psi_x^2-\phi_x^2)^2} - \frac{\beta}{2}\frac{\psi_{tx}\psi_x-\phi_{tx}\phi_x}{\psi_x^2-\phi_x^2}, \\ s_2 =& \frac{\beta}{2}\frac{\psi_{xx}\phi_{xx}(\psi_x^2+\phi_x^2)-\psi_x\phi_x(\psi_{xx}^2+\phi_{xx}^2)}{(\psi_x^2-\phi_x^2)^2} - \frac{\beta}{4}\frac{\phi_{xxx}\psi_x-\psi_{xxx}\phi_x}{\psi_x^2-\phi_x^2} \\ &+ \frac{\beta}{2}\frac{(\psi_{xx}\phi_t+\phi_{xx}\psi_t)(\psi_x^2+\phi_x^2)-2\psi_x\phi_x(\psi_{xx}\psi_t+\phi_{xx}\phi_t)}{(\psi_x^2-\phi_x^2)^2} - \frac{\beta}{2}\frac{\phi_{tx}\psi_x-\psi_{tx}\phi_x}{\psi_x^2-\phi_x^2}. \end{aligned} \tag{5e}$$

Further, the Schwarzian form of the field ψ and ϕ is obtained as follows:

$$\begin{aligned} C_t &= CC_x + DD_x - \frac{1}{4}S_x - C_{xx}, \\ D_t &= CD_x + DC_x - \frac{1}{4}T_x - D_{xx}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} C &= \frac{\psi_t\psi_x-\phi_t\phi_x}{\psi_x^2-\phi_x^2}, \quad D = \frac{\phi_t\psi_x-\psi_t\phi_x}{\psi_x^2-\phi_x^2}, \\ S &= \frac{\psi_{xxx}\psi_x-\phi_{xxx}\phi_x}{\psi_x^2-\phi_x^2} - \frac{3}{2}\frac{(\psi_{xx}^2+\phi_{xx}^2)(\psi_x^2+\phi_x^2)-4\psi_x\phi_x\psi_{xx}\phi_{xx}}{(\psi_x^2-\phi_x^2)^2}, \\ T &= \frac{\phi_{xxx}\psi_x-\psi_{xxx}\phi_x}{\psi_x^2-\phi_x^2} - 3\frac{\psi_{xx}\phi_{xx}(\psi_x^2+\phi_x^2)-\psi_x\phi_x(\psi_{xx}^2+\phi_{xx}^2)}{(\psi_x^2-\phi_x^2)^2}. \end{aligned}$$

The Schwarzian form Equation (6) is invariant under the Möbius transformation:

$$\psi \to \frac{(a+b\psi)(c+d\psi)-bd\phi^2}{(c+d\psi)^2-(d\phi)^2}, \quad \phi \to \frac{b\phi(c+d\psi)-d\phi(a+b\psi)}{(c+d\psi)^2-(d\phi)^2}, \quad (a,b,c,d \in \mathbb{C}),$$

which means the field ψ and ϕ possesses six point symmetries, $\sigma^\psi = a_1$, $\sigma^\phi = a_2$, $\sigma^\psi = b_1\psi$, $\sigma^\phi = b_1\phi$, $\sigma^\psi = c_1(\psi^2+\phi^2)$, $\sigma^\phi = 2c_1\psi\phi$, where a_1, a_2 , b_1 and c_1 are arbitrary constants.

Actually, the residuals p_0, q_0, r_1, and s_1 are nonlocal symmetries of Equation (2) with p_1, q_1, r_2, and s_2 as solutions. Then, we introduce dependent variables f, g, h, and k with the relations $f = \psi_x$, $g = \phi_x$, $h = f_x$, and $k = g_x$ to obtain a local symmetry.

Substituting Equation (5) into Equation (4), one has

$$\begin{aligned} p &= -\frac{\psi_x\psi-\phi_x\phi}{2(\psi^2-\phi^2)} + \frac{\psi_x(\psi_{xx}+2\psi_t)-\phi_x(\phi_{xx}+2\phi_t)}{4(\psi_x^2-\phi_x^2)}, \\ q &= -\frac{\phi_x\psi-\psi_x\phi}{2(\psi^2-\phi^2)} + \frac{\psi_x(\phi_{xx}+2\phi_t)-\phi_x(\psi_{xx}+2\psi_t)}{4(\psi_x^2-\phi_x^2)}, \end{aligned} \tag{7}$$

$$\begin{aligned} r &= -\frac{\beta}{2}\frac{(\psi_x^2+\phi_x^2)(\psi^2+\phi^2)-4\psi_x\phi_x}{(\psi^2-\phi^2)^2}+\frac{\beta}{2}\frac{\psi_{xx}\psi-\phi_{xx}\phi}{\psi^2-\phi^2}+r_2,\\ s &= -\beta\frac{\psi_x\phi_x(\psi^2+\phi^2)-\psi\phi(\psi_x^2+\phi_x^2)}{(\psi^2-\phi^2)^2}+\frac{\beta}{2}\frac{\phi_{xx}\psi-\psi_{xx}\phi}{\psi^2-\phi^2}+s_2. \end{aligned} \tag{8}$$

Then, one can get the localized Lie point symmetry

$$\sigma^p = -\frac{f}{2}, \quad \sigma^q = -\frac{g}{2}, \tag{9a}$$

$$\sigma^r = \frac{\beta h}{2}, \quad \sigma^s = \frac{\beta k}{2}, \tag{9b}$$

$$\sigma^f = -2(f\psi+g\phi), \quad \sigma^g = -2(g\psi+f\phi), \tag{9c}$$

$$\sigma^h = -2(f^2+g^2)-2(h\psi+k\phi), \quad \sigma^k = -4fg-2(k\psi+h\phi), \tag{9d}$$

$$\sigma^\psi = -\psi^2-\phi^2, \quad \sigma^\phi = -2\psi\phi, \tag{9e}$$

and the corresponding Lie point symmetry vector takes the following form:

$$\begin{aligned} \overrightarrow{V} &= -\frac{f}{2}\partial_p+\frac{\beta h}{2}\partial_r-2(f\psi+g\phi)\partial_f-2(f^2+g^2+h\psi+k\phi)\partial_h-(\psi^2+\phi^2)\partial_\psi,\\ \overrightarrow{W} &= -\frac{g}{2}\partial_q+\frac{\beta k}{2}\partial_s-2(g\psi+f\phi)\partial_g-2(2fg+k\psi+h\phi)\partial_k-2\psi\phi\partial_\phi. \end{aligned} \tag{10}$$

Next we will give the Bäcklund symmetry theorem, which is obtained by using a finite transformation of the Lie point symmetry Equation (10).

Theorem 1. *If $\{p,q,r,s,f,g,h,k,\psi,\phi\}$ is a solution of the extended system Equation (9), then so is $\{\overline{p},\overline{q},\overline{r},\overline{s},\overline{f},\overline{g},\overline{h},\overline{k},\overline{\psi},\overline{\phi}\}$ with*

$$\overline{p} = p-\frac{\epsilon}{2}\frac{f(\epsilon\psi+1)-\epsilon g\phi}{(\epsilon\psi+1)^2-\epsilon^2\phi^2}, \quad \overline{q} = q-\frac{\epsilon}{2}\frac{g(\epsilon\psi+1)-\epsilon f\phi}{(\epsilon\psi+1)^2-\epsilon^2\phi^2}, \tag{11a}$$

$$\overline{r} = r+\frac{A((\epsilon\psi+1)^2+\epsilon^2\phi^2)-2B(\epsilon\psi+1)\epsilon\phi}{2(\epsilon\psi+\epsilon\phi+1)^2(\epsilon\psi-\epsilon\phi+1)^2}, \quad \overline{s} = s+\frac{B((\epsilon\psi+1)^2+\epsilon^2\phi^2)-2A(\epsilon\psi+1)\epsilon\phi}{2(\epsilon\psi+\epsilon\phi+1)^2(\epsilon\psi-\epsilon\phi+1)^2}, \tag{11b}$$

$$\overline{\psi} = \frac{\psi(\epsilon\psi+1)-\epsilon\phi^2}{(\epsilon\psi+1)^2-\epsilon^2\phi^2}, \quad \overline{\phi} = \frac{\phi(\epsilon\psi+1)-\epsilon\psi\phi}{(\epsilon\psi+1)^2-\epsilon^2\phi^2}, \tag{11c}$$

$$\overline{f} = \frac{f((\epsilon\psi+1)^2+\epsilon^2\phi^2)-2g(\epsilon\psi+1)\epsilon\phi}{(\epsilon\psi+\epsilon\phi+1)^2(\epsilon\psi-\epsilon\phi+1)^2}, \quad \overline{g} = \frac{g((\epsilon\psi+1)^2+\epsilon^2\phi^2)-2f(\epsilon\psi+1)\epsilon\phi}{(\epsilon\psi+\epsilon\phi+1)^2(\epsilon\psi-\epsilon\phi+1)^2}, \tag{11d}$$

$$\begin{aligned} \overline{h} &= \frac{C_0((\epsilon\psi+1)^3+3\epsilon^2\phi^2(\epsilon\psi+1))-D_0(3\epsilon\phi(\epsilon\psi+1)^2+\epsilon^3\phi^3)}{(\epsilon\psi+\epsilon\phi+1)^3(\epsilon\psi-\epsilon\phi+1)^3},\\ \overline{k} &= \frac{D_0((\epsilon\psi+1)^3+3\epsilon^2\phi^2(\epsilon\psi+1))-C_0(3\epsilon\phi(\epsilon\psi+1)^2+\epsilon^3\phi^3)}{(\epsilon\psi+\epsilon\phi+1)^3(\epsilon\psi-\epsilon\phi+1)^3}, \end{aligned} \tag{11e}$$

where

$$\begin{aligned} &A = \beta h\epsilon-\beta\epsilon^2(f^2+g^2)+\beta\epsilon^2(h\psi+k\phi), \quad B = \beta k\epsilon-2\beta\epsilon^2 fg+\beta\epsilon^2(h\phi+k\psi),\\ &C_0 = -2\epsilon(f^2+g^2)+\epsilon(h\psi+k\phi)+h, \quad D_0 = -4\epsilon fg+\epsilon(k\psi+h\phi)+k, \end{aligned}$$

and ϵ is an arbitrary group parameter.

Proof. According to Lie's first theorem on vector Equation (10) with the corresponding initial value problem:

$$\frac{d\overline{p}}{d\epsilon} = -\frac{\overline{f}}{2}, \quad \frac{d\overline{q}}{d\epsilon} = -\frac{\overline{g}}{2}, \tag{12a}$$

$$\frac{d\overline{r}}{d\epsilon} = \frac{\beta\overline{h}}{2}, \quad \frac{d\overline{s}}{d\epsilon} = \frac{\beta\overline{k}}{2}, \tag{12b}$$

$$\frac{d\overline{f}}{d\epsilon} = -2(\overline{\psi}\overline{f} + \overline{\phi}\overline{g}), \quad \frac{d\overline{g}}{d\epsilon} = -2(\overline{\psi}\overline{g} + \overline{\phi}\overline{f}), \tag{12c}$$

$$\frac{d\overline{h}}{d\epsilon} = -2(\overline{f}^2 + \overline{g}^2 + \overline{\psi}\overline{h} + \overline{\phi}\overline{k}), \quad \frac{d\overline{k}}{d\epsilon} = -2(2\overline{f}\overline{g} + \overline{\psi}\overline{k} + \overline{\phi}\overline{h}), \tag{12d}$$

$$\frac{d\overline{\psi}}{d\epsilon} = -(\overline{\psi}^2 + \overline{\phi}^2), \quad \frac{d\overline{\phi}}{d\epsilon} = -2\overline{\psi}\overline{\phi}, \tag{12e}$$

$$\begin{aligned} &\overline{p}(0) = p, \quad \overline{q}(0) = q, \quad \overline{r}(0) = r, \quad \overline{s}(0) = s, \quad \overline{f}(0) = f, \\ &\overline{g}(0) = g, \quad \overline{h}(0) = h, \quad \overline{k}(0) = k, \quad \overline{\psi}(0) = \psi, \quad \overline{\phi}(0) = \phi, \end{aligned} \tag{12f}$$

one can find the solutions of the above system is Equation (11). Therefore, we have completed the proof of Theorem 1. □

3. Bäcklund Transformations of Strongly Coupled Burgers System Related to Multiple Residual Symmetries

In the strongly coupled B–B system Equation (2), the original residual symmetry have forms:

$$\sigma^p = -\frac{1}{2}\psi_x, \quad \sigma^q = 2b\phi_x, \quad \sigma^r = \frac{\beta}{2}\psi_{xx}, \quad \sigma^s = \frac{\beta}{2}\phi_{xx},$$

which is related to the solution of the Equation (6). Then, according to the linear property of symmetry equations, the multiple residual symmetries are expressed in terms of any linear superposition of symmetries

$$\begin{aligned} &\sigma_n^p = \sum_{i=1}^{n} c_i\psi_{i,x}, \quad \sigma_n^q = \sum_{i=1}^{n} c_i\phi_{i,x}, \\ &\sigma_n^r = \sum_{i=1}^{n} d_i\psi_{i,xx}, \quad \sigma_n^s = \sum_{i=1}^{n} d_i\phi_{i,xx}, \quad (n = 1, 2, 3, \ldots), \end{aligned} \tag{13}$$

where ψ_i and ϕ_i $i = 1, 2, 3, \ldots$, are different solutions of Equation (6). The symmetry Equation (13) should be localized to a Lie point symmetry by introducing more variables. In fact, one can find the finite transformation group of the symmetry Equation (13).

Theorem 2. *The symmetry Equation* (13) *is localized to the Lie point symmetry*

$$\sigma^p = -\frac{1}{2}\sum_{j=1}^{n} c_j f_j, \quad \sigma^q = -\frac{1}{2}\sum_{j=1}^{n} c_j g_j, \tag{14a}$$

$$\sigma^r = \frac{\beta}{2}\sum_{j=1}^{n} c_j h_j, \quad \sigma^s = \frac{\beta}{2}\sum_{j=1}^{n} c_j k_j, \tag{14b}$$

$$\begin{aligned} \sigma^{\psi_i} &= -c_i(\psi_i^2 + \phi_i^2) - \sum_{j\neq i}^{n} c_j(\psi_j\psi_i + \phi_j\phi_i), \\ \sigma^{\phi_i} &= -2c_i\psi_i\phi_i - \sum_{j\neq i}^{n} c_j(\psi_j\phi_i + \phi_j\psi_i), \end{aligned} \tag{14c}$$

$$\begin{aligned}\sigma^{f_i} &= -2c_i(f_i\psi_i + g_i\phi_i) - \sum_{j\neq i}^{n} c_j(f_j\psi_i + g_j\phi_i + f_i\psi_j + g_i\phi_j),\\ \sigma^{g_i} &= -2c_i(g_i\psi_i + f_i\phi_i) - \sum_{j\neq i}^{n} c_j(f_j\phi_i + g_j\psi_i + g_i\psi_j + f_i\phi_j),\end{aligned} \tag{14d}$$

$$\begin{aligned}\sigma^{h_i} &= -2c_i(f_i^2 + g_i^2) - 2c_i(h_i\psi_i + k_i\phi_i) - \sum_{j\neq i}^{n} c_j(h_j\psi_i + k_j\phi_i + 2f_jf_i + 2g_jg_i + h_i\psi_j + k_i\phi_j),\\ \sigma^{k_i} &= -4c_if_ig_i - 2c_i(k_i\psi_i + h_i\phi_i) - \sum_{j\neq i}^{n} c_j(h_j\phi_i + k_j\psi_i + 2f_jg_i + 2g_jf_i + k_i\psi_j + h_i\phi_j),\end{aligned} \tag{14e}$$

where $\{p, q, r, s, f_i, g_i, h_i, k_i, \psi_i, \phi_i\}$ $(i = 1, 2, \ldots, n)$ *is a solution of the enlarged system*

$$\begin{aligned}p_t - \frac{1}{2}(\beta-1)p_{xx} - 2pp_x - 2qq_x - \frac{1}{2}r_x = 0,\\ q_t - \frac{1}{2}(\beta-1)q_{xx} - 2pq_x - 2qp_x - \frac{1}{2}s_x = 0,\end{aligned} \tag{15a}$$

$$\begin{aligned}r_t + \frac{1}{2}(\beta-1)r_{xx} + \beta(\frac{\beta}{2}-1)p_{xxx} - 2(pr+qs)_x = 0,\\ s_t + \frac{1}{2}(\beta-1)s_{xx} + \beta(\frac{\beta}{2}-1)q_{xxx} - 2(ps+qr)_x = 0,\end{aligned} \tag{15b}$$

$$\begin{aligned}p &= \frac{\psi_{i,x}(\psi_{i,xx} + 2\psi_{i,t}) - \phi_{i,x}(\phi_{i,xx} + 2\phi_{i,t})}{4(\psi_{i,x}^2 - \phi_{i,x}^2)},\\ q &= \frac{\psi_{i,x}(\phi_{i,xx} + 2\phi_{i,t}) - \phi_{i,x}(\psi_{i,xx} + 2\psi_{i,t})}{4(\psi_{i,x}^2 - \phi_{i,x}^2)},\end{aligned} \tag{15c}$$

$$\begin{aligned}r =& \frac{\beta}{4}\frac{(\psi_{i,xx}^2 + \phi_{i,xx}^2)(\psi_{i,x}^2 + \phi_{i,x}^2) - 4\psi_{i,x}\phi_{i,x}\psi_{i,xx}\phi_{i,xx}}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} - \frac{\beta}{4}\frac{\psi_{i,xxx}\psi_{i,x} - \phi_{i,xxx}\phi_{i,x}}{\psi_{i,x}^2 - \phi_{i,x}^2}\\ &+ \frac{\beta}{2}\frac{(\psi_{i,xx}\psi_{i,t} + \phi_{i,xx}\phi_{i,t})(\psi_{i,x}^2 + \phi_{i,x}^2) - 2\psi_{i,x}\phi_{i,x}(\psi_{i,xx}\phi_{i,t} + \phi_{i,xx}\psi_{i,t})}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} - \frac{\beta}{2}\frac{\psi_{i,tx}\psi_{i,x} - \phi_{i,tx}\phi_{i,x}}{\psi_{i,x}^2 - \phi_{i,x}^2},\\ s =& \frac{\beta}{2}\frac{\psi_{i,xx}\phi_{i,xx}(\psi_{i,x}^2 + \phi_{i,x}^2) - \psi_{i,x}\phi_{i,x}(\psi_{i,xx}^2 + \phi_{i,xx}^2)}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} - \frac{\beta}{4}\frac{\phi_{i,xxx}\psi_{i,x} - \psi_{i,xxx}\phi_{i,x}}{\psi_{i,x}^2 - \phi_{i,x}^2}\\ &+ \frac{\beta}{2}\frac{(\psi_{i,xx}\phi_{i,t} + \phi_{i,xx}\psi_{i,t})(\psi_{i,x}^2 + \phi_{i,x}^2) - 2\psi_{i,x}\phi_{i,x}(\psi_{i,xx}\psi_{i,t} + \phi_{i,xx}\phi_{i,t})}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} - \frac{\beta}{2}\frac{\phi_{i,tx}\psi_{i,x} - \psi_{i,tx}\phi_{i,x}}{\psi_{i,x}^2 - \phi_{i,x}^2},\end{aligned} \tag{15d}$$

$$f_i = \psi_{i,x}, \quad g_i = \phi_{i,x}, \tag{15e}$$

$$h_i = f_{i,x}, \quad k_i = g_{i,x}, \tag{15f}$$

$(i = 1, 2, \ldots, n)$.

Proof. The extended system Equation (15) has the following linearized form:

$$\begin{aligned}\sigma_t^p - \frac{1}{2}\beta\sigma_{xx}^p + \frac{1}{2}\sigma_{xx}^p - 2p\sigma_x^p - 2q\sigma_x^q - 2\sigma^p p_x - 2\sigma^q q_x - \frac{1}{2}\sigma_x^r = 0,\\ \sigma_t^q - \frac{1}{2}\beta\sigma_{xx}^q + \frac{1}{2}\sigma_{xx}^q - 2p\sigma_x^q - 2q\sigma_x^p - 2\sigma^p q_x - 2\sigma^q p_x - \frac{1}{2}\sigma_x^s = 0,\end{aligned} \tag{16a}$$

$$\begin{aligned}
&\sigma_t^r - \beta\sigma_{xxx}^p + \frac{1}{2}\beta^2\sigma_{xxx}^p - \frac{1}{2}\sigma_{xx}^r + \frac{1}{2}\beta\sigma_{xx}^r - 2p_x\sigma^r - 2q_x\sigma^s - 2\sigma_x^p r - 2\sigma_x^q s \\
&- 2p\sigma_x^r - 2q\sigma_x^s - 2\sigma^p r_x - 2\sigma^q s_x = 0, \\
&\sigma_t^s - \beta\sigma_{xxx}^q + \frac{1}{2}\beta^2\sigma_{xxx}^q - \frac{1}{2}\sigma_{xx}^s + \frac{1}{2}\beta\sigma_{xx}^s - 2p_x\sigma^s - 2q_x\sigma^r - 2\sigma_x^p s - 2\sigma_x^q r \\
&- 2p\sigma_x^s - 2q\sigma_x^r - 2\sigma^p s_x - 2\sigma^q r_x = 0,
\end{aligned} \tag{16b}$$

$$\begin{aligned}
\sigma^p &= \frac{\psi_{i,x}(\sigma_{xx}^{\psi_i} + 2\sigma_t^{\psi_i}) - \phi_{i,x}(\sigma_{xx}^{\phi_i} + 2\sigma_t^{\phi_i})}{4(\psi_{i,x}^2 - \phi_{i,x}^2)} - \frac{A_0(\psi_{i,x}^2 + \phi_{i,x}^2) - 2B_0\psi_{i,x}\phi_{i,x}}{4(\psi_{i,x}^2 - \phi_{i,x}^2)^2}, \\
\sigma^q &= \frac{\psi_{i,x}(\sigma_{xx}^{\phi_i} + 2\sigma_t^{\phi_i}) - \phi_{i,x}(\sigma_{xx}^{\psi_i} + 2\sigma_t^{\psi_i})}{4(\psi_{i,x}^2 - \phi_{i,x}^2)} - \frac{B_0(\psi_{i,x}^2 + \phi_{i,x}^2) - 2A_0\psi_{i,x}\phi_{i,x}}{4(\psi_{i,x}^2 - \phi_{i,x}^2)^2},
\end{aligned} \tag{16c}$$

$$\begin{aligned}
\sigma^r = &-\frac{\beta}{2}\frac{A_1(\psi_{i,x}^3 + 3\phi_{i,x}^2\psi_{i,x}) - B_1(\phi_{i,x}^3 + 3\psi_{i,x}^2\phi_{i,x})}{(\psi_{i,x}^2 - \phi_{i,x}^2)^3} + \frac{\beta}{2}\frac{A_2(\psi_{i,x}^2 + \phi_{i,x}^2) - 2B_2\psi_{i,x}\phi_{i,x})}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} \\
&+ \frac{\beta}{4}\frac{A_3(\psi_{i,x}^2 - \phi_{i,x}^2) - 2B_3\psi_{i,x}\phi_{i,x}}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} - \beta\frac{A_5(\psi_i^3 + 3\phi_i^2\psi_i) - B_5(\phi_i^3 + 3\psi_i^2\phi_i)}{(\psi_i^2 - \phi_i^2)^3} \\
&- \frac{\beta}{4}\frac{A_4\psi_{i,x} - B_4\phi_{i,x}}{\psi_{i,x}^2 - \phi_{i,x}^2}, \\
\sigma^s = &-\frac{\beta}{2}\frac{B_1(\psi_{i,x}^3 + 3\phi_{i,x}^2\psi_{i,x}) - A_1(\phi_{i,x}^3 + 3\psi_{i,x}^2\phi_{i,x})}{(\psi_{i,x}^2 - \phi_{i,x}^2)^3} + \frac{\beta}{2}\frac{B_2(\psi_{i,x}^2 + \phi_{i,x}^2) - 2A_2\psi_{i,x}\phi_{i,x})}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} \\
&+ \frac{\beta}{4}\frac{B_3(\psi_{i,x}^2 - \phi_{i,x}^2) - 2A_3\psi_{i,x}\phi_{i,x}}{(\psi_{i,x}^2 - \phi_{i,x}^2)^2} - \beta\frac{B_5(\psi_i^3 + 3\phi_i^2\psi_i) - A_5(\phi_i^3 + 3\psi_i^2\phi_i)}{(\psi_i^2 - \phi_i^2)^3} \\
&- \frac{\beta}{4}\frac{B_4\psi_{i,x} - A_4\phi_{i,x}}{\psi_{i,x}^2 - \phi_{i,x}^2},
\end{aligned} \tag{16d}$$

$$\sigma_x^{\psi_i} = \sigma^{f_i}, \quad \sigma_x^{\phi_i} = \sigma^{g_i}, \tag{16e}$$

$$\sigma_x^{f_i} = \sigma^{h_i}, \quad \sigma_x^{g_i} = \sigma^{k_i}, \ \ i = 1, 2, \ldots, n, \tag{16f}$$

where

$$\begin{aligned}
A_0 &= \psi_{i,xx}\sigma_x^{\psi_i} + \phi_{i,xx}\sigma_x^{\phi_i} + 2\psi_{i,t}\sigma_x^{\psi_i} + 2\phi_{i,t}\sigma_x^{\phi_i}, \\
B_0 &= \psi_{i,xx}\sigma_x^{\phi_i} + \phi_{i,xx}\sigma_x^{\psi_i} + 2\psi_{i,t}\sigma_x^{\phi_i} + 2\phi_{i,t}\sigma_x^{\psi_i}, \\
A_1 &= (\psi_{i,xx}^2 + \phi_{i,xx}^2)\sigma_x^{\psi_i} + 2\psi_{i,xx}\phi_{i,xx}\sigma_x^{\phi_i}, \\
B_1 &= (\psi_{i,xx}^2 + \phi_{i,xx}^2)\sigma_x^{\phi_i} + 2\psi_{i,xx}\phi_{i,xx}\sigma_x^{\psi_i}, \\
A_2 &= \psi_{i,xx}\sigma_{xx}^{\psi_i} + \phi_{i,xx}\sigma_{xx}^{\phi_i} + \psi_{i,xx}\sigma_t^{\psi_i} + \phi_{i,xx}\sigma_t^{\phi_i} + \psi_{i,t}\sigma_{xx}^{\psi_i} + \phi_{i,t}\sigma_{xx}^{\phi_i} + \psi_{i,xt}\sigma_x^{\psi_i} + \phi_{i,xt}\sigma_x^{\phi_i}, \\
B_2 &= \psi_{i,xx}\sigma_{xx}^{\phi_i} + \phi_{i,xx}\sigma_{xx}^{\psi_i} + \psi_{i,xx}\sigma_t^{\phi_i} + \phi_{i,xx}\sigma_t^{\psi_i} + \psi_{i,t}\sigma_{xx}^{\phi_i} + \phi_{i,t}\sigma_{xx}^{\psi_i} + \psi_{i,xt}\sigma_x^{\phi_i} + \phi_{i,xt}\sigma_x^{\psi_i}, \\
A_3 &= \psi_{i,xxx}\sigma_x^{\psi_i} + \phi_{i,xxx}\sigma_x^{\phi_i}, \quad B_3 = \phi_{i,xxx}\sigma_x^{\psi_i} + \psi_{i,xxx}\sigma_x^{\phi_i}, \\
A_4 &= \sigma_{xxx}^{\psi_i} + 2\sigma_{xt}^{\psi_i}, \quad B_4 = \sigma_{xxx}^{\phi_i} + 2\sigma_{xt}^{\phi_i}, \\
A_5 &= (\psi_{i,xx}\psi_{i,t} + \phi_{i,xx}\phi_{i,t})\sigma_x^{\psi_i} + (\psi_{i,xx}\phi_{i,t} + \phi_{i,xx}\psi_{i,t})\sigma_x^{\phi_i}, \\
B_5 &= (\psi_{i,xx}\phi_{i,t} + \phi_{i,xx}\psi_{i,t})\sigma_x^{\psi_i} + (\psi_{i,xx}\psi_{i,t} + \phi_{i,xx}\phi_{i,t})\sigma_x^{\phi_i}.
\end{aligned}$$

Let us first consider the special case: for any fixed $m, c_m \neq 0$, while $c_j = 0$, $j \neq m$ in Equation (13). In this case, the localized symmetry for $\{p, q, r, s, f_m, g_m, h_m, k_m, \psi_m, \phi_m\}$ can be obtained from Equation (9):

$$\sigma^p = -\frac{1}{2}c_m f_m, \quad \sigma^q = -\frac{1}{2}c_m g_m, \tag{17a}$$

$$\sigma^r = \frac{\beta}{2}c_m h_m, \quad \sigma^s = \frac{\beta}{2}c_m k_m, \tag{17b}$$

$$\sigma^{\psi_m} = -c_m(\psi_m^2 + \phi_m^2), \quad \sigma^{\phi_m} = -2c_m\psi_m\phi_m, \tag{17c}$$

$$\sigma^{f_m} = -2c_m(f_m\psi_m + g_m\phi_m), \quad \sigma^{g_m} = -2c_m(g_m\psi_m + f_m\phi_m), \tag{17d}$$

$$\sigma^{h_m} = -2c_m(f_m^2 + g_m^2) - 2c_m(h_m\psi_m + k_m\phi_m), \quad \sigma^{k_m} = -4c_m f_m g_m - 2c_m(k_m\psi_m + h_m\phi_m). \tag{17e}$$

In Equation (15c), p and q can be eliminated by taking $i = m$ and $i = j$, respectively, and then we obtain

$$\begin{aligned} \psi_{j,xx} &= \frac{A_6\psi_{m,x} - B_6\phi_{m,x}}{\psi_{m,x}^2 - \phi_{m,x}^2}, \\ \phi_{j,xx} &= \frac{B_6\psi_{m,x} - A_6\phi_{m,x}}{\psi_{m,x}^2 - \phi_{m,x}^2}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} A_6 &= \psi_{j,x}\psi_{m,xx} + \phi_{j,x}\phi_{m,xx} + 2\psi_{j,x}\psi_{m,t} + 2\phi_{j,x}\phi_{m,t} - 2\psi_{m,x}\psi_{j,t} - 2\phi_{m,x}\phi_{j,t}, \\ B_6 &= \psi_{j,x}\phi_{m,xx} + \phi_{j,x}\psi_{m,xx} + 2\psi_{j,x}\phi_{m,t} + 2\phi_{j,x}\psi_{m,t} - 2\psi_{m,x}\phi_{j,t} - 2\phi_{m,x}\psi_{j,t}. \end{aligned}$$

Substituting Equation (17a) into Equation (16c) with $i = j$ and vanishing $\psi_{j,xx}$ and $\phi_{j,xx}$ through Equation (18), we have

$$\begin{aligned} \sigma^{\psi_j} &= -c_m(\psi_m\psi_j + \phi_m\phi_j), \\ \sigma^{\phi_j} &= -c_m(\psi_m\phi_j + \phi_m\psi_j). \end{aligned} \tag{19}$$

The symmetries for f_j, g_j, h_j, and k_j can be obtained from Equation (19) as follows:

$$\begin{aligned} \sigma^{f_j} &= -c_m(\psi_m f_j + \phi_m g_j + f_m\psi_j + g_m\phi_j), \\ \sigma^{g_j} &= -c_m(\psi_m g_j + \phi_m f_j + f_m\phi_j + g_m\psi_j), \end{aligned} \tag{20}$$

$$\begin{aligned} \sigma^{h_j} &= -c_m(h_j\psi_m + k_j\phi_m + 2f_m f_j + 2g_m g_j + h_m\psi_j + k_m\phi_j), \\ \sigma^{k_j} &= -c_m(h_j\phi_m + k_j\psi_m + 2f_m g_j + 2g_m f_j + k_m\psi_j + h_m\phi_j). \end{aligned} \tag{21}$$

Further, Equation (14) can be obtained by taking a linear combination of the above results for $m = 1, 2, \ldots, n$. Therefore, we have completed the proof of Theorem 2. □

According to Lie's first theorem, the initial value problem of the Lie point symmetry Equation (14) has the following form:

$$\frac{dP(\epsilon)}{d\epsilon} = -\frac{1}{2}\sum_{j=1}^{n} c_j F_j(\epsilon), \quad \frac{dQ(\epsilon)}{d\epsilon} = -\frac{1}{2}\sum_{j=1}^{n} c_j G_j(\epsilon), \tag{22a}$$

$$\frac{dR(\epsilon)}{d\epsilon} = \frac{1}{2}\beta\sum_{j=1}^{n} c_j H_j(\epsilon), \quad \frac{dS(\epsilon)}{d\epsilon} = \frac{1}{2}\beta\sum_{j=1}^{n} c_j k_j(\epsilon), \tag{22b}$$

$$\begin{aligned} \frac{d\Psi(\epsilon)}{d\epsilon} &= -c_i(\Psi_i^2(\epsilon) + \Phi_i^2(\epsilon)) - \sum_{j\neq i}^{n} c_j(\Psi_j(\epsilon)\Psi_i(\epsilon) + \Phi_j(\epsilon)\Phi_i(\epsilon)), \\ \frac{d\Phi(\epsilon)}{d\epsilon} &= -2c_i\Psi_i(\epsilon)\Phi_i(\epsilon) - \sum_{j\neq i}^{n} c_j(\Psi_j(\epsilon)\Phi_i(\epsilon) + \Phi_j(\epsilon)\Psi_i(\epsilon)), \end{aligned} \tag{22c}$$

$$\begin{aligned}\frac{dF(\epsilon)}{d\epsilon} =& -2c_i(F_i(\epsilon)\Psi_i(\epsilon)+G_i(\epsilon)\Phi_i(\epsilon))\\ &-\sum_{j\neq i}^{n}c_j(F_j(\epsilon)\Psi_i(\epsilon)+G_j(\epsilon)\Phi_i(\epsilon)+F_i(\epsilon)\Psi_j(\epsilon)+G_i(\epsilon)\Phi_j(\epsilon)),\\ \frac{dG(\epsilon)}{d\epsilon} =& -2c_i(F_i(\epsilon)\Phi_i(\epsilon)+G_i(\epsilon)\Psi_i(\epsilon))\\ &-\sum_{j\neq i}^{n}c_j(F_i(\epsilon)\Phi_j(\epsilon)+G_i(\epsilon)\Psi_j(\epsilon)+F_j(\epsilon)\Phi_i(\epsilon)+G_j(\epsilon)\Psi_i(\epsilon)),\end{aligned} \tag{22d}$$

$$\begin{aligned}\frac{dH(\epsilon)}{d\epsilon} =& -2c_i(F_i^2(\epsilon)+G_i^2(\epsilon))-2c_i(H_i(\epsilon)\Psi_i(\epsilon)+K_i(\epsilon)\Phi_i(\epsilon))\\ &-\sum_{j\neq i}^{n}c_j(H_i(\epsilon)\Psi_j(\epsilon)+K_i(\epsilon)\Phi_j(\epsilon)+2F_j(\epsilon)F_i(\epsilon)+2G_j(\epsilon)G_i(\epsilon)+H_j(\epsilon)\Psi_i(\epsilon)+K_j(\epsilon)\Phi_i(\epsilon)),\\ \frac{dK(\epsilon)}{d\epsilon} =& -4c_iF_i(\epsilon)G_i(\epsilon)-2c_i(H_i(\epsilon)\Phi_i(\epsilon)+K_i(\epsilon)\Psi_i(\epsilon))\\ &-\sum_{j\neq i}^{n}c_j(H_i(\epsilon)\Phi_j(\epsilon)+K_i(\epsilon)\Psi_j(\epsilon)+2F_j(\epsilon)G_i(\epsilon)+2G_j(\epsilon)F_i(\epsilon)+H_j(\epsilon)\Phi_i(\epsilon)+K_j(\epsilon)\Psi_i(\epsilon)),\end{aligned} \tag{22e}$$

$$\begin{aligned}&P(0)=p,\ \ Q(0)=q,\ \ R(0)=r,\ \ S(0)=s,\ \ \Psi_i(0)=\psi_i,\ \ \Phi_i(0)=\phi_i,\\ &F_i(0)=f_i,\ \ G_i(0)=g_i,\ \ H_i(0)=h_i,\ \ K_i(0)=k_i,\end{aligned} \tag{22f}$$

$i=1,2,\ldots,n$.

Then, one can get the following N-th Bäcklund transformation for the extended system Equation (15) by solving Equation (22).

Theorem 3. *If $\{p,q,r,s,f_i,g_i,h_i,k_i,\psi_i,\phi_i\}$ is a solution of the coupled system Equation (15), then so is $\{P(\epsilon),Q(\epsilon),R(\epsilon),S(\epsilon),F_i(\epsilon),G_i(\epsilon),H_i(\epsilon),K_i(\epsilon),\Psi_i(\epsilon),\Phi_i(\epsilon)\}$, $(i=1,2,\ldots,n)$, where*

$$P(\epsilon)=p-\frac{1}{4}(\ln(M+N)+\ln(M-N))_x,\quad Q(\epsilon)=q-\frac{1}{4}(\ln(M+N)-\ln(M-N))_x, \tag{23a}$$

$$R(\epsilon)=r-\frac{\beta}{4}(\ln(M+N)+\ln(M-N))_{xx},\quad S(\epsilon)=s-\frac{\beta}{4}(\ln(M+N)-\ln(M-N))_{xx}, \tag{23b}$$

$$\Psi_i(\epsilon)=\frac{N_iN-M_iM}{M^2-N^2},\quad \Phi_i(\epsilon)=\frac{M_iN-N_iM}{M^2-N^2}, \tag{23c}$$

$$F_i(\epsilon)=\Psi_{i,x}(\epsilon),\quad G_i(\epsilon)=\Phi_{i,x}(\epsilon), \tag{23d}$$

$$H_i(\epsilon)=\Psi_{i,xx}(\epsilon),\quad K_i(\epsilon)=\Phi_{i,xx}(\epsilon), \tag{23e}$$

where

$$M=\begin{vmatrix} c_1\epsilon\psi_1+1 & c_1\epsilon A_{1,2} & \cdots & c_1\epsilon A_{1,j} & \cdots & c_1\epsilon A_{1,n}\\ c_2\epsilon A_{1,2} & c_2\epsilon\psi_2+1 & \cdots & c_2\epsilon A_{2,j} & \cdots & c_2\epsilon A_{2,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ c_j\epsilon A_{1,j} & c_j\epsilon A_{2,j} & \cdots & c_j\epsilon\psi_j+1 & \cdots & c_j\epsilon A_{j,n}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ c_n\epsilon A_{1,n} & c_n\epsilon A_{2,n} & \cdots & c_n\epsilon A_{j,n} & \cdots & c_n\epsilon\psi_n+1 \end{vmatrix},$$

$$N = \begin{vmatrix} c_1\epsilon\phi_1+1 & c_1\epsilon B_{1,2} & \cdots & c_1\epsilon B_{1,j} & \cdots & c_1\epsilon B_{1,n} \\ c_2\epsilon B_{1,2} & c_2\epsilon\phi_2+1 & \cdots & c_2\epsilon B_{2,j} & \cdots & c_2\epsilon B_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_j\epsilon B_{1,j} & c_j\epsilon B_{2,j} & \cdots & c_j\epsilon\phi_j+1 & \cdots & c_j\epsilon B_{j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_n\epsilon B_{1,n} & c_n\epsilon B_{2,n} & \cdots & c_n\epsilon B_{j,n} & \cdots & c_n\epsilon\phi_n+1 \end{vmatrix},$$

$$M_i = \begin{vmatrix} c_1\epsilon\psi_1+1 & c_1\epsilon A_{1,2} & \cdots & c_1\epsilon A_{1,i-1} & c_1\epsilon A_{1,i} & c_1\epsilon A_{1,i+1} & \cdots & c_1\epsilon A_{1,n} \\ c_2\epsilon A_{1,2} & c_2\epsilon\psi_2+1 & \cdots & c_2\epsilon A_{2,i-1} & c_2\epsilon A_{2,i} & c_2\epsilon A_{2,i+1} & \cdots & c_2\epsilon A_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1}\epsilon A_{1,i-1} & c_{i-1}\epsilon A_{2,i-1} & \cdots & c_{i-1}\epsilon\psi_{i-1}+1 & c_{i-1}\epsilon A_{i-1,i} & c_{i-1}\epsilon A_{i-1,i+1} & \cdots & c_{i-1}\epsilon A_{i-1,n} \\ A_{1,i} & A_{2,i} & \cdots & A_{i,i-1} & \psi_i & A_{i,i+1} & \cdots & A_{i,n} \\ c_{i+1}\epsilon A_{1,i+1} & c_{i+1}\epsilon A_{2,i+1} & \cdots & c_{i+1}\epsilon A_{i-1,i+1} & c_{i+1}\epsilon A_{i,i+1} & c_{i+1}\epsilon\psi_{i+1}+1 & \cdots & c_{i+1}\epsilon A_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ c_n\epsilon A_{1,n} & c_n\epsilon A_{2,n} & \cdots & c_n\epsilon A_{i-1,n} & c_n\epsilon A_{i,n} & c_n\epsilon A_{i+1,n} & \cdots & c_n\epsilon\psi_n+1 \end{vmatrix},$$

$$N_i = \begin{vmatrix} c_1\epsilon\phi_1+1 & c_1\epsilon B_{1,2} & \cdots & c_1\epsilon B_{1,i-1} & c_1\epsilon B_{1,i} & c_1\epsilon B_{1,i+1} & \cdots & c_1\epsilon B_{1,n} \\ c_2\epsilon B_{1,2} & c_2\epsilon\phi_2+1 & \cdots & c_2\epsilon B_{2,i-1} & c_2\epsilon B_{2,i} & c_2\epsilon B_{2,i+1} & \cdots & c_2\epsilon B_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{i-1}\epsilon B_{1,i-1} & c_{i-1}\epsilon B_{2,i-1} & \cdots & B_{i-1}\epsilon\phi_{i-1}+1 & c_{i-1}\epsilon B_{i-1,i} & c_{i-1}\epsilon B_{i-1,i+1} & \cdots & c_{i-1}\epsilon B_{i-1,n} \\ B_{1,i} & B_{2,i} & \cdots & B_{i,i-1} & \phi_i & B_{i,i+1} & \cdots & B_{i,n} \\ c_{i+1}\epsilon B_{1,i+1} & c_{i+1}\epsilon B_{2,i+1} & \cdots & c_{i+1}\epsilon B_{i-1,i+1} & c_{i+1}\epsilon B_{i,i+1} & c_{i+1}\epsilon\phi_{i+1}+1 & \cdots & c_{i+1}\epsilon B_{i+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ c_n\epsilon B_{1,n} & c_n\epsilon B_{2,n} & \cdots & c_n\epsilon B_{i-1,n} & c_n\epsilon B_{i,n} & c_n\epsilon B_{i+1,n} & \cdots & c_n\epsilon\phi_n+1 \end{vmatrix},$$

with

$$A_{i,j} = \frac{1}{2}(\psi_i\psi_j + \phi_i\phi_j + \psi_i\phi_j + \phi_i\psi_j)^{\frac{1}{2}} + \frac{1}{2}(\psi_i\psi_j + \phi_i\phi_j - \psi_i\phi_j - \phi_i\psi_j)^{\frac{1}{2}},$$
$$B_{i,j} = \frac{1}{2}(\psi_i\psi_j + \phi_i\phi_j + \psi_i\phi_j + \phi_i\psi_j)^{\frac{1}{2}} - \frac{1}{2}(\psi_i\psi_j + \phi_i\phi_j - \psi_i\phi_j - \phi_i\psi_j)^{\frac{1}{2}}.$$

From any seed solution of the strongly coupled B–B system, one can get an infinite number of new solutions because n is an arbitrary positive integer.

4. Conclusions

To sum up, a new strongly coupled B–B system was first introduced. Then, the Schwarzian form of the strongly coupled B–B system was obtained by using the truncated Painlevé expansion, and the corresponding residual was the nonlocal symmetry. To localize the residual symmetry, we introduced a suitable enlarged system. According to Lie's first theorem, the ralated finite transformation was found. Further, the N-th Bäcklund transformation of the strongly coupled Burgers system was obtained by localizing the linear superposition of multiple residual symmetries, and the N-th Bäcklund transformation was expressed by determinants in a compact form.

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References

1. Gardner, C.S.; Greene, J.M.; Kruskal, M.D.; Miura, R.M. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett.* **1967**, *19*, 1095. [CrossRef]
2. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2012.
3. Bluman, G.W.; Cheviakov, A.F.; Anco, S.C. *Applications of Symmetry Methods to Partial Differential Equations*; Springer: New York, NY, USA, 2010.
4. Lie, S. *Vorlesungen ber Differentialgleichungen mit Bekannten Infinitesimalen Transformationen*; BG Teubner: Leipzig, Germany, 1891.
5. Huang, L.; Chen, Y. Nonlocal symmetry and similarity reductions for the Drinfeld-Sokolov-Satsuma-Hirota system. *Appl. Math. Lett.* **2017**, *64*, 177. [CrossRef]
6. Li, Y.; Chen, J.; Chen, Y.; Lou, S.Y. Darboux transformations via lie point symmetries: Kdv equation. *Chin. Phys. Lett.* **2014**, *31*, 010201. [CrossRef]
7. Jia, M. Lie point symmetry algebras and finite transformation groups of the general Broer-Kaup system. *Chin. Phys.* **2007**, *16*, 1534.
8. Zhang, C.; Chen, A. Bilinear form and new multi-soliton solutions of the classical Boussinesq–Burgers system. *Appl. Math. Lett.* **2016**, *58*, 133. [CrossRef]
9. Wang, Y. CTE method to the interaction solutions of Boussinesq–Burgers equations. *Appl. Math. Lett.* **2014**, *38*, 100. [CrossRef]
10. Zhang, Y.; Feng, B. A few Lie algebras and their applications for generating integrable hierarchies of evolution types. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 3045. [CrossRef]
11. Zhang, Y. Similarity solutions and the computation formulas of a nonlinear fractional-order generalized heat equation. *Mod. Phys. Lett. B* **2019**, *33*, 1950122. [CrossRef]
12. Liu, X.; Yu, J.; Ren, B. Bäcklund transformations for the Burgers equation via localization of residual symmetries. *Chin. Phys. B* **2014**, *23*, 110203. [CrossRef]
13. Liu, X.; Yu, J.; Lou, Z. New Bäcklund transformations of the (2+1)-dimensional Burgers system related to residual symmetry. *Eur. Phys. J. Plus* **2018**, *133*, 89. [CrossRef]
14. Esipov, S.E. Coupled Burgers equations: A model of polydispersive sedimentation. *Phys. Rev. E* **1995**, *52*, 3711. [CrossRef] [PubMed]
15. Wang, P.; Tian, B.; Liu, W.J.; Lu, X.; Jiang, Y. Lax pair, Bäcklund transformation and multi-soliton solutions for the Boussinesq–Burgers equations from shallow water waves. *Appl. Math. Comput.* **2011**, *218*, 1726. [CrossRef]
16. Weiss, J.; Tabor, M.; Carnevale, G. The Painlevé property for partial differential equations. *J. Math. Phys.* **1983**, *24*, 522. [CrossRef]
17. Conte, R. Invariant Painlevé analysis of partial differential equations. *Phys. Lett. A* **1989**, *140*, 383. [CrossRef]
18. Gao, X.; Lou, S.; Tang, X. Bosonization, singularity analysis, nonlocal symmetry reductions and exact solutions of supersymmetric KdV equation. *J. High Energy Phys.* **2013**, *2013*, 29. [CrossRef]
19. Lou, S. Residual symmetries and Bäcklund transformations. *arXiv* **2013**, arXiv:1308.1140.
20. Lou, S.; Hu, X.; Chen, Y. Nonlocal symmetries related to Bäcklund transformation and their applications. *J. Phys. A Math. Theor.* **2012**, *45*, 155209. [CrossRef]
21. Zhu, N.; Liu, Z.; Zhao, K. On the Boussinesq–Burgers equations driven by dynamic boundary conditions. *J. Differ. Equ.* **2018**, *264*, 2287. [CrossRef]
22. Zeng, Y.; Ma, W.; Lin, R. Integration of the soliton hierarchy with self-consistent sources. *J. Math. Phys.* **2000**, *41*, 5453. [CrossRef]
23. Zhu, Q.; Wang, Q.; Fu, J.; Zhang, Z. New second-order finite difference scheme for the problem of contaminant in groundwater flow. *J. Appl. Math.* **2012**, *2012*, 575493. [CrossRef]
24. Zhu, Q.; Fei, J.; Ma, Z. Residual symmetry analysis for novel localized excitations of a (2+1)-dimensional general Korteweg-de Vries system. *Zeitschrift für Naturforschung A* **2017**, *72*, 795. [CrossRef]

25. Yan, Z. Nonlocal general vector nonlinear Schrödinger equations: Integrability, PT symmetribility, and solutions. *Appl. Math. Lett.* **2016**, *62*, 101. [CrossRef]
26. Yan, Z. Integrable PT-symmetric local and nonlocal vector nonlinear Schrödinger equations: A unified two-parameter model. *Appl. Math. Lett.* **2015**, *47*, 61–68. [CrossRef]
27. Chen, J.; Chen, Y.; Feng, B.F.; Maruno, K.I. Rational solutions to two-and one-dimensional multicomponent Yajima-Oikawa systems. *Phys. Lett. A* **2015**, *379*, 1510. [CrossRef]
28. Fei, J.; Zhu, Q.; Ma, Z. Residual symmetry and Bäcklund transformations of model equations for shallow water waves. *Waves Random Complex Media* **2019**, *17*, 1–8. [CrossRef]
29. Fei, J.; Cao, W. Explicit soliton-cnoidal wave interaction solutions for the (2+1)-dimensional negative-order breaking soliton equation. *Waves Random Complex Media* **2018**, *10*, 1. [CrossRef]
30. Strachan, I.A.B.; Zuo, D. Integrability of the Frobenius algebra-valued Kadomtsev-Petviashvili hierarchy. *J. Math. Phys.* **2015**, *56*, 113509. [CrossRef]
31. Zuo, D. The Frobenius-Virasoro algebra and Euler equations. *J. Geom. Phys.* **2014**, *86*, 203. [CrossRef]
32. Li, C.; He, J. The extended Z_N-Toda hierarchy. *Theor. Math. Phys.* **2015**, *185*, 1614. [CrossRef]
33. Li, C. Gauge transformation and symmetries of the commutative multicomponent BKP hierarchy. *J. Phys. A Math. Theor.* **2015**, *49*, 015203. [CrossRef]
34. Wang, H.; Li, C. Bäcklund transformation of Frobenius Painlevé equations. *Mod. Phys. Lett. B* **2018**, *32*, 1850181. [CrossRef]

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