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A Note on the Degenerate Type of Complex Appell Polynomials

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Abstract: In this paper, complex Appell polynomials and their degenerate-type polynomials are considered as an extension of real-valued polynomials. By treating the real value part and imaginary part separately, we obtained useful identities and general properties by convolution of sequences. To justify the obtained results, we show several examples based on famous Appell sequences such as Euler polynomials and Bernoulli polynomials. Further, we show that the degenerate types of the complex Appell polynomials are represented in terms of the Stirling numbers of the first kind.

Keywords: Appell polynomials; complex Appell polynomials; degenerate type of Appell polynomials; Stirling numbers

1. Introduction

Appell polynomials are very frequently used in various problems in pure and applied mathematics related to functional equations in differential equations, approximation theories, interpolation problems, summation methods, quadrature rules, and their multidimensional extensions (see [1–11]). Also, for a further general account of the study in Appell polynomials, a number of applications can be found (see [4] and references therein).

The sequence of complex Appell polynomials $\{A_n(z)\}_{n=0}^{\infty}$ can be obtained by either of the following equivalent conditions:

$$\frac{d}{dz} A_n(z) = n A_{n-1}(z), \quad A_0(z) \neq 0, \quad z = x + iy \in \mathbb{C}, \quad n \in \mathbb{N}, \quad (1)$$

or the following formal equality

$$A(t)e^{zt} = \sum_{n=0}^{\infty} A_n(z) \frac{t^n}{n!}, \quad (2)$$

where $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$, ($a_0 \neq 0$) is a formal power series with coefficients a_n called by Appell numbers.

There is a large number of classical sequences of polynomials in Appell polynomials and a list of famous classical Appell polynomials is shown as the Bernoulli polynomials, the Euler polynomials, the Hermite polynomials, the Genocchi polynomials, the generalized Bernoulli polynomials, the generalized Euler polynomials, etc. (see [2,3] for more examples).

Many authors have obtained useful results by considering the Appell polynomials of a complex variable by splitting complex-valued polynomials into real and imaginary values: Analytic properties of the sequence of complex Hermite polynomials are studied in [12] and their orthogonality relation is

established in [13,14]. Also, authors in [15] show the representation of the real and imaginary parts in the complex Appell polynomials in terms of the Chebyshev polynomials of the first and second kind.

With the help of the research of complex Appell polynomials, their degenerate versions have been also extensively studied looking for useful identities, as well as their related properties, since Carlitz introduced degenerate formulas of special numbers and polynomials in [16,17]. Further, there have been studies of various degenerate numbers and polynomials by means of degenerate types of generating functions, combinatorial methods, umbral calculus, and differential equations. For example, several authors have studied the degenerate types of Appell polynomials, such as Bernoulli and Euler polynomials (see [18–23]) and their complex version [24], degenerate gamma functions, degenerate Laplace transforms [25], and their modified ones [26].

The research for degenerate versions of known special numbers and polynomials brought many valuable identities and properties into mathematics. In the future, we hope the results of the degenerate types of complex Appell polynomials can be further applicable to many different problems in various areas.

The aim of this paper is to introduce Appell polynomials of a complex variable and their degenerate formulas and provide some of their properties and examples. Also, we study some further properties of the degenerate type of Appell polynomials and show that degenerate cosine- and sine-Appell polynomials can be expressed by the Stirling numbers of the first kind.

The paper is organized as follows. In Section 2, we recall the complex Appell polynomials with cosine- and sine-Appell polynomials and present some properties and their relations. Section 3 introduces the degenerate version of complex Appell polynomials and provides some expressions, properties, and examples. Finally, Section 4 contains the conclusion of this study.

2. Complex Appell Polynomials

In this section, we introduce the cosine-Appell polynomials and sine-Appell polynomials by splitting complex Appell polynomials into real \Re and imaginary \Im parts, and present some properties, which can apply to any Appell-type polynomial, as mentioned in the introduction.

Definition 1. For $n \in \mathbb{N} \cup \{0\}$, we define the cosine-Appell polynomials $A_n^{(c)}(x, y)$ and the sine-Appell polynomials $A_n^{(s)}(x, y)$ by the generating functions respectively,

$$\begin{aligned} A(t)e^{xt} \cos(yt) &= \sum_{n=0}^{\infty} A_n^{(c)}(x, y) \frac{t^n}{n!}, \\ A(t)e^{xt} \sin(yt) &= \sum_{n=0}^{\infty} A_n^{(s)}(x, y) \frac{t^n}{n!}. \end{aligned} \tag{3}$$

The definition in (3) with the fact (2) implies that

$$A_n(z) = \sum_{k=0}^n \binom{n}{k} a_k z^{n-k} = A_n^{(c)}(x, y) + iA_n^{(s)}(x, y). \tag{4}$$

Also, it is easily observed that for $\bar{z} = x - iy$,

$$\begin{aligned} \sum_{n=0}^{\infty} A_n(z) \frac{t^n}{n!} &= A(t)e^{zt} = A(t)e^{xt} (\cos(yt) + i \sin(yt)), \\ \sum_{n=0}^{\infty} A_n(\bar{z}) \frac{t^n}{n!} &= A(t)e^{\bar{z}t} = A(t)e^{xt} (\cos(yt) - i \sin(yt)), \end{aligned}$$

which show

$$A(t)e^{xt} \cos(yt) = \sum_{n=0}^{\infty} \left(\frac{A_n(z) + A_n(\bar{z})}{2} \right) \frac{t^n}{n!},$$

$$A(t)e^{xt} \sin(yt) = \sum_{n=0}^{\infty} \left(\frac{A_n(z) - A_n(\bar{z})}{2i} \right) \frac{t^n}{n!},$$

and

$$A_n^{(c)}(x, y) = \frac{A_n(z) + A_n(\bar{z})}{2}, \quad A_n^{(s)}(x, y) = \frac{A_n(z) - A_n(\bar{z})}{2i}.$$

Further, noting that $A_n^{(c)}(x, y) = \Re(A_n(z))$ and $A_n^{(s)}(x, y) = \Im(A_n(z))$, $z = x + iy$ for $n \geq 0$, it can be checked that the cosine-Appell polynomials and the sine-Appell polynomials satisfy the following properties:

- (i) $A_n(\bar{z}) = \overline{A_n(z)}$,
- (ii) $A_n(z) = \sum_{k=0}^n \binom{n}{k} (iy)^{n-k} A_k(x)$,
- (iii) $A_n(z) = \sum_{k=0}^n \binom{n}{k} A_{n-k}(x) i^k y^k$ and $A_n(\bar{z}) = \sum_{k=0}^n \binom{n}{k} A_{n-k}(x) (-1)^k i^k y^k$.
- (iv) $A_n^{(c)}(x, y) = A_n^{(c)}(x, -y)$,
- (v) $A_n^{(c)}(x, 0) = A_n(x)$,
- (vi) $A_n^{(s)}(x, 0) = 0$.

The above properties are easily proved by the comparison of coefficients after polynomial expansion of the generating functions and we omit the proofs here for lack of space.

We next investigate further properties of complex Appell polynomials.

Theorem 1. For $n, m \in \mathbb{N} \cup \{0\}$, $z = x + iy$, the following product of complex Appell polynomials is established:

$$A_n(z) \cdot A_m(\bar{z}) = \left(A_n^{(c)}(x, y) A_m^{(c)}(x, -y) - A_n^{(s)}(x, y) A_m^{(s)}(x, -y) \right) + i \left(A_n^{(c)}(x, y) A_m^{(s)}(x, -y) + A_n^{(s)}(x, y) A_m^{(c)}(x, -y) \right).$$

Proof. The product of the identities for $A_n(z)$ and $A_m(\bar{z})$ from (4) shows the desired identity. \square

Corollary 1. For $n \in \mathbb{N} \cup \{0\}$, $z = x + iy$, we have the identity,

$$A_n(z) \cdot A_n(\bar{z}) = \left(A_n^{(c)}(x, y) \right)^2 + \left(A_n^{(s)}(x, y) \right)^2.$$

Lemma 1. For $n \in \mathbb{N} \cup \{0\}$, $z = x + iy$, the real \Re and imaginary \Im parts of complex Appell polynomials satisfy that

$$\Re(A_n(z)) = A_n^{(c)}(x, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k y^{2k} A_{n-2k}(x), \tag{5}$$

$$\Im(A_n(z)) = A_n^{(s)}(x, y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k y^{2k+1} A_{n-2k-1}(x). \tag{6}$$

Proof. Considering that $\cos(yt) = \frac{1}{2}(e^{iyt} + e^{-iyt})$, we have

$$A(t)e^{xt} \cos(yt) = A(t)e^{xt} \left(\frac{e^{iyt} + e^{-iyt}}{2} \right).$$

Thus, using identity for the binomials convolution of sequences $\{A_n(x)\}_{n=0}^\infty$ and $\{\frac{1+(-1)^n}{2}t^n y^n\}_{n=0}^\infty$, we get identity (5). Similarly, one can show identity (6) by considering $\sin(yt) = \frac{1}{2i}(e^{iyt} - e^{-iyt})$. \square

Remark 1. Note that the sequences $\{A_n^{(c)}(x, y)\}_{n=0}^\infty$ and $\{A_n^{(s)}(x, y)\}_{n=0}^\infty$ can be explicitly determined when $A(t)$ is given, namely, complex Euler polynomials $E_n(z)$ for $A(t) = \frac{2}{e^t+1}$ and complex Bernoulli polynomials $B_n(z)$ for $A(t) = \frac{t}{e^t-1}$,

$$\frac{2}{e^t+1} e^{zt} = \sum_{n=0}^\infty E_n(z) \frac{t^n}{n!} \tag{7}$$

and

$$\frac{t}{e^t-1} e^{zt} = \sum_{n=0}^\infty B_n(z) \frac{t^n}{n!}, \tag{8}$$

respectively. For example, the first four consecutive polynomials are listed as in Tables 1 and 2.

Table 1. Expressions of the first four $E_n^{(c)}(x, y)$ and $E_n^{(s)}(x, y)$.

n	0	1	2	3
$E_n^{(c)}(x, y)$	1	$x - \frac{1}{2}$	$x^2 - x - y^2$	$\frac{1}{4}(2x - 1)(2x^2 - 2x - 6y^2 - 1)$
$E_n^{(s)}(x, y)$	0	y	$2xy - y$	$3x^2y - 3xy - y^3$

Table 2. Expressions of the first four $B_n^{(c)}(x, y)$ and $B_n^{(s)}(x, y)$.

n	0	1	2	3
$B_n^{(c)}(x, y)$	1	$x - \frac{1}{2}$	$\frac{1}{6}(6x^2 - 6x - 6y^2 + 1)$	$\frac{1}{2}(2x - 1)(x^2 - x - 3y^2)$
$B_n^{(s)}(x, y)$	0	y	$2xy - y$	$\frac{1}{2}(6x^2y - 6xy - 2y^3 + y)$

Lemma 2. Let n be a nonnegative integer and $z = x + iy$. Then, the complex Appell polynomials satisfy the following identities:

$$\sum_{k=0}^n \binom{n}{k} A_k(z) A_{n-k}(\bar{z}) = \sum_{k=0}^n \binom{n}{k} A_k(x) A_{n-k}(x), \quad \bar{z} = x - iy,$$

$$\sum_{k=0}^n \binom{n}{k} A_k(z) A_{n-k}(-\bar{z}) = \sum_{k=0}^n \binom{n}{k} A_k(iy) A_{n-k}(iy).$$

Proof. The right side of the first identity we get directly by the 2-fold binomial convolution of sequence $\{A_n(x)\}_{n=0}^\infty$ (thus using the square of exponential generating function $(A(t)e^{xt})^2$). Alternatively, rewriting $(A(t)e^{xt})^2$ as the product of two exponential generating functions by this way: $(A(t)e^{(x+iy)t})(A(t)e^{(x-iy)t})$, we obtain the left side of the first identity by the binomial convolution of sequences $\{A_n(z)\}$ and $\{A_n(\bar{z})\}$. The second identity we obtain in a similar argument. \square

Remark 2. In particular, if we consider $A(t) = 1$ in Definition 1, Equation (3) shows that

$$\Re(A(t)e^{zt}) = \Re(e^{zt}) = e^{xt} \cos(yt) = \sum_{n=0}^{\infty} C_n(z) \frac{t^n}{n!}$$

and

$$\Im(A(t)e^{zt}) = \Im(e^{zt}) = e^{xt} \sin(yt) = \sum_{n=0}^{\infty} S_n(z) \frac{t^n}{n!}$$

for some sequences $C_n(z)$ and $S_n(z)$. As $A_n(x) = x^n$ for $A(t) = 1$, we have from identities (5) and (6)

$$C_n(z) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} (-1)^m y^{2m} x^{n-2m},$$

$$S_n(z) = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2m+1} (-1)^m y^{2m+1} x^{n-2m-1}.$$

The sequences $C_n(z)$ and $S_n(z)$ satisfy the following formulas.

Theorem 2. For $z_1, z_2 \in \mathbb{C}$, the following identities are established:

$$C_n(z_1 \pm z_2) = \sum_{k=0}^n \binom{n}{k} (C_{n-k}(z_1)C_k(\pm z_2) - S_{n-k}(z_1)S_k(\pm z_2)), \tag{9}$$

$$S_n(z_1 \pm z_2) = \sum_{k=0}^n \binom{n}{k} (S_{n-k}(z_1)C_k(\pm z_2) + C_{n-k}(z_1)S_k(\pm z_2)). \tag{10}$$

Proof. If we denote $z_l = x_l + iy_l \in \mathbb{C}$, ($l = 1, 2$), the trigonometric identities yield that

$$\Re(e^{(z_1 \pm z_2)t}) = e^{(x_1 \pm x_2)t} \cos((y_1 \pm y_2)t) = e^{x_1 t} \cos(y_1 t) e^{\pm x_2 t} \cos(\pm y_2 t) - e^{x_1 t} \sin(y_1 t) e^{\pm x_2 t} \sin(\pm y_2 t), \tag{11}$$

$$\Im(e^{(z_1 \pm z_2)t}) = e^{(x_1 \pm x_2)t} \sin((y_1 \pm y_2)t) = e^{x_1 t} \sin(y_1 t) e^{\pm x_2 t} \cos(\pm y_2 t) + e^{x_1 t} \cos(y_1 t) e^{\pm x_2 t} \sin(\pm y_2 t). \tag{12}$$

As the right hand side of (11) exponential generating functions for $C_n(z_1 \pm z_2)$, we have the first formula (9) by the difference between the binomial convolutions of sequences $\{C_n(z_1)\}_{n=0}^{\infty}$ and $\{C_n(\pm z_2)\}_{n=0}^{\infty}$ and the binomial convolutions of sequences $\{S_n(z_1)\}_{n=0}^{\infty}$ and $\{S_n(\pm z_2)\}_{n=0}^{\infty}$. The second formula (10) we get in a similar way by using (12). \square

The following two subsequent theorems show that the complex Appell polynomials can be split into $C_n(z)$ and $S_n(z)$ and their relations.

Theorem 3. For $n \in \mathbb{N} \cup \{0\}$, the Appell-type polynomials satisfy the following relations with $C_n(z)$ and $S_n(z)$,

$$A_n(z) = \sum_{k=0}^n \binom{n}{k} a_{n-k} (C_k(z) + iS_k(z)). \tag{13}$$

Proof. As $A(t)e^{xt} (\cos(yt) + i \sin(yt))$ exponential generating functions for $A_n(z)$, $z = x + iy$, we have directly equation (13) by the binomial convolution of sequences $\{a_n\}_{n=0}^{\infty}$ and $\{C_n(z) + iS_n(z)\}_{n=0}^{\infty}$. \square

Theorem 4. For $k > 0$, the cosine-Appell polynomials and the sine-Appell polynomials satisfy the following properties,

$$\begin{aligned}
 A_n^{(c)}(k \pm x, y) &= \sum_{l=0}^n \binom{n}{l} A_l(k) (\pm 1)^{n-l} C_{n-l}(x + iy) \\
 &= \sum_{l=0}^n \binom{n}{l} \left(\sum_{j=0}^l \binom{l}{j} a_j k^{l-j} \right) (\pm 1)^{n-l} C_{n-l}(x + iy),
 \end{aligned}
 \tag{14}$$

$$\begin{aligned}
 A_n^{(s)}(k \pm x, y) &= \sum_{l=0}^n \binom{n}{l} A_l(k) (\pm 1)^{n-l} S_{n-l}(x + iy) \\
 &= \sum_{l=0}^n \binom{n}{l} \left(\sum_{j=0}^l \binom{l}{j} a_j k^{l-j} \right) (\pm 1)^{n-l} S_{n-l}(x + iy).
 \end{aligned}
 \tag{15}$$

Proof. As $A(t)e^{kt}e^{\pm xt} \cos(\pm yt)$ exponential generating function for $A_n^{(c)}(k \pm x, y)$, we get the first line in formula (14) by the binomial convolution of sequences $\{A_n(k)\}_{n=0}^\infty$ and $\{C_n(x + iy)(\pm 1)^n\}_{n=0}^\infty$. The second line of (14) follows from formula (5). Similarly, identity (15) can be proved. \square

Next, the derivatives of $A_n^{(c)}(x, y)$ and $A_n^{(s)}(x, y)$ show that the sequence $\{A_n(z)\}_{n=0}^\infty$ satisfies the condition (1).

Theorem 5. For all $n \in \mathbb{N} \cup \{0\}$ and $z = x + iy$, the sequence $\{A_n(z)\}_{n=0}^\infty$ is verified by a sequence of complex Appell polynomials in terms of $A_n^{(c)}(x, y)$ and $A_n^{(s)}(x, y)$.

Proof. Noting that $\frac{\partial}{\partial x} A_0^{(c)}(x, y) = 0$, the derivative of the cosine-Appell polynomials satisfies

$$\sum_{n=1}^\infty \frac{\partial}{\partial x} A_n^{(c)}(x, y) \frac{t^n}{n!} = \frac{\partial}{\partial x} (A(t)e^{xt} \cos(yt)) = A(t)te^{xt} \cos(yt) = \sum_{n=1}^\infty (nA_{n-1}^{(c)}(x, y)) \frac{t^n}{n!},$$

which implies that

$$\frac{\partial}{\partial x} A_n^{(c)}(x, y) = nA_{n-1}^{(c)}(x, y).
 \tag{16}$$

Similarly, it can be seen that

$$\frac{\partial}{\partial x} A_n^{(s)}(x, y) = nA_{n-1}^{(s)}(x, y), \quad \frac{\partial}{\partial y} A_n^{(c)}(x, y) = -nA_{n-1}^{(s)}(x, y), \quad \frac{\partial}{\partial y} A_n^{(s)}(x, y) = nA_{n-1}^{(c)}(x, y).
 \tag{17}$$

By using (16) and (17), it is easily shown that

$$\frac{d}{dz} A_n(z) = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (A_n^{(c)}(x, y) + iA_n^{(s)}(x, y)) = n (A_{n-1}^{(c)}(x, y) + iA_{n-1}^{(s)}(x, y)) = nA_{n-1}(z).$$

\square

3. Degenerate Type of Complex Appell Polynomials

In this section, we introduce the degenerate type of complex Appell polynomials based on the non-degenerate ones given in Definition 1 and study some of their properties. To do this, we first recall and introduce several definitions, some notations, and basic properties.

Let us recall several definitions: the degenerate exponential function $e_\lambda^x(t)$ for $\lambda \in \mathbb{R} \setminus \{0\}$ is defined by (see [27–29])

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}. \quad (18)$$

It is noted that $\lim_{\lambda \rightarrow 0} e_\lambda^x(t) = e^{xt}$.

The falling factorial sequence is defined by (see [30–34])

$$(x)_0 = 1, \quad (x)_n = x(x-1) \cdots (x-n+1), \quad (n \geq 1). \quad (19)$$

Similarly, the λ -falling factorial sequence is given by (see [27])

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda) \cdots (x-(n-1)\lambda), \quad (n \geq 1). \quad (20)$$

Then, it can be easily checked that the factorial sequences in (19) and (20) satisfy $\lim_{\lambda \rightarrow 1} (x)_{n,\lambda} = (x)_n$ and $\lim_{\lambda \rightarrow 0} (x)_{n,\lambda} = x^n$ for all $n \geq 0$.

Further, the λ -binomial expansion is defined by (see [27])

$$e_\lambda^x(t) := (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{l=0}^{\infty} \binom{x}{l}_\lambda t^l = \sum_{l=0}^{\infty} (x)_{l,\lambda} \frac{t^l}{l!}, \quad (21)$$

in which the λ -binomial coefficient satisfies

$$\binom{x}{l}_\lambda = \frac{(x)_{l,\lambda}}{l!} = \frac{x(x-\lambda)(x-2\lambda) \cdots (x-(l-1)\lambda)}{l!}.$$

Definition 2. Let us assume that $A_\lambda(t) := \sum_{k=0}^{\infty} a_{k,\lambda} \frac{t^k}{k!}$ for some sequence $\{a_{k,\lambda}\}_{k \geq 0}$ and satisfy $\lim_{\lambda \rightarrow 0} A_\lambda(t) = A(t)$. Then we define the degenerate type of complex Appell polynomials by the generating function

$$A_\lambda(t) e_\lambda^z(t) = \sum_{n=0}^{\infty} A_n(z; \lambda) \frac{t^n}{n!}, \quad \lambda \in \mathbb{R} \setminus \{0\}. \quad (22)$$

Remark 3. If $A_\lambda(t)$ is considered as $\frac{t}{e_\lambda(t)-1}$ and $\frac{2}{e_\lambda(t)+1}$ for $e_\lambda(t) := e_\lambda^1(t)$, then one can have the sequences $\{E_n(z; \lambda)\}_{n=0}^{\infty}$ and $\{B_n(z; \lambda)\}_{n=0}^{\infty}$ of degenerate types of complex Euler polynomials and complex Bernoulli polynomials

$$\frac{2}{e_\lambda(t)+1} e_\lambda^z(t) = \sum_{n=0}^{\infty} E_n(z; \lambda) \frac{t^n}{n!} \quad (23)$$

and

$$\frac{t}{e_\lambda(t)-1} e_\lambda^z(t) = \sum_{n=0}^{\infty} B_n(z; \lambda) \frac{t^n}{n!}, \quad (24)$$

respectively.

Letting $x = i$ in Equation (18), we find, for $\lambda \in \mathbb{R} \setminus \{0\}$

$$e_\lambda^i(t) = (1 + \lambda t)^{\frac{i}{\lambda}},$$

and we can formulate the following degenerate Euler formula given by

$$e_\lambda^i(t) := \cos_\lambda(t) + i \sin_\lambda(t), \tag{25}$$

where

$$\cos_\lambda(t) := \cos\left(\frac{1}{\lambda} \log(1 + \lambda t)\right), \quad \sin_\lambda(t) := \sin\left(\frac{1}{\lambda} \log(1 + \lambda t)\right).$$

Thus, the complex value of the degenerate exponential function is split into the real and imaginary values.

Note that

$$\lim_{\lambda \rightarrow 0} e_\lambda^i = e^{it}, \quad \lim_{\lambda \rightarrow 0} \cos_\lambda(t) = \cos(t), \quad \lim_{\lambda \rightarrow 0} \sin_\lambda(t) = \sin(t).$$

Similarly, $\forall y \in \mathbb{R}$, we can put

$$e_\lambda^{iy}(t) := \cos_\lambda^{(y)}(t) + i \sin_\lambda^{(y)}(t),$$

where

$$\cos_\lambda^{(y)}(t) := \cos\left(\frac{y}{\lambda} \log(1 + \lambda t)\right), \quad \sin_\lambda^{(y)}(t) := \sin\left(\frac{y}{\lambda} \log(1 + \lambda t)\right), \tag{26}$$

so that

$$\lim_{\lambda \rightarrow 0} e_\lambda^{iy}(t) = \cos(yt) + i \sin(yt).$$

Now, using the degenerate functions (26), we define the following polynomials.

Definition 3. For a nonnegative integer n , let us define the degenerate cosine-Appell polynomials $A_n^{(c)}(x, y; \lambda)$ and the degenerate sine-Appell polynomials $A_n^{(s)}(x, y; \lambda)$ by the generating functions, respectively, as follows:

$$A_\lambda(t) e_\lambda^x(t) \cos_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} A_n^{(c)}(x, y; \lambda) \frac{t^n}{n!}, \quad A_\lambda(t) e_\lambda^x(t) \sin_\lambda^{(y)}(t) = \sum_{n=0}^{\infty} A_n^{(s)}(x, y; \lambda) \frac{t^n}{n!}. \tag{27}$$

From Definitions 2 and 3 with property (21), one can see that for $z = x + iy$,

$$A_n(z; \lambda) = \sum_{l=0}^n \binom{n}{l} a_{l,\lambda}(z)_{n-l,\lambda} = A_n^{(c)}(x, y; \lambda) + i A_n^{(s)}(x, y; \lambda). \tag{28}$$

Also, the following property can be stated.

Lemma 3. For $n \geq 0$ and $z = x + iy$, let $A_n^{(c)}(x, y; \lambda)$ and $A_n^{(s)}(x, y; \lambda)$ be the degenerate cosine-Appell and sine-Appell polynomials defined in (27). Then, we have

$$A_n^{(c)}(x, y; \lambda) = \frac{A_n(z; \lambda) + A_n(\bar{z}; \lambda)}{2}, \quad A_n^{(s)}(x, y; \lambda) = \frac{A_n(z; \lambda) - A_n(\bar{z}; \lambda)}{2i}.$$

Proof. We first note that from (26) and (27), $A_n^{(c)}(x, -y; \lambda) = A_n^{(c)}(x, y; \lambda)$ and $A_n^{(s)}(x, -y; \lambda) = -A_n^{(s)}(x, y; \lambda)$, since $\cos_\lambda^{(y)}(t) = \cos_\lambda^{(-y)}(t)$ and $\sin_\lambda^{(y)}(t) = -\sin_\lambda^{(-y)}(t)$. Thus, $A_n(\bar{z}; \lambda) = \overline{A_n(z; \lambda)}$ from (28), so that the desired identities are easily obtained. \square

Remark 4. It is noted that the sequences $\{A_n^{(c)}(x, y; \lambda)\}_{n=0}^{\infty}$ and $\{A_n^{(s)}(x, y; \lambda)\}_{n=0}^{\infty}$ can be explicitly determined when $A_\lambda(t)$ is specified. For example, for $A_n(z; \lambda) = E_n(z; \lambda)$ and $A_n(z; \lambda) = B_n(z; \lambda)$ as defined in (23) and (24)

respectively, the first four consecutive polynomials can be listed as in Tables 3 and 4. One can check that from Tables 1 and 2, $\lim_{\lambda \rightarrow 0} E_n^{(c)}(x, y; \lambda) = E_n^{(c)}(x, y)$, $\lim_{\lambda \rightarrow 0} E_n^{(s)}(x, y; \lambda) = E_n^{(s)}(x, y)$, $\lim_{\lambda \rightarrow 0} B_n^{(c)}(x, y; \lambda) = B_n^{(c)}(x, y)$, and $\lim_{\lambda \rightarrow 0} B_n^{(s)}(x, y; \lambda) = B_n^{(s)}(x, y)$.

Table 3. Expressions of the first four $E_n^{(c)}(x, y; \lambda)$ and $E_n^{(s)}(x, y; \lambda)$.

n	0	1	2	3
$E_n^{(c)}(x, y; \lambda)$	1	$x - \frac{1}{2}$	$-\lambda x + \frac{1}{2}\lambda + x^2 - x - y^2$	$\lambda^2(2x - 1) - 3\lambda(x^2 - x - y^2) + \frac{1}{4}(2x - 1)(2x^2 - 2x - 6y^2 - 1)$
$E_n^{(s)}(x, y; \lambda)$	0	y	$-y\lambda + 2xy - y$	$2\lambda^2y - \lambda(6x - 3)y + 3x^2y - 3xy - y^3$

Table 4. Expressions of the first four $B_n^{(c)}(x, y; \lambda)$ and $B_n^{(s)}(x, y; \lambda)$.

n	0	1	2	3
$B_n^{(c)}(x, y; \lambda)$	1	$x + \frac{\lambda-1}{2}$	$\frac{1}{6}(-\lambda^2 + 6x^2 - 6x - 6y^2 + 1)$	$\frac{1}{4}(\lambda - 2x + 1)(\lambda^2 + \lambda(2x - 1) - 2x^2 + 2x + 6y^2)$
$B_n^{(s)}(x, y; \lambda)$	0	y	$2xy - y$	$\frac{1}{2}y(\lambda(3 - 6x) + 6x^2 - 6x - 2y^2 + 1)$

Lemma 4. Let n be a non-negative integer. The degenerate cosine and sine functions, namely $\cos_\lambda^{(y)}(t)$ and $\sin_\lambda^{(y)}(t)$, are the exponential generating functions of the sequences

$$\left\{ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k y^{2k} \lambda^{n-2k} S_1(n, 2k) \right\}_{n=0}^{\infty} \tag{29}$$

and

$$\left\{ \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k y^{2k} \lambda^{n-2k-1} S_1(n, 2k+1) \right\}_{n=0}^{\infty}, \tag{30}$$

respectively, where $S_1(n, m)$ are the Stirling polynomials of the first kind, which satisfy (for details see [35–37])

$$x(x - 1) \cdots (x - n + 1) = \sum_{m=0}^n S_1(n, m) x^m = (x)_n.$$

Proof. We show the proof for $\cos_\lambda^{(y)}(t)$ only, as the proof for $\sin_\lambda^{(y)}(t)$ can be done similarly:

$$\begin{aligned} \cos_\lambda^{(y)}(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{y}{\lambda}\right)^{2k} \log^{2k}(1 + \lambda t) \\ &= \sum_{k=0}^{\infty} (-1)^k y^{2k} \lambda^{-2k} \sum_{n=2k}^{\infty} S_1(n, 2k) \frac{(\lambda t)^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k y^{2k} \lambda^{n-2k} S_1(n, 2k) \right) \frac{t^n}{n!}, \end{aligned}$$

where we use the well-known identity (see [35,37])

$$\sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} = \frac{1}{m!} \log^m(1 + t).$$

□

We next give an expression of $A_n^{(c)}(x, y; \lambda)$ and $A_n^{(s)}(x, y; \lambda)$ in terms of Stirling numbers of the first kind.

Theorem 6. *Let n be a non-negative integer. Then, the following identities hold:*

$$A_n^{(c)}(x, y; \lambda) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=2m}^n \binom{n}{k} A_{n-k}(x; \lambda) (-1)^m y^{2m} \lambda^{k-2m} S_1(k, 2m),$$

$$A_n^{(s)}(x, y; \lambda) = \begin{cases} 0, & \text{when } n = 0, \\ \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=2m+1}^n \binom{n}{k} A_{n-k}(x; \lambda) (-1)^m y^{2m+1} \lambda^{k-2m-1} S_1(k, 2m + 1), & \text{when } n \in \mathbb{N}. \end{cases}$$

Proof. The identity for $A_n^{(c)}(x, y; \lambda)$ we get easily by the binomial convolution of the sequence $\{A_n(x; \lambda)\}_{n=0}^{\infty}$ and the sequence in Formula (29). Similarly, the identity for $A_n^{(s)}(x, y; \lambda)$ we obtain by the binomials convolution of sequences $\{A_n(x; \lambda)\}_{n=0}^{\infty}$ and (30). □

Lemma 5. *If we assume that $\frac{1}{A_\lambda(t)} = \sum_{n=0}^{\infty} \tilde{a}_{n,\lambda} \frac{t^n}{n!}$ for some sequence $\{\tilde{a}_n\}$, then for $n \geq 0$, we have*

$$\sum_{k=0}^n \binom{n}{k} \tilde{a}_{n-k,\lambda} A_k^{(c)}(x; \lambda) = \sum_{k=0}^n \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k} (x)_{n-k,\lambda} (-1)^m y^{2m} \lambda^{k-2m} S_1(k, 2m),$$

$$\sum_{k=0}^n \binom{n}{k} \tilde{a}_{n-k,\lambda} A_k^{(s)}(x; \lambda) = \sum_{k=1}^n \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{n}{k} (x)_{n-k,\lambda} (-1)^m y^{2m+1} \lambda^{k-2m-1} S_1(k, 2m + 1).$$

Proof. We show the proof of the first formula only, as the proof of the second one can be done similarly. We will consider product $e_\lambda^x(t) \cos_\lambda^{(y)}(t)$. Firstly, using the binomial convolution of sequences $\{(x)_{n,\lambda}\}_{n=0}^{\infty}$ and (29), we have that $e_\lambda^x(t) \cos_\lambda^{(y)}(t)$ is the exponential generating function of the sequence

$$\left\{ \sum_{k=0}^n \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{k} (x)_{n-k,\lambda} (-1)^m y^{2m} \lambda^{k-2m} S_1(k, 2m) \right\}_{n=0}^{\infty}.$$

Secondly, by rewriting the product $e_\lambda^x(t) \cos_\lambda^{(y)}(t)$ as

$$\frac{1}{A_\lambda(t)} \left(A_\lambda(t) e_\lambda^x(t) \cos_\lambda^{(y)}(t) \right)$$

and using the binomial convolution of sequences $\{\tilde{a}_{n,\lambda}\}_{n=0}^{\infty}$ and $\{A_n(x;\lambda)\}_{n=0}^{\infty}$, we get that $e_{\lambda}^x(t) \cos_{\lambda}^{(y)}(t)$ is the exponential generating function of the sequence

$$\left\{ \sum_{k=0}^n \binom{n}{k} \tilde{a}_{n-k,\lambda} A_k^{(c)}(x;\lambda) \right\}_{n=0}^{\infty},$$

thus, the first identity is proved. \square

Example 1. Type 2 degenerate Euler polynomials $E_n^{(2)}(z;\lambda)$ are defined by the generating functions (see [26])

$$\frac{2}{e_{\lambda}^{\frac{1}{2}}(t) + e_{\lambda}^{-\frac{1}{2}}(t)} e_{\lambda}^z(t) = \sum_{n=0}^{\infty} E_n^{(2)}(z;\lambda) \frac{t^n}{n!}.$$

One can check that the identities in Lemma 5 are established, considering that $\tilde{a}_{n,\lambda} = \frac{1}{2} \left(\left(\frac{1}{2}\right)_{n,\lambda} + \left(-\frac{1}{2}\right)_{n,\lambda} \right)$.

Finally, we show that the degenerate types of cosine- and sine-Appell polynomials, $A_n^{(c)}(x,y;\lambda)$ and $A_n^{(s)}(x,y;\lambda)$, are represented by the Stirling numbers of the first kind $S_1(n,m)$.

Theorem 7. For $n \in \mathbb{N} \cup \{0\}$, the degenerate type of cosine-Appell polynomials is satisfied by

$$A_n^{(c)}(x,y;\lambda) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=2m}^n \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} a_{l,\lambda}(x)_{n-k-l,\lambda} (-1)^m \lambda^{k-2m} y^{2m} S_1(k,2m).$$

In particular, for $n \in \mathbb{N}$, the degenerate type of sine-Appell polynomials holds true:

$$A_n^{(s)}(x,y;\lambda) = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{k=2m+1}^n \sum_{l=0}^{n-k} \binom{n}{k} \binom{n-k}{l} a_{l,\lambda}(x)_{n-k-l,\lambda} (-1)^m \lambda^{k-2m-1} y^{2m+1} S_1(k,2m+1).$$

Proof. We prove the first identity only, as the second one can be proved similarly. By the binomials convolution of sequences $\{A_n(x;\lambda)\}_{n=0}^{\infty}$ and (29) and using identity (28) we obtain the first identity. \square

Example 2.

1. If $A_{\lambda}(t) = \frac{2}{e_{\lambda}(t)+1}$, then we have the sequence $\{E_n(z;\lambda)\}_{n=0}^{\infty}$ of the degenerate Euler polynomials defined in (23). Thus, the degenerate cosine-Euler polynomials and sine-Euler polynomials can be obtained by (see [22] (Theorem 3))

$$E_n^{(c)}(x,y;\lambda) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=2m}^n \binom{n}{k} E_{n-k}(x;\lambda) (-1)^m \lambda^{k-2m} y^{2m} S_1(k,2m),$$

$$E_n^{(s)}(x,y;\lambda) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=2m+1}^n \binom{n}{k} E_{n-k}(x;\lambda) (-1)^m \lambda^{k-2m-1} y^{2m+1} S_1(k,2m+1),$$

replacing $A_n(x;\lambda) = \sum_{l=0}^n \binom{n}{l} a_{l,\lambda}(x)_{n-l,\lambda}$ by $E_n(x;\lambda)$ in Theorem 7.

2. When $A(t) = \frac{t}{e_{\lambda}(t)-1}$, the sequence $\{B_n(z; \lambda)\}_{n=0}^{\infty}$ of degenerate Bernoulli polynomials defined in (24) can be considered. Hence, the degenerate cosine-Euler polynomials and sine-Euler polynomials can be obtained by (see [22] (Theorem 7))

$$B_n^{(c)}(x, y; \lambda) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=2m}^n \binom{n}{k} B_{n-k}(x; \lambda) (-1)^m \lambda^{k-2m} y^{2m} S_1(k, 2m),$$

$$B_n^{(s)}(x, y; \lambda) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{k=2m+1}^n \binom{n}{k} B_{n-k}(x; \lambda) (-1)^m \lambda^{k-2m-1} y^{2m+1} S_1(k, 2m+1),$$

replacing $A_n(x; \lambda) = \sum_{l=0}^n \binom{n}{l} a_{l,\lambda}(x)_{n-l,\lambda}$ by $B_n(x; \lambda)$ in Theorem 7.

4. Conclusions

In this paper, we study the general properties and identities of the complex Appell polynomials by treating the real and imaginary parts separately, which provide the cosine-Appell and sine-Appell polynomials. These presented results can be applied in any complex Appell-type polynomials such as complex Bernoulli polynomials and complex Euler polynomials. Further, we consider the degenerate version of complex Appell polynomials by splitting them into the degenerate cosine-Appell and sine-Appell polynomials and present some of their properties, relations, and examples. Finally, we show that any degenerate cosine-Appell- and sine-Appell-type polynomial can be expressed in terms of the Stirling numbers of the first kind.

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