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A Multi-Stage Homotopy Perturbation Method for the Fractional Lotka-Volterra Model

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Abstract: In this work, we propose an efficient multi-stage homotopy perturbation method to find an analytic solution to the fractional Lotka-Volterra model. We obtain its order of accuracy, and we study the stability of the system. Moreover, we present several examples to show of the effectiveness of this method, and we conclude that the value of the derivative order plays an important role in the trajectories velocity.

Keywords: predator-prey model; anomalous dynamical system; Caputo fractional derivative

1. Introduction

The well-known predator-prey model was introduced by Lotka [1] and Volterra [2]. The resulting model is the classical Lotka-Volterra system of ordinary differential equations, which is proposed to provide a model for two-species competition. This model has potential applications in various fields of science like biology, sociology, medicine, history, economics, and ecology, among others. In addition, competition between more than two species has been studied in the field of mathematical biology using Lotka-Volterra-type systems, describing multispecies population dynamics. Many interesting results on the dynamical behavior for the solutions have been found in [3–7], such as the existence and uniqueness of solutions, asymptotic behavior, and bifurcations of the system.

On the other hand, fractional differential equations have been highly attractive to many scientists due to the fact that these equations have many applications in a wide range of phenomena (see [8–13]). In comparison with standard derivatives of integer order, the fractional order derivatives are characterized by their memory; i.e., the rate of change of a function near a point is affected by the history in the time domain of definition rather than just near the point itself.

In this note, we study the fractional Lotka-Volterra model

$$\begin{aligned} D^{\alpha}x(t) &= ax(t) - bx(t)y(t), \\ D^{\beta}y(t) &= -cy(t) + dx(t)y(t), \end{aligned} \tag{1}$$

where a, b, c, and d are non-negative constants, α, $\beta \in (0,1]$, and D^{α}, and D^{β} is the Caputo fractional derivative. Javeed et al. [14] use the homotopy perturbation method (HPM) (see [15]) to solve fractional partial differential equations. Rafei et al. [16], using HPM, found an approximation to the solution for the classic Lotka-Volterra model ($\alpha = \beta = 1$). In addition, Kadem and Baleanu [17] use HPM to find an analytic approximate solution for the coupled Lotka-Volterra equations. On the other hand, HPM was implemented by Chowdhury et al. [18] to subintervals of equal length from a partition of total time evolution. This algorithm is known as the multistage homotopy perturbation method (MHPM).

Moreover, the model (1) was studied by Das and Gupta [19] applying HPM, where the parameters a, b, c, y d depend on time. However, its solution is valid only for a small time interval. Other numerical methods have been used to study nonlinear fractional differential equations (see [20–22]); for example, Pilar et al. [23], using Diethelm's numerical algorithm, find solutions when $a = b = c = d = 1$ and $\alpha = r\beta, r \in \mathbb{N}$.

In this work, we propose to implement an MHPM to the fractional Lotka-Volterra model defined in (1) in order to obtain an analytical solution for the model, and an expression for the truncation error is given. Through Lemma 2, the stability of the system is guaranteed, which is analogous to the results obtained by Ahmed et al. [8] and Elsadany et al. [9]. Finally, several examples are presented to show the effectiveness of this method.

2. Preliminaries

In this section, we give some basic definitions and properties of fractional calculus theory that are used in this work.

Definition 1. *The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ is defined by*

$$(I_a^\alpha x)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau,$$

where Γ is the gamma function.

It can be directly verified that

$$(I_0^\alpha t^{r\alpha}) = \frac{\Gamma(1+r\alpha)}{\Gamma(1+(1+r)\alpha)}\, t^{(1+r)\alpha}, \tag{2}$$

where $r \geq 0$.

Definition 2. *The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ is defined by*

$$(D_a^\alpha x)(t) = (I_a^{m-\alpha} D_a^m x)(t),$$

where $D = d/dt$, $m = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $m = \alpha$ for $\alpha \in \mathbb{N}$.

Lemma 1. *Let $\alpha \in (0, 1]$ and $x(t)$ be a differentiable function in $\mathbb{R}$. Then $(I_a^\alpha D_a^\alpha x)(t) = x(t) - x(a)$.*

Proof. The proof can be found in [24]. □

Lemma 2. *We consider the fractional-order system*

$$\begin{aligned} D^{\alpha_1} x(t) &= f_1(x(t), y(t)), \\ D^{\alpha_2} y(t) &= f_2(x(t), y(t)), \end{aligned} \tag{3}$$

where $\alpha_1 \neq \alpha_2$. Suppose that m is the least common multiple of the denominators u_i of α_i, where $\alpha_i = v_i/u_i, v_i, u_i \in \mathbb{Z}^+$ for $i = 1, 2$, and we set $\gamma = \frac{1}{m}$. System (3) is asymptotically stable at the equilibrium point (x^*, y^*) if

$$|arg(\lambda)| > \gamma\frac{\pi}{2}$$

for all roots λ of the following equation $\det(\mathrm{diag}([\lambda^{m\alpha_1}, \lambda^{m\alpha_2}]) - J) = 0$, and $J = \partial f/\partial x$ is the Jacobian matrix evaluated in (x^*, y^*), where $f = [f_1, f_2]^T$.

Proof. The proof can be found in [25]. □

3. Main Result

Theorem 1. *Let $t_0 = 0 < t_1 < t_2 < \cdots < t_m = T$ be a partition of $[0, T]$, such that*

$$t_k = k\frac{T}{m}, \text{ for each } k = 1, \cdots, m,$$

and let $(x_k(t), y_k(t))$, where $t \in [t_{k-1}, t_k]$, be solutions for the following initial value problems:

$$\begin{aligned} D^{\alpha}_{t_{k-1}} x_k(t) &= a x_k(t) - b x_k(t) y_k(t), \\ D^{\beta}_{t_{k-1}} y_k(t) &= -c y_k(t) + d x_k(t) y_k(t), \end{aligned} \tag{4}$$

with $x_{0k} = x_k(t_k)$ and $y_{0k} = y_k(t_k)$, for $k = 0, 1, \cdots, m$. Therefore, the solution for the system (1) is

$$\begin{aligned} x(t) &= \sum_{k=1}^{m} I_{[t_{k-1}, t_k]} \sum_{j=0}^{\infty} v_{jk}, \\ y(t) &= \sum_{k=1}^{m} I_{[t_{k-1}, t_k]} \sum_{j=0}^{\infty} w_{jk}, \end{aligned} \tag{5}$$

where

$$I_{[t_{k-1}, t_k]} = \begin{cases} 1, & \text{if } t \in [t_{k-1}, t_k], \\ 0, & \text{if } t \notin [t_{k-1}, t_k]. \end{cases}$$

Proof. In order to solve (4) for each k, we apply the homotopy perturbation method (see [15]). First, we define the homotopy

$$\begin{aligned} (1-p)\big(D^{\alpha}_{t_{k-1}} v_k - D^{\alpha}_{t_{k-1}} x_k(t_k)\big) + p\big(D^{\alpha}_{t_{k-1}} v_k - a v_k + b v_k w_k\big) = 0, \\ (1-p)\big(D^{\beta}_{t_{k-1}} w_k - D^{\beta}_{t_{k-1}} y_k(t_k)\big) + p\big(D^{\beta}_{t_{k-1}} w_k + c v_k - d v_k w_k\big) = 0, \end{aligned} \tag{6}$$

where $p \in [0, 1]$. Let us suppose that the solution to (4) is given by

$$\begin{aligned} v_k &= v_{0k} + p v_{1k} + p^2 v_{2k} + p^3 v_{3k} + \cdots, \\ w_k &= w_{0k} + p w_{1k} + p^2 w_{2k} + p^3 w_{3k} + \cdots, \end{aligned} \tag{7}$$

where $v_{ik}, w_{ik}, i \in \mathbb{N}$ are functions to be determined. Therefore, the solution will be

$$\begin{aligned} x_k &= \lim_{p \to 1} v_k = v_{0k} + v_{1k} + v_{2k} + v_{3k} + \cdots, \\ y_k &= \lim_{p \to 1} w_k = w_{0k} + w_{1k} + w_{2k} + w_{3k} + \cdots. \end{aligned}$$

In order to find the functions v_{ik} and w_{ik}, since the Caputo's fractional derivative of a constant is zero, $D^{\alpha}_{t_{k-1}} x_k(t_k) = 0$ and $D^{\beta}_{t_{k-1}} y_k(t_k) = 0$, we note that from the system (6), it follows that

$$\begin{aligned} D^{\alpha}_{t_{k-1}} v_k &= p(a v_k - b v_k w_k), \\ D^{\beta}_{t_{k-1}} w_k &= p(-c v_k + d v_k w_k). \end{aligned} \tag{8}$$

Substituting (7) in the above system, we get

$$\begin{aligned}
\sum_{i=0}^{\infty} p^i D_{t_{k-1}}^{\alpha} v_{ik} &= ap\sum_{i=0}^{\infty} p^i v_{ik} - bp\bigg(v_{0k}w_{0k} + p[v_{0k}w_{1k} + v_{1k}w_{0k}] \\
&\quad + \cdots + p^n \sum_{j=0}^{n} v_{jk}w_{(n-j)k} \cdots\bigg), \\
\sum_{i=0}^{\infty} p^i D_{t_{k-1}}^{\alpha} w_{ik} &= -cp\sum_{i=0}^{\infty} p^i w_{ik} + dp\bigg(v_{0k}w_{0k} + p[v_{0k}w_{1k} + v_{1k}w_{0k}] \\
&\quad + \cdots + p^n \sum_{j=0}^{n} v_{jk}w_{(n-j)k} \cdots\bigg).
\end{aligned}$$

Then, matching the coefficients, we obtain the following differential equations for v_{ik} and w_{ik}:

$$\begin{aligned}
D_{t_{k-1}}^{\alpha} v_{0k} &= 0, \\
D_{t_{k-1}}^{\alpha} v_{1k} &= av_{0k} - bv_{0k}w_{0k}, \\
D_{t_{k-1}}^{\alpha} v_{2k} &= av_{1k} - b(v_{0k}w_{1k} + v_{1k}w_{0k}), \\
D_{t_{k-1}}^{\alpha} v_{3k} &= av_{2k} - b(v_{0k}w_{2k} + v_{1k}w_{1k} + v_{2k}w_{0k}), \\
&\cdots \\
D_{t_{k-1}}^{\alpha} v_{(n+1)k} &= av_{nk} - b\sum_{j=0}^{n} v_{jk}w_{(n-j)k},
\end{aligned}$$

and

$$\begin{aligned}
D_{t_{k-1}}^{\alpha} w_{0k} &= 0, \\
D_{t_{k-1}}^{\alpha} w_{1k} &= -cw_{0k} + dv_{0k}w_{0k}, \\
D_{t_{k-1}}^{\alpha} w_{2k} &= -cw_{1k} + d(v_{0k}w_{1k} + v_{1k}w_{0k}), \\
D_{t_{k-1}}^{\alpha} w_{3k} &= -cw_{2k} + d(v_{0k}w_{2k} + v_{1k}w_{1k} + v_{2k}w_{0k}), \\
&\cdots \\
D_{t_{k-1}}^{\alpha} w_{(n+1)k} &= -cw_{nk} + d\sum_{j=0}^{n} v_{jk}w_{(n-j)k}.
\end{aligned}$$

Applying Lemma 1, we get explicit expressions for v_{ik} y w_{ik}. For example,

$$\begin{aligned}
v_{1k} &= \frac{v_{0k}(a - bw_{0k})}{\Gamma(1+\alpha)}(t - t_{k-1})^{\alpha}, \\
v_{2k} &= \frac{v_{0k}(a - bw_{0k})^2}{\Gamma(1+2\alpha)}(t - t_{k-1})^{2\alpha} - \frac{bv_{0k}w_{0k}(-c + dv_{0k})}{\Gamma(1+\alpha+\beta)}(t - t_{k-1})^{\alpha+\beta}, \\
v_{3k} &= \frac{v_{0k}(a - bw_{0k})^3}{\Gamma(1+3\alpha)}(t - t_{k-1})^{3\alpha} + \frac{bv_{0k}w_{0k}(-a + bw_{0k})}{\Gamma(1+2\alpha+\beta)}\bigg[-c + 2dv_{0k} \\
&\quad + \frac{\Gamma(1+\alpha+\beta)(-c + dv_{0k})}{\Gamma(1+\alpha)\Gamma(1+\beta)}\bigg](t - t_{k-1})^{2\alpha+\beta} \\
&\quad - \frac{(c - dv_{0k})^2 w_{0k} b v_{0k}}{\Gamma(1+\alpha+2\beta)}(t - t_{k-1})^{\alpha+2\beta},
\end{aligned}$$

and

$$\begin{aligned}
w_{1k} &= \frac{w_{0k}(-c+dv_{0k})}{\Gamma(1+\beta)}(t-t_{k-1})^{\beta}, \\
w_{2k} &= \frac{w_{0k}(-c+dv_{0k})^2}{\Gamma(1+2\beta)}(t-t_{k-1})^{2\beta} + \frac{dv_{0k}w_{0k}(a-bw_{0k})}{\Gamma(1+\alpha+\beta)}(t-t_{k-1})^{\alpha+\beta}, \\
w_{3k} &= \frac{w_{0k}(-c+dv_{0k})^3}{\Gamma(1+3\beta)}(t-t_{k-1})^{3\beta} + \frac{dv_{0k}w_{0k}(-c+dv_{0k})}{\Gamma(1+\alpha+2\beta)} \\
&\quad \left[\frac{\Gamma(1+\alpha+\beta)(a-bw_{0k})}{\Gamma(1+\alpha)\Gamma(1+\beta)} + a - 2bw_{0k}\right](t-t_{k-1})^{\alpha+2\beta} \\
&\quad + \frac{dv_{0k}w_{0k}(a-bw_{0k})^2}{\Gamma(1+2\alpha+\beta)}(t-t_{k-1})^{2\alpha+\beta}.
\end{aligned}$$

Therefore, from the above calculations, Equation (5) follows. □

Now, if we only take the first n terms for the series in (5), the order of accuracy for the solution is $O((\Delta t)^{n\beta+\alpha})$ and $O((\Delta t)^{n\alpha+\beta})$, when $\Delta t \to 0$, for $x(t)$ and $y(t)$, respectively. That is,

$$\begin{aligned}
x(t) &= \sum_{k=1}^{m} I_{[t_{k-1},t_k]} \sum_{j=0}^{n} v_{jk} + O((\Delta t)^{n\beta+\alpha}), \\
y(t) &= \sum_{k=1}^{m} I_{[t_{k-1},t_k]} \sum_{j=0}^{n} w_{jk} + O((\Delta t)^{n\alpha+\beta}).
\end{aligned}$$

Here, Δt is the partition length.

Convergence

In order to demonstrate the efficiency of our method, we carry on an analysis of the series convergence for various values of the truncation parameters n and m that control the approximation accuracy. Here, the time step is $\Delta t = T/m$, where T is the total length of the time interval (see Figure 1). We take $T = 1$ as a reference value, and $\alpha = \beta$.

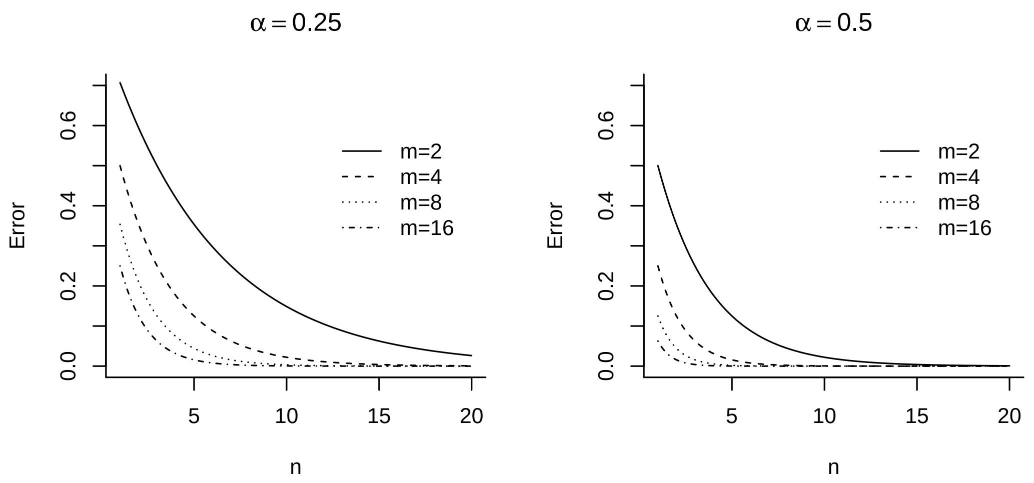


Figure 1. *Cont.*

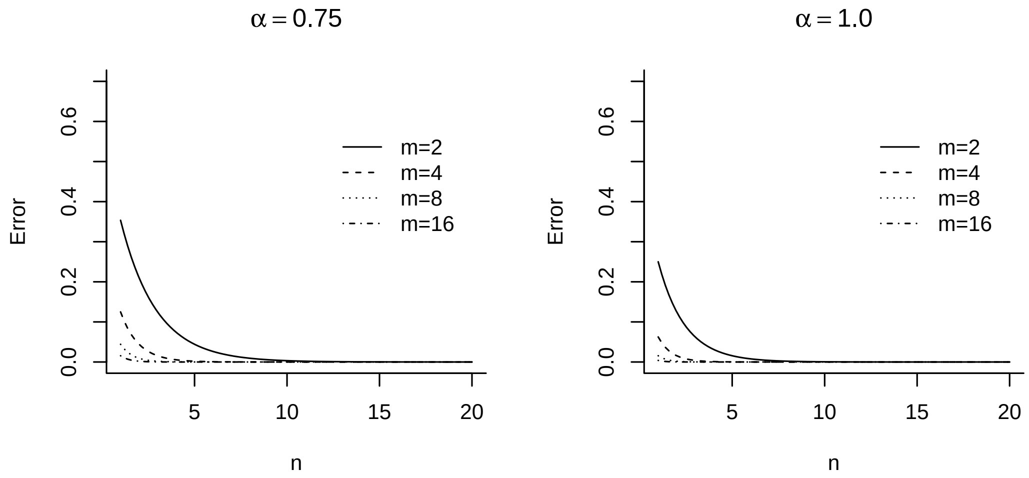


Figure 1. Convergence error.

4. Examples

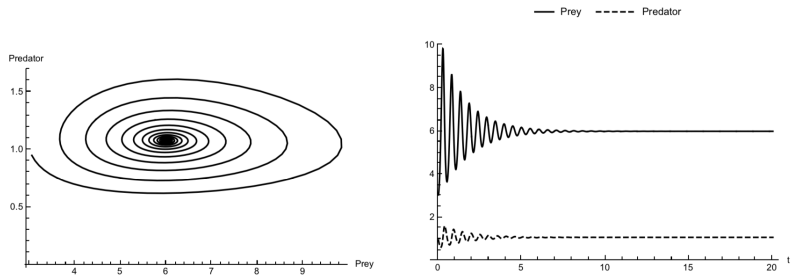
In order to illustrate the proposed method, we present some examples. Let $(x_0, y_0) = (3, 1)$ be the initial condition, $a = 0.4$, $b = 0.37$, $c = 0.3$, $d = 0.05$, $m = 200$, and $n = 50$ be the parameter values, $t \in [0, 20]$, and $\Delta t = 0.1$. We consider the following cases for α and β.

Case 1. *$\alpha = \beta$ taking the values 0.25, 0.5, 0.75, and 1.*

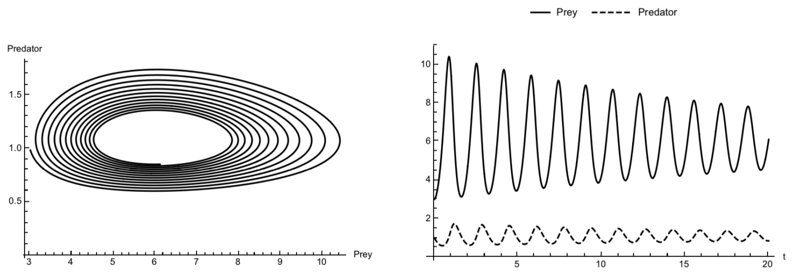
Case 2. *$\alpha = 0.25$ fixed and β taking the values $0.5, 0.75, 1$.*

5. Conclusions

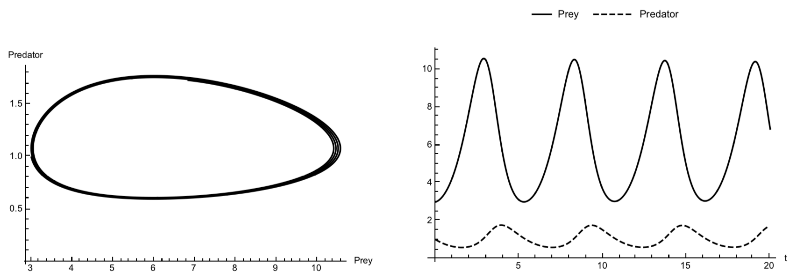
In this work, we found an analytic solution for a fractional Lotka-Volterra model using the multistage homotopy perturbation method. The effectiveness of this method is proven with different values for the parameters m, n, and α. In Figure 1, we observe that for a given error, we can choose the partition size and the number of terms for the solution series. We note that when α is getting close to zero, bigger values for m and n are needed in order to obtain a smaller error. From Figure 2, we conclude that the velocity trajectories depend of the fractional derivative order; if the order is close to zero, the trajectories get closer to the critical point faster. Moreover, we can observe that if α and β tend to one, the trajectories tend to be periodic, as in the classical case. In Figure 3, we take α not equal to β. A similar behavior to Case 1 is observed for the trajectories. In addition, lets point out that in the fractional Lotka-Volterra model, the solutions are not periodic. Then, the symmetry present in the ordinary case is broken when a fractional derivative is considered; this asymmetry is a typical feature of the fractional order systems of differential equations. Therefore, the model studied complements and generalizes the symmetric one. Finally, note that the behavior of the solutions depicted in Figures 1 and 2 is consistent with the theoretical result given in Lemma 2.



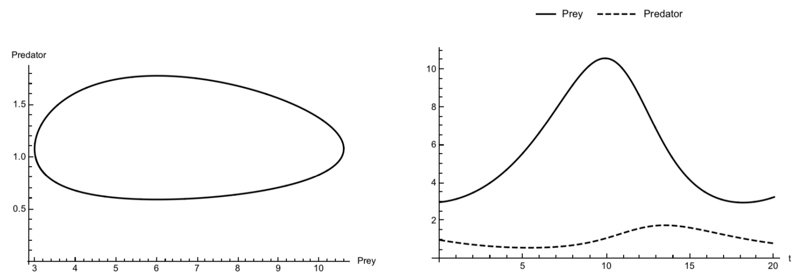
(**a**)$\alpha = \beta = 0.25$



(**b**)$\alpha = \beta = 0.50$

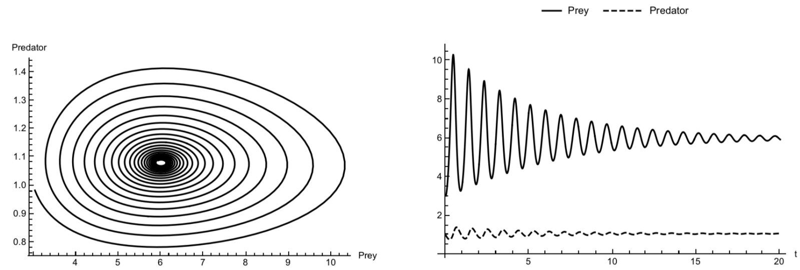


(**c**)$\alpha = \beta = 0.75$

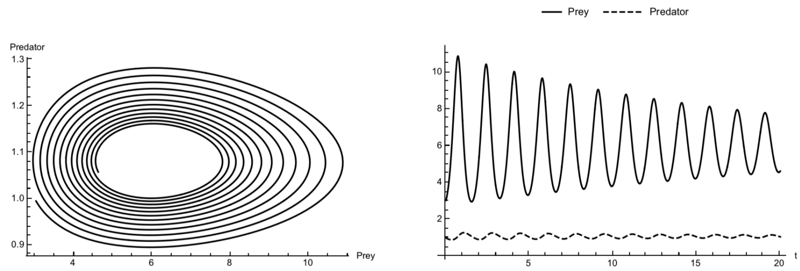


(**d**)$\alpha = \beta = 1$

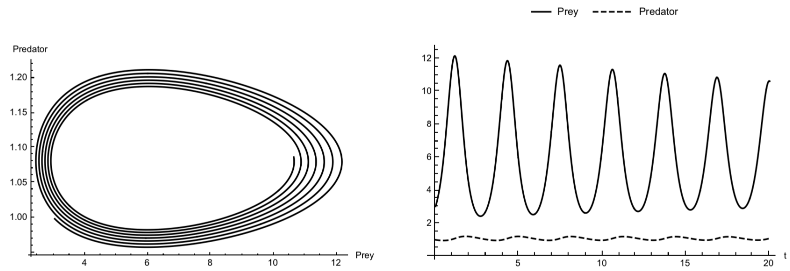
Figure 2. Case 1.



(a)$\alpha = 0.25$, $\beta = 0.5$



(b)$\alpha = 0.25$, $\beta = 0.75$



(c)$\alpha = 0.25$, $\beta = 1$

Figure 3. Case 2.

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Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this paper.

Abbreviations

The following abbreviations are used in this manuscript:

HPM	Homotopy perturbation method
MHPM	Multistage homotopy perturbation method

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