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A Sharp Oscillation Criterion for a Linear Differential Equation with Variable Delay

Ábel Garab

Institute of Mathematics, University of Klagenfurt, Universitätsstraße 65–67, 9020 Klagenfurt am Wörthersee, Austria; abel.garab@aau.at

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Abstract: We consider linear differential equations with variable delay of the form

$$x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \geq t_0,$$

where $p\colon [t_0, \infty) \to [0, \infty)$ and $\tau\colon [t_0, \infty) \to (0, \infty)$ are continuous functions, such that $t - \tau(t) \to \infty$ (as $t \to \infty$). It is well-known that, for the oscillation of all solutions, it is necessary that

$$B := \limsup_{t\to\infty} A(t) \geq \frac{1}{e} \quad \text{holds, where} \quad A(t) := \int_{t-\tau(t)}^{t} p(s)\, ds.$$

Our main result shows that, if the function A is slowly varying at infinity (in additive form), then under mild additional assumptions on p and τ, condition $B > 1/e$ implies that all solutions of the above delay differential equation are oscillatory.

Keywords: oscillation; delay differential equation; variable delay; deviating argument; non-monotone argument; slowly varying function

MSC: 34K11; 34K06; 26A12

1. Introduction and Preliminary Results

Consider the following linear differential equation with variable delay:

$$x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \geq t_0, \tag{1}$$

where $p\colon [t_0, \infty) \to [0, \infty)$ and $\tau\colon [t_0, \infty) \to (0, \infty)$ are continuous functions, such that $t - \tau(t) \to \infty$ (as $t \to \infty$). Note that $t - \tau(t)$ is not assumed to be nondecreasing. Let $t_{-1} = \inf\{s - \tau(s) : s \in [t_0, \infty)\}$ and note that $t_{-1} \in (-\infty, t_0)$ holds. Then, a continuous function $x\colon [t_{-1}, \infty) \to \mathbb{R}$ is called a *solution* of Equation (1), if it is continuously differentiable on $[t_0, \infty)$ and satisfies Equation (1) there.

Such equations, and, in general, delay differential equations with either constant or variable delay arise naturally in a multitude of models from biology, physics, engineering, chemistry and economy. For an extensive introduction to the theory of delay differential equations, we refer to the books [1,2], whereas for more on their applications we recommend the reader to study [3,4].

This paper is concerned with the oscillatory behaviour of Equation (1). By convention, a solution is called *oscillatory* if it has arbitrary large zeros and is *nonoscillatory* otherwise. Results on oscillation of retarded first order equations already appeared in the works of Johann Bernoulli [5]. The first systematic study of oscillatory and nonoscillatory behaviour of Equation (1) goes back to Myshkis [6]. He showed that, in case the functions τ and p are bounded, then

$$\inf_{t\in[t_0,\infty)} \tau(t) \inf_{t\in[t_0,\infty)} p(t) > \frac{1}{e} \tag{2}$$

implies that all solutions of Equation (1) are oscillatory, whereas condition

$$\sup_{t\in[t\_0,\infty)} \tau(t) \sup_{t\in[t\_0,\infty)} p(t) \leq \frac{1}{e} \tag{3}$$

guarantees the existence of a nonoscillatory solution.

Since then, the question of oscillation has received much attention and many results have been published providing sufficient conditions guaranteeing that all solutions are oscillatory and others that establish the existence of a nonoscillatory solution. For more details, we refer the interested reader to monographs [7–9] and to the survey papers [10,11]. Here, we only point out some results that are most relevant from our perspective.

Ladas, Lakshmikantham and Papadakis [12] proved that all solutions of Equation (1) are oscillatory, provided

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\,ds > 1, \qquad t-\tau(t) \text{ is nondecreasing,} \quad \text{and} \quad p(t) > 0 \text{ for all } t \geq t_0. \tag{4}$$

The following important contribution is due to Koplatadze and Chanturija [13]. For the proof, see also e.g., Theorem 2.1.1 of [9].

Theorem 1 ([13])**.**

(i) *If*

$$\liminf_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\,ds > \frac{1}{e}, \tag{5}$$

then all solutions of Equation (1) *are oscillatory.*

(ii) *If*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\,ds < \frac{1}{e}, \tag{6}$$

or, more generally, if

$$\int_{t-\tau(t)}^{t} p(s)\,ds \leq \frac{1}{e} \quad \textit{for all large } t, \tag{7}$$

then Equation (1) *has a nonoscillatory solution.*

After these central results, many works have focused on filling the gap between Conditions (2) and (3), as well as between the necessary and the sufficient conditions given by Theorem 1 and Condition (4). For more on such results, see, e.g., the recent survey by Moremedi and Stavroulakis [10].

It is worth mentioning that, in case the functions τ and p are constant, then both Conditions (5) and (2) reduce to condition $\tau p > 1/e$, which is in this case not only sufficient, but—in view of Inequality (3)—also necessary for the oscillation of all solutions. Another immediate corollary of Theorem 1 is that, if $\tau(t)$ is constant $\tau > 0$, and p is τ-periodic, then $\int_{t-\tau(t)}^{t} p(s)\,ds$ is constant and Condition (7) is sharp.

Motivated by these facts, Pituk [14] recently proved that, for constant delay τ, there is a class of functions p, for which the 'almost necessary' condition $\tau \limsup_{t\to\infty} p(t) > 1/e$ is sufficient for the oscillation of all solutions of Equation (1). More precisely, he showed in Theorem 1 of [14] that, if p is slowly varying at infinity with $\liminf_{t\to\infty} p(t) > 0$, then

$$\tau \limsup_{t\to\infty} p(t) > \frac{1}{e} \tag{8}$$

implies that all solutions of Equation (1) are oscillatory, where a function $f\colon [t_0, \infty) \to \mathbb{R}$ is called *slowly varying at infinity* if, for every $s \geq 0$,

$$f(t+s) - f(t) \to 0 \quad \text{as } t \to \infty. \tag{9}$$

In a subsequent paper, Pituk, Stavroulakis, and the present author [15] generalized the above result and gave a class of functions p—broader than τ-periodic—for which Condition (6) is 'almost sharp'. More precisely, the following theorem was proved.

Theorem 2 ([15])**.** *Let the function τ in Equation (1) be constant, and function p be nonnegative, bounded and uniformly continuous. Assume further that the function $t \mapsto \int_{t-\tau}^{t} p(s)\,ds$ is slowly varying at infinity. Then,*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > 0 \quad \text{and} \quad \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > \frac{1}{e} \tag{10}$$

imply that all solutions of Equation (1) are oscillatory.

The purpose of this paper is to show that Theorem 2 remains valid in case of variable delay, provided τ is uniformly continuous and bounded. The proof is similar to that of Theorem 2; nevertheless, some technical difficulties also arise due to the variable delay.

In the next section, we present our main theorems and give some hints to support applicability of the results. Then, in Section 3, we provide an illustrative example. Section 4 is devoted to conclusions.

2. Results

The following theorem is our main result.

Theorem 3. *For some positive numbers M and κ, let $p\colon [t_0,\infty) \to [0, M]$ and $\tau\colon [t_0,\infty) \to (0,\kappa]$ be uniformly continuous functions, and suppose that the function*

$$A\colon [t_0+\kappa, \infty) \to [0,\infty), \qquad A(t) := \int_{t-\tau(t)}^{t} p(s)\,ds \tag{11}$$

is slowly varying at infinity. Then,

$$\liminf_{t\to\infty} A(t) > 0 \quad \text{and} \quad \limsup_{t\to\infty} A(t) > \frac{1}{e} \tag{12}$$

imply that all solutions of Equation (1) are oscillatory.

Before we prove the theorem, we make some comments, mainly to support applicability of the result.

From Theorem 1, it is apparent that condition $\limsup_{t\to\infty} A(t) \geq 1/e$ is necessary for the oscillation of all solutions, so Theorem 3 is sharp in this sense. Example 9 of [15] showed that the slowly varying assumption is important: even in the constant delay case, the theorem does not hold if we omit that assumption.

We remark that uniform continuity of p and τ are guaranteed, if they are globally Lipschitz continuous, which is the case if they are differentiable with their derivatives bounded on (t_0, ∞).

Let us also devote some comments to functions that are slowly varying at infinity—we shall call them *slowly varying* for brevity.

The class of slowly varying functions was studied already by Karamata [16] in a multiplicative form. For more information about slowly varying functions and their characterization, we refer the reader to the monograph by Seneta [17]. In particular, for the relation between the two terminologies, see the remark below Theorem 1.2 in Chapter 1 of [17].

Here, let us mention only one characterization of slowly varying functions given by Pituk [14] (in the additive form, see Formula (9)): a continuous function $f\colon [t_0, \infty) \to \mathbb{R}$ is slowly varying if and only if there exists $t_1 \geq t_0$, such that f can be written in the form

$$f(t) = c(t) + d(t), \quad \text{for all } t \geq t_1, \tag{13}$$

where $c\colon [t_1, \infty) \to \mathbb{R}$ is a continuous function which tends to some finite limit as $t \to \infty$, and $d\colon [t_1, \infty) \to \mathbb{R}$ is a continuously differentiable function for which $\lim_{t\to\infty} d'(t) = 0$ holds.

The next lemma will be essential in our proof.

Lemma 1 ([13])**.** *Suppose that $p\colon [t_0, \infty) \to [0, \infty)$ is a continuous function satisfying*

$$\liminf_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\, ds > 0.$$

If x is an eventually positive solution of Equation (1), then, for all sufficiently large T,

$$\sup_{t\geq T} \frac{x(t-\tau(t))}{x(t)} < \infty.$$

Proof of Theorem 3. Assume to the contrary that x is an eventually positive solution and all assumptions of the theorem hold (if the solution x is eventually negative, then take the solution $-x$).

By virtue of Lemma 1, there exists $T \geq t_0 + \kappa$ such that $x(t) > 0$ holds for all $t \in T - \kappa$ and

$$K := \sup_{t\geq T} \frac{x(t-\tau(t))}{x(t)} < \infty. \tag{14}$$

Then, there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subset [T, \infty)$, such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty} A(t_n) = \limsup_{t\to\infty} A(t) =: B.$$

Let us introduce the following sequence of functions:

$$y_n(t) := \frac{x(t_n + t)}{x(t_n)}, \qquad p_n(t) := p(t_n + t) \quad \textit{and} \quad \tau_n(t) := \tau(t_n + t) \quad \text{for all } t \geq -\kappa \text{ and } n \in \mathbb{N}. \tag{15}$$

Then, applying (1) leads to the equation

$$\begin{aligned} y_n'(t) &= \frac{x'(t_n+t)}{x(t_n)} = \frac{-p(t_n+t)x(t_n+t-\tau(t_n+t))}{x(t_n)} \\ &= -p(t_n+t)y_n(t-\tau(t_n+t)) \qquad (16) \\ &= -p_n(t)y_n(t-\tau_n(t)). \qquad (17) \end{aligned}$$

Now, we would like to pass to the limit by applying the Arzelà–Ascoli theorem for the above sequences of functions $\{y_n\}_{n\in\mathbb{N}}$, $\{p_n\}_{n\in\mathbb{N}}$ and $\{\tau_n\}_{n\in\mathbb{N}}$, hence we need to establish their uniform boundedness and equicontinuity. Uniform boundedness, respectively equicontinuity of $\{p_n\}_{n\in\mathbb{N}}$ and $\{\tau_n\}_{n\in\mathbb{N}}$ follow from the boundedness, respectively uniform-continuity of functions p and τ.

It remains to check these properties for $\{y_n\}_{n\in\mathbb{N}}$. For this, note that by virtue of Equation (1) and Equation (14) we obtain that the inequality

$$x'(t_n+t) = -p(t_n+t)\frac{x(t_n+t-\tau(t_n+t))}{x(t_n+t)}x(t_n+t) \geq -KMx(t_n+t)$$

holds for all $t \geq 0$ and $n \in \mathbb{N}$. This immediately implies

$$y_n'(t) = \frac{x'(t_n+t)}{x(t_n)} \geq -\frac{KMx(t_n+t)}{x(t_n)} = -KMy_n(t).$$

As y_n is positive on $[-\kappa, \infty)$, we obtain inequalities

$$-KM \leq \frac{y_n'(t)}{y_n(t)} \leq 0 \quad \text{for all } t \geq 0 \text{ and } n \in \mathbb{N}. \qquad (18)$$

Integration leads to

$$-KMt \leq \ln\frac{y_n(t)}{y_n(0)} \leq 0 \quad \text{for all } t \geq 0 \text{ and } n \in \mathbb{N}. \qquad (19)$$

Taking into account that $y_n(0) = 1$ for all $n \in \mathbb{N}$, we obtain that

$$e^{-KMt} \leq y_n(t) \leq 1 \qquad (20)$$

holds for all $t \geq 0$ and $n \in \mathbb{N}$. Now, Inequalities (20) and (18) imply that $\{y_n\}_{n\in\mathbb{N}}$ and $\{y_n'\}_{n\in\mathbb{N}}$ are uniformly bounded on $[0, \infty)$. Furthermore, the uniform boundedness of $\{y_n'\}$ yields that functions y_n are globally Lipschitz continuous with a common Lipschitz constant, and consequently $\{y_n\}_{n\in\mathbb{N}}$ is uniformly equicontinuous.

In view of the above, by the Arzelà–Ascoli theorem, we may assume (by passing to a subsequence without changing notation) that the limits

$$y(t) := \lim_{n\to\infty} y_n(t), \qquad q(t) := \lim_{n\to\infty} p_n(t) \quad \text{and} \quad \sigma(t) := \lim_{n\to\infty} \tau_n(t) \qquad (21)$$

exist and are continuous on $[0, \infty)$, and the convergence is uniform on every bounded subinterval of $[0, \infty)$. Note that

$$e^{-KMt} \leq y(t) \leq 1 \qquad (22)$$

also holds for all $t \geq 0$ and $n \in \mathbb{N}$.

Furthermore, from Equation (16), together with the uniform continuity of functions p and τ and the uniform equicontinuity of $\{y_n\}_{n\in\mathbb{N}}$, we obtain that $\{y_n'\}_{n\in\mathbb{N}}$ is also equicontinuous on $[\kappa,\infty)$. Recall that the sequence $\{y_n'\}_{n\in\mathbb{N}}$ is uniformly bounded on $[0,\infty)$. Hence, according to the Arzelà–Ascoli theorem, we may assume (after passing to a subsequence if necessary) that the limit $\lim_{n\to\infty} y_n'(t)$ exists for all $t\in[\kappa,\infty)$, and the convergence is uniform on all bounded subintervals of $[\kappa,\infty)$. This combined with the fact that $\lim_{n\to\infty} y_n(\kappa) = y(\kappa)$ yields (see, e.g., Theorem 7.17 of [18]) that

$$y'(t) = \lim_{n\to\infty} y_n'(t)$$

holds for all $t \geq \kappa$. By virtue of Equation (17),

$$y'(t) = -\lim_{n\to\infty} p_n(t) y_n(t-\tau_n(t)) \tag{23}$$

is satisfied for all $t \geq \kappa$. From Equation (21) and the (uniform) equicontinuity of $\{y_n\}_{n\in\mathbb{N}}$, one can easily derive that

$$\lim_{n\to\infty} y_n(t-\tau_n(t)) = y(t-\sigma(t))$$

holds for all $t \geq \kappa$. Thus, Inequality (22) impies that y is a positive solution of equation

$$y'(t) = -q(t)y(t-\sigma(t)). \tag{24}$$

As a final step, we will apply Theorem 1 (i) to show that every solution of Equation (24) is oscillatory, which is a contradiction. Thus, we need to verify that Equation (24) fulfils the hypotheses imposed on Equation (1) and that Inequality (5) holds.

First, observe that $q(t)\in[0,M]$ and $\sigma(t)\in[0,\kappa]$ for all $t\geq\kappa$ follow immediately from their definitions and from the assumptions on p and τ, respectively. Note that we have not yet shown that $\sigma(t)$ is positive for all t.

Next, we prove that Inequality (5) is satisfied. For this, let us fix $t \geq \kappa$ and note that, since p_n converges uniformly to q on the interval $[t-\sigma(t), t]$, we obtain

$$\int_{t-\sigma(t)}^{t} q(s)\,ds = \lim_{n\to\infty}\int_{t-\sigma(t)}^{t} p_n(s)\,ds = \lim_{n\to\infty}\left(\int_{t-\tau_n(t)}^{t} p_n(s)\,ds + \int_{t-\sigma(t)}^{t-\tau_n(t)} p_n(s)\,ds\right).$$

The functions p_n are uniformly bounded, and $\tau_n(t)\to\sigma(t)$, as $n\to\infty$, so the limit of the last integral vanishes. This in turn leads to

$$\begin{aligned}\int_{t-\sigma(t)}^{t} q(s)\,ds &= \lim_{n\to\infty}\int_{t-\tau(t_n+t)}^{t} p(t_n+s)\,ds\\ &= \lim_{n\to\infty}\int_{t_n+t-\tau(t_n+t)}^{t_n+t} p(u)\,du\\ &= \lim_{n\to\infty} A(t_n+t) = \lim_{n\to\infty} A(t_n) = B > \frac{1}{e}.\end{aligned}$$

Here, the last inequality and the last equality hold by assumption, whereas the last but one equality follows from the slowly varying property of A. Hence, $\int_{t-\sigma(t)}^{t} q(s)\,ds$ is constant B, and thus Inequality (5) holds.

The only condition that still needs to be verified is that σ is positive for all $t \geq \kappa$. Notice that this follows immediately from the above formulas: since

$$0 < B = \int_{t-\sigma(t)}^{t} q(s)\,ds \leq M\sigma(t)$$

holds for all $t \geq \kappa$, thus $\sigma(t) \geq B/M$ for all $t \geq \kappa$.

Therefore, Theorem 1 (i) can be applied for Equation (24) with $\tau := \sigma$, $t_0 := \kappa$ and $p := q$ to obtain that every solution of Equation (24) is oscillatory, which contradicts Inequality (22). □

The following lemma may be helpful to verify the slowly varying property of A without having to evaluate it.

Lemma 2. *For some $t_0 \in \mathbb{R}$ and positive number κ, let $p\colon [t_0, \infty) \to \mathbb{R}$ be bounded and locally integrable, and $\tau\colon [t_0, \infty) \to [-\kappa, \kappa]$ be any function. If both p and τ are slowly varying at infinity, then so is the function*

$$A\colon [t_0 + \kappa, \infty) \to \mathbb{R}, \qquad A(t) := \int_{t-\tau(t)}^{t} p(s)\,ds.$$

To prove this lemma, we first need to state the following result (see Lemma 1.1 of [17]).

Lemma 3. *If $p\colon [t_0, \infty) \to \mathbb{R}$ is Lebesgue measurable and slowly varying at infinity, then, for all finite interval I, $\sup_{s\in I} |p(t+s) - p(t)| \to 0$, as $t \to \infty$.*

Proof of Lemma 2. For $t \geq t_0 + \kappa$, we have

$$A(t) = \int_{t-\tau(t)}^{t} p(s)\,ds = \int_{-\tau(t)}^{0} p(t+u)\,du = \int_{-\tau(t)}^{0} p(t+u) - p(t)\,du + \tau(t)p(t). \tag{25}$$

From this and the triangle inequality, we obtain that, for any fixed $r \in \mathbb{R}$, the inequalities

$$\begin{aligned} |A(t+r) - A(t)| \leq{} & \left| \int_{-\tau(t+r)}^{0} p(t+r+u) - p(t+r)\,du \right| + \left| \int_{-\tau(t)}^{0} p(t+u) - p(t)\,du \right| \\ & + \big|\tau(t+r)p(t+r) - \tau(t)p(t)\big| \\ \leq{} & \int_{-\kappa}^{\kappa} |p(t+r+u) - p(t+r)|\,du + \int_{-\kappa}^{\kappa} |p(t+u) - p(t)|\,du \\ & + \big|\tau(t+r)p(t+r) - \tau(t)p(t)\big| \\ \leq{} & 2\kappa \left(\sup_{u\in[-\kappa,\kappa]} |p(t+r+u) - p(t+r)| + \sup_{u\in[-\kappa,\kappa]} |p(t+u) - p(t)| \right) \\ & + |p(t+r)||\tau(t+r) - \tau(t)| + |\tau(t)||p(t+r) - p(t)| \end{aligned}$$

hold. Now, if we let $t \to \infty$, then the last two suprema vanish due to Lemma 3 and because p is slowly varying. On the other hand, the last two terms also tend to 0, thanks to boundedness and to the slowly varying property of functions τ and p.

Therefore, $\lim_{t\to\infty} A(t+r) - A(t) = 0$ holds for all $r \geq 0$. □

Note that, for A to be slowly varying, it is not sufficient to assume merely that at least one of p and τ is slowly varying. This is the case even under the additional assumptions of Theorem 3 on p and τ. This can be readily seen by considering examples $p \equiv 1$ and $\tau(t) = 2 + \sin t$, and $\tau \equiv \pi$ and $p(t) := 2 + \sin t$, respectively. In both cases, function A will be 2π-periodic, but nonconstant, so it cannot be slowly varying.

Our last theorem is a corollary of Lemma 2 and Theorem 3, and it gives another generalization of Theorem 1 of [14] in case p is bounded.

Theorem 4. *For some positive numbers M and κ let $p\colon [t_0, \infty) \to [0, M]$ and $\tau\colon [t_0, \infty) \to (0, \kappa]$ be continuous and slowly varying at infinity. Then Condition (12) implies that all solutions of Equation (1) are oscillatory.*

Proof. First, Lemma 2 infers that function A from Equation (11) is slowly varying. As already noted after Theorem 4 of [15], the slowly varying property together with continuity implies uniform continuity. Hence, p and τ are uniformly continuous, so Theorem 3 applies, which finishes the proof. □

Let us briefly consider the case when p is unbounded, and slowly varying. If we further assume that $p(t) > 0$ holds for large t, and τ is such that there exists some $\tau_0 \in (0, \kappa]$, for which $\liminf_{t\to\infty} \tau(t) \geq \tau_0$ holds and $t - \tau(t)$ is nondecreasing (note that Theorem 1 of [14] meets these assumptions), then, using the slowly varying property of p, it can be easily shown that $\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\, ds = \infty$. In particular, Condition (4) is fulfilled, which yields that all solutions are oscillatory regardless of Condition (12).

3. Example

Before concluding the paper, let us consider the following example, which may look a bit artificial. This is because our intention was to design it in such a way that—hopefully—no other known results could guarantee the oscillation of all solutions. Obviously, it is not possible to be aware of all the related results, and to check whether they are applicable; nevertheless, we shall exclude applicability of many classical, as well as many recent theorems.

Consider the equation

$$x'(t) + \left(\frac{1}{2\pi e} + \delta \sin\sqrt{t}\right) x\big(t - \big(2\pi + \varepsilon\cos\sqrt{t}\big)\big) = 0, \qquad t \geq 0, \tag{26}$$

where $\delta \in (0, \frac{1}{2\pi e})$ and $\varepsilon \in (0, 2\pi)$ are small positive constants that will be determined later. Functions p and τ are clearly positive and bounded, so Equation (26) is a special case of Equation (1) with

$$p(t) = \frac{1}{2\pi e} + \delta \sin\sqrt{t}, \qquad \tau(t) = 2\pi + \varepsilon\cos\sqrt{t} \quad \text{and} \quad t_0 = 0.$$

Note that the functions $\sin\sqrt{t}$ and $\cos\sqrt{t}$ are slowly varying at infinity, since their derivatives vanish there (see Equation (13)). This in turn yields that both p and τ are slowly varying, and, thus, in view of Lemma 2, A is slowly varying as well.

On the other hand, a direct calculation shows that

$$A(t) = \frac{2\pi + \varepsilon\cos\sqrt{t}}{2\pi e} + \delta \int_{t-\tau(t)}^{t} \sin\sqrt{s}\, ds.$$

This immediately implies

$$\frac{2\pi - \varepsilon}{2\pi e} - \delta(2\pi + \varepsilon) \leq \liminf_{t\to\infty} A(t) \leq \limsup_{t\to\infty} A(t) \leq \frac{2\pi + \varepsilon}{2\pi e} + \delta(2\pi + \varepsilon). \tag{27}$$

Now, by setting $t_n = (2n\pi)^2$ and $t'_n = ((2n+1)\pi)^2$ for all $n \in \mathbb{N}$, we obtain that

$$A(t_n) = \frac{2\pi + \varepsilon}{2\pi e} + \delta \int_{t_n - \tau(t_n)}^{t_n} \sin\sqrt{s}\, ds \geq \frac{2\pi + \varepsilon}{2\pi e} - \delta(2\pi + \varepsilon)$$

and

$$A(t'_n) = \frac{2\pi - \varepsilon}{2\pi e} + \delta \int_{t'_n - \tau(t'_n)}^{t'_n} \sin\sqrt{s}\, ds \leq \frac{2\pi - \varepsilon}{2\pi e} + \delta(2\pi + \varepsilon)$$

hold for all $n \in \mathbb{N}$. These together with Inequalities (27) yield the estimates

$$\frac{2\pi + \varepsilon}{2\pi e} - \delta(2\pi + \varepsilon) \leq \limsup_{t\to\infty} A(t) \leq \frac{2\pi + \varepsilon}{2\pi e} + \delta(2\pi + \varepsilon)$$

and

$$\frac{2\pi - \varepsilon}{2\pi e} - \delta(2\pi + \varepsilon) \leq \liminf_{t\to\infty} A(t) \leq \frac{2\pi - \varepsilon}{2\pi e} + \delta(2\pi + \varepsilon).$$

Finally, for $\gamma > 0$, let $\varepsilon := \varepsilon(\gamma) := 4\pi e\gamma$ and $\delta := \delta(\gamma) := \frac{\gamma}{2\pi+\varepsilon}$. Then, the above estimates take the form

$$\frac{1}{e} + \gamma \leq \limsup_{t\to\infty} A(t) \leq \frac{1}{e} + 3\gamma \quad \text{and} \quad \frac{1}{e} - 3\gamma \leq \liminf_{t\to\infty} A(t) \leq \frac{1}{e} - \gamma.$$

It is now easy to see that, for all $\gamma \in \left(0, \frac{1}{3e}\right)$, all assumptions of Theorem 3 (and also of Theorem 4) are fulfilled, and therefore all solutions are oscillatory. Note also that, since $\limsup_{t\to\infty} A(t) \to \frac{1}{e}$ as $\gamma \to 0^+$, and $\liminf_{t\to\infty} A(t) < \frac{1}{e}$ for all $\gamma \in \left(0, \frac{1}{3e}\right)$, by choosing $\gamma > 0$ small enough we can rule out the application of Conditions (4), (5) and various other sufficient conditions for the oscillation of all solutions of Equation (26) (see e.g., conditions (C_3)–(C_{12}) from [10]). Since function τ is nonconstant, therefore neither Condition (8) nor Theorem 2 can be applied to guarantee oscillation.

4. Conclusions

It has been known for almost forty years that, for the oscillation of all solutions of equation

$$x'(t) + p(t)x(t - \tau(t)) = 0, \qquad t \geq t_0,$$

it is necessary that $\limsup_{t\to\infty} A(t) \geq 1/e$ holds, where $A(t) := \int_{t-\tau(t)}^{t} p(s)\, ds$ (see [13]). In our main result (see Theorem 3), we showed that, if the function A is slowly varying at infinity (see Formula (9)), then, under mild additional assumptions on p and τ, the 'almost necessary' condition $\limsup_{t\to\infty} A(t) > 1/e$ is sufficient for the oscillation of all solutions.

In Theorem 4, we formulated a corollary of Theorem 3. The advantage of this theorem is that its assumptions can be verified more easily.

The applicability and novelty of our results were demonstrated in Section 3.

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