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Some Fixed Point Theorems for Quadratic Quasicontractive Mappings

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Abstract: In this paper, we introduce the notion of quadratic quasicontractive mapping and prove two generalizations of some classical fixed point theorems. Furthermore, we present some examples to support our main results.

Keywords: fixed point; Edelstein's theorem; Greguš's theorem

1. Introduction

In 1962, Edelstein [1] proved the following fixed point theorem.

Theorem 1. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then, T has a unique fixed point.*

In 1973, Hardy and Rogers [2] extended Theorem 1. They proved the following theorem.

Theorem 2. *Let (X, d) be a compact metric space and let $T : X \to X$ be a mapping satisfying inequality*

$$d(Tx, Ty) < A \cdot d(x, Tx) + B \cdot d(y, Ty) + C \cdot d(x, y) \tag{1}$$

for all $x, y \in X$ and $x \neq y$, where A, B, C are positive and $A + B + C = 1$. Then, T has a unique fixed point.

Other generalizations of Theorem 1 have appeared in recent years, see [3–8].

Let X be a Banach space and C a closed convex subset of X. Greguš [9] proved the following theorem.

Theorem 3. *Let $T : X \to X$ be a mapping satisfying inequality*

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\| \tag{2}$$

for all $x, y \in C$, where $0 < a < 1$, $b \geq 0$, $c \geq 0$ and $a + b + c = 1$. Then, T has a unique fixed point.

Many theorems that are closely related to Greguš's Theorem can be found in [10–21]. In this paper, we will prove two generalizations of Theorem 1, Theorem 2, and Theorem 3.

2. Main Results

Before stating the main results, we introduce the following type of quasicontraction.

Definition 1. *A mapping $T : X \to X$ of a metric space X into itself is said to be a quadratic quasicontractive if there exists $a \in \left(0, \frac{1}{2}\right)$ such that*

$$d^2\left(Tx, Ty\right) \leq a \cdot d^2\left(x, Tx\right) + a \cdot d^2\left(y, Ty\right) + \left(1 - 2a\right) \cdot d^2\left(x, y\right) \tag{3}$$

for all $x, y \in X$ and a strict quadratic quasicontraction if in Relation (3) we have the strict inequality for all $x, y \in X$ with $x \neq y$.

Lemma 1. *If $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha, \beta, \gamma \geq 0$, $a \in \left(0, \frac{1}{2}\right)$ and $b \in (0, 1)$, then*

$$(i)\; b\alpha + (1-b)\,\beta \leq \sqrt{b\alpha^2 + (1-b)\,\beta^2}, \tag{4}$$

$$(ii)\, a\alpha + a\beta + (1-2a)\,\gamma \leq \sqrt{a\alpha^2 + a\beta^2 + (1-2a)\,\gamma^2}. \tag{5}$$

Proof. (i) Inequality (4) is equivalent to

$$b^2\alpha^2 + 2b\,(1-b)\,\alpha\beta + (1-b)^2\,\beta^2 \leq b\alpha^2 + (1-b)\,\beta^2 \tag{6}$$

or

$$b\,(1-b)\,(\alpha-\beta)^2 \geq 0, \tag{7}$$

which is obvious.

(ii) We have by (i)

$$\begin{aligned} a\alpha + a\beta + (1-2a)\,\gamma &= 2a \cdot \frac{\alpha+\beta}{2} + (1-2a)\,\gamma \\ \leq &\sqrt{2a \cdot \left(\frac{\alpha+\beta}{2}\right)^2 + (1-2a)\,\gamma^2} \leq \sqrt{2a \cdot \frac{\alpha^2+\beta^2}{2} + (1-2a)\,\gamma^2} \\ = &\sqrt{a\alpha^2 + a\beta^2 + (1-2a)\,\gamma^2}. \end{aligned}$$

□

Remark 1. *If T satisfies Inequality (1), then T is a strict quadratic quasicontraction. Indeed, suppose that T satisfies Inequality (1). Then, we have by symmetry*

$$d\left(Tx, Ty\right) < A \cdot d\left(y, Ty\right) + B \cdot d\left(x, Tx\right) + C \cdot d\left(x, y\right). \tag{8}$$

By Inequalities (1) and (8), we obtain that

$$d\left(Tx, Ty\right) < \frac{A+B}{2} \cdot \left[d\left(x, Tx\right) + d\left(y, Ty\right)\right] + C \cdot d\left(x, y\right) \tag{9}$$

and $\frac{A+B}{2} + \frac{A+B}{2} + C = A + B + C = 1$.

By Inequality (9) and Lemma 1 taking $\alpha = d\left(x, Tx\right)$, $\beta = d\left(y, Ty\right)$ and $\gamma = d\left(x, y\right)$, we obtain

$$d^2\left(Tx, Ty\right) < \frac{A+B}{2} \cdot d^2\left(x, Tx\right) + \frac{A+B}{2} \cdot d^2\left(y, Ty\right) + C \cdot d^2\left(x, y\right), \tag{10}$$

hence T satisfies Inequality (3).

Remark 2. *We denote by*

$$E_n(x,y) = a \cdot d^n(x,Tx) + a \cdot d^n(y,Ty) + (1-2a) \cdot d^n(x,y),$$

for $n \in \{1,2\}$, $x,y \in X$.

By Lemma 1, we have that $d^2(Tx,Ty) < E_2(x,y)$ if $d(Tx,Ty) < E_1(x,y)$.

The following example shows that not every strict quadratic quasicontraction satisfies Inequality (1).

Example 1. *Let $X = [-1,1]$, $d(x,y) = |x-y|$ and $T : X \to X$, $Tx = 0$ for $-1 \le x \le \frac{1}{2}$ and $Tx = -1$ for $\frac{1}{2} < x \le 1$. Then, T satisfies Inequality (3) but does not verify Inequality (1).*

If $x,y \in \left[-1,\frac{1}{2}\right]$ or $x,y \in \left(\frac{1}{2},1\right]$, then $d(Tx,Ty) = 0$ and Inequality (3) is obvious.

If $x \in \left[-1,\frac{1}{2}\right]$ and $y \in \left(\frac{1}{2},1\right]$, then $d(Tx,Ty) = 1$ and

$$\begin{aligned} E_2(x,y) &= \frac{4}{9} \cdot d^2(x,Tx) + \frac{4}{9} \cdot d^2(y,Ty) + \frac{1}{9} \cdot d^2(x,y) \\ &= \frac{4}{9}x^2 + \frac{4}{9}(y+1)^2 + \frac{1}{9}(y-x)^2 \\ &\ge \frac{4}{9}x^2 + \frac{4}{9}\left(\frac{1}{2}+1\right)^2 + \frac{1}{9}\left(\frac{1}{2}-x\right)^2 \\ &= 1 + \frac{4}{9}x^2 + \frac{1}{9}\left(\frac{1}{2}-x\right)^2 > 1. \end{aligned}$$

Hence, Inequality (3) holds with $a = \frac{4}{9}$.

For $x = 0$ and $y = \frac{3}{4}$, we have $d(Tx,Ty) = 1$ and

$$\begin{aligned} E_1(x,y) &= a \cdot d(x,Tx) + a \cdot d(y,Ty) + (1-2a) \cdot d(x,y) \\ &= \frac{7a}{4} + \frac{3(1-2a)}{4} = \frac{a+3}{4} < 1, \end{aligned}$$

so Inequality (1) is not satisfied.

Theorem 4. *Let (X,d) be a compact metric space and let $T : X \to X$ be a strict quadratic quasicontraction. Then, T has a unique fixed point $v \in X$. Moreover, if T is continuous, then, for each $x \in X$, the sequence of iterates $\{T^n x\}$ converges to v.*

Proof. Taking $y = Tx$ in Inequality (3), we have for all $x \in X$ with $x \ne Tx$

$$d^2\left(Tx, T^2x\right) < a \cdot d^2(x,Tx) + a \cdot d^2\left(Tx,T^2x\right) + (1-2a) \cdot d^2(x,Tx).$$

This implies $d\left(Tx,T^2x\right) < d(x,Tx)$.

Let $\beta = \inf\{d(x,Tx) : x \in X\}$. By compactness of X, there exists a sequence $\{x_n\} \subset X$ such that $x_n \to u \in X$, $Tx_n \to v \in X$ and $\beta = \lim\limits_{n\to\infty} d(x_n,Tx_n) = d(u,v)$.

If there exists a subsequence $\left\{x_{n(k)}\right\}$ of $\{x_n\}$ such that $x_{n(k)} = v$ for every $k \in \mathbb{N}$, then $u = v$ and $Tv = v$. Otherwise, there exists $N \in \mathbb{N}$ such that $x_n \ne v$ for every $n \ge N$. Taking $x = x_n$ and $y = v$ in Inequality (3), we obtain

$$d^2(Tx_n,Tv) < a \cdot d^2(x_n,Tx_n) + a \cdot d^2(v,Tv) + (1-2a) \cdot d^2(x_n,v).$$

As $n \to \infty$, we get

$$d^2(v, Tv) \leq a \cdot d^2(u, v) + a \cdot d^2(v, Tv) + (1-2a) \cdot d^2(u, v).$$

This implies $d(v, Tv) \leq d(u, v) = \beta$. By definition of β, we have $d(v, Tv) = \beta$.

If $\beta > 0$, since $d(T^2 v, Tv) < d(v, Tv) = \beta$, we have a contradiction. Therefore, $\beta = 0$, so $u = v$.

If w is another fixed point of T, by Inequality (3), we have

$$d^2(Tv, Tw) < a \cdot d^2(v, Tv) + a \cdot d^2(w, Tw) + (1-2a) \cdot d^2(v, w),$$

where

$$d^2(v, w) < (1-2a) \cdot d^2(v, w),$$

which is a contradiction.

Now suppose T is continuous. Take any $x_0 \in X$ and define a sequence $\{x_n = T^n x_0\}$. If there exists $N \in \mathbb{N} \cup \{0\}$ such that $x_N = v$, then $x_n = v$ for all $n \geq N$ and then $x_n \to v$. Otherwise, we have $x_n \neq v$ for all $n \in \mathbb{N} \cup \{0\}$.

Since v is unique, we have $x_n \neq x_{n+1}$ for every $n \in \mathbb{N} \cup \{0\}$. Therefore, $d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) < d(x_n, x_{n-1})$ for every $n \in \mathbb{N}$, so sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and positive. Let $b = \lim\limits_{n\to\infty} d(x_{n+1}, x_n)$. The assumption that $b > 0$ leads to the contradiction. By compactness of X, sequence $\{x_n\}$ contains a subsequence $\left\{x_{n(k)}\right\}$ such that $x_{n(k)} \to z \in X$ as $k \to \infty$.

Because T is continuous, we have

$$0 < b = \lim_{n\to\infty} d\left(x_{n(k)+1}, x_{n(k)}\right) = d(Tz, z),$$

and

$$0 < b = \lim_{n\to\infty} d\left(x_{n(k)+2}, x_{n(k)+1}\right) = d\left(T^2 z, Tz\right).$$

Then, we get $d\left(T^2 z, Tz\right) = d(Tz, z) = b > 0$, which is a contradiction. Thus, $b = 0$.

Since

$$d^2(x_{n+1}, v) = d^2(Tx_n, Tv) < a \cdot d^2(x_n, Tx_n) + a \cdot d^2(v, Tv) + (1-2a) \cdot d^2(x_n, v),$$

we obtain

$$c_{n+1}^2 < (1-2a) \cdot c_n^2 + a \cdot b_n^2,$$

where $c_n = d(x_n, v)$ and $b_n = d(x_n, x_{n+1})$.

Since $d(x_n, v) \leq d(x_{n+1}, v) + d(x_n, x_{n+1})$, we get $c_n \leq c_{n+1} + b_n$, hence

$$c_{n+1}^2 < (1-2a) \cdot (c_{n+1} + b_n)^2 + a \cdot b_n^2.$$

This implies

$$\left(c_{n+1} - \frac{1-2a}{2a} \cdot b_n\right)^2 < \left[1 - a + \left(\frac{1-2a}{2a}\right)^2\right] \cdot b_n^2.$$

Taking the limit as $n \to \infty$, we obtain $\lim\limits_{n\to\infty} c_{n+1} = 0$, hence $x_n \to v$. □

Remark 3. *In Example 1, X is a compact metric space and T is a strict quadratic quasicontraction and asymptotic regular.*

In the following example, T is a strict quadratic quasicontraction and not asymptotic regular.

Example 2. *Let* $X = [-2,-1] \cup \{0\} \cup [1,2]$, $d(x,y) = |x-y|$ *and* $T : X \to X$,

$$Tx = \begin{cases} \frac{1-x}{2}, & \text{if } x \in [-2,-1), \\ 0, & \text{if } x \in \{-1,0\}, \\ \frac{-1-x}{2}, & \text{if } x \in [1,2]. \end{cases}$$

Then, T is not asymptotic regular and satisfies the hypothesis of Theorem (4).

It is obvious that (X,d) is a compact metric space. By induction, it is easy to prove that

$$T^n 2 = (-1)^n \cdot \frac{2^n+1}{2^n} \text{ for every } n \geq 1.$$

Thus,

$$d\left(T^n 2, T^{n+1} 2\right) = \frac{2^{n+1}+1}{2^{n+1}} + \frac{2^n+1}{2^n} > 2,$$

so T is not asymptotic regular.

If $x, y \in [-2,-1)$, $x \neq y$, then

$$d^2(Tx,Ty) = \frac{(x-y)^2}{4} < \frac{1}{4} \text{ and}$$

$$E_2(x,y) = a \cdot \left(\frac{1-3x}{2}\right)^2 + a \cdot \left(\frac{1-3y}{2}\right)^2 + (1-2a)(y-x)^2 > 4a + 4a = 8a.$$

For $a > \frac{1}{32}$, we have $d^2(x,y) < E_2(x,y)$.

If $x \in [-2,-1)$, $y = -1$, then

$$d(Tx,Ty) = \frac{1-x}{2} \text{ and}$$

$$E_1(x,y) = a \cdot \left(\frac{1-3x}{2}\right) + a + (1-2a)(x+1) = \frac{2-7a}{2} \cdot x + \frac{2-a}{2}.$$

Taking $a \geq \frac{3}{7}$, we have $\frac{1-a}{7a-3} > -1 > x$, so $(7a-3) \cdot x < 1-a$, then $1-x < (2-7a) \cdot x + 2 - a$. Hence, $d(Tx,Ty) < E_1(x,y)$ and by Remark (2) we get $d^2(x,y) < E_2(x,y)$.

If $x \in [-2,-1)$, $y = 0$, then

$$d(Tx,Ty) = \frac{1-x}{2} \text{ and}$$

$$E_1(x,y) = a \cdot \frac{1-3x}{2} + (1-2a) \cdot (-x) = \frac{a-2}{2} \cdot x + \frac{a}{2}.$$

Since $x < -1$ and $a < \frac{1}{2}$, we have $(1-a) \cdot x < a-1$, so $1-x < (a-2) \cdot x + a$. Thus, $d(Tx,Ty) < E_1(x,y)$, and then $d^2(x,y) < E_2(x,y)$.

If $x \in [-2,-1)$, $y \in [1,2]$, then

$$d(Tx,Ty) = \frac{1-x}{2} - \frac{-1-y}{2} = \frac{2+y-x}{2} \text{ and}$$

$$E_1(x,y) = a \cdot \frac{1-3x}{2} + a \cdot \frac{1+3y}{2} + (1-2a) \cdot (y-x) = \frac{2 \cdot a + (2-a) \cdot (y-x)}{2}.$$

Since $y - x > 2$ and $a < 1$, we have $2a - 2 > (a-1) \cdot (y-x)$, so $2 + y - x < 2 \cdot a + (2-a) \cdot (y-x)$. Thus, $d(Tx,Ty) < E_1(x,y)$, and then $d^2(x,y) < E_2(x,y)$.

If $x, y \in \{-1,0\}$, $x \neq y$, we have $d^2(x,y) = 0 < E_2(x,y)$.

If $x = -1$, $y \in [1,2]$, then

$$d(Tx,Ty) = \frac{1+y}{2} \text{ and}$$

$$E_1(x,y) = a + a \cdot \frac{1+3y}{2} + (1-2a) \cdot (y+1) = \frac{(2-a)(y+1)}{2}.$$

Since $y \geq 1$ and $a < \frac{1}{2}$, we have $1 + y < (2-a)(y+1)$. Thus, $d(Tx, Ty) < E_1(x,y)$, and then $d^2(x,y) < E_2(x,y)$.

If $x = 0$, $y \in [1,2]$, then

$$d(Tx, Ty) = \frac{1+y}{2} \text{ and}$$

$$E_1(x,y) = a \cdot \frac{1+3y}{2} + (1-2a) \cdot y = \frac{(2-a)y + a}{2}.$$

For $y > 1$, we have $1 - a < (1-a)y$, so $1 + y < (2-a)y + a$. Thus, $d(Tx, Ty) < E_1(x,y)$, and then $d^2(x,y) < E_2(x,y)$. For $y = 1$, we have $d(Tx, Ty) = 1 = E_1(x,y)$, but $E_2(x,y) = 4a + 1 - 2a = 1 + 2a > d^2(Tx, Ty)$.

If $x, y \in [1,2]$, $x \neq y$, then

$$d(Tx, Ty) = \frac{|y-x|}{2} \leq \frac{1}{2} \text{ and}$$

$$E_1(x,y) = a \cdot \frac{1+3x}{2} + a \cdot \frac{1+3y}{2} + (1-2a) \cdot |y-x| \geq 4a.$$

Taking $a > \frac{1}{8}$, we get $d(Tx, Ty) < E_1(x,y)$, and then $d^2(x,y) < E_2(x,y)$.

We note that, for $a = \frac{4}{9}$, we have that T is a strict quadratic quasicontraction.

Lemma 2. *Let C be a nonempty closed subset of a complete metric space (X, d) and let $T : C \to C$ be a quadratic quasicontraction mapping. Assume that there exist constants $a, b \in \mathbb{R}$ such that $0 \leq a < 1$ and $b > 0$. If for arbitrary $x \in C$ there exists $u \in C$ such that $d(u, Tu) \leq a \cdot d(x, Tx)$ and $d(u, x) \leq b \cdot d(x, Tx)$, then T has a unique fixed point.*

Proof. Let $x_0 \in C$ be an arbitrary point. Consider a sequence $\{x_n\} \subset C$ satisfying

$$d(Tx_{n+1}, x_{n+1}) \leq a \cdot d(Tx_n, x_n),$$
$$d(x_{n+1}, x_n) \leq b \cdot d(Tx_n, x_n), n = 0, 1, 2, \ldots$$

Since

$$d(x_{n+1}, x_n) \leq b \cdot d(Tx_n, x_n) \leq b \cdot a \cdot d(Tx_{n-1}, x_{n-1}) \leq \ldots \leq b \cdot a^n \cdot d(Tx_0, x_0), \tag{11}$$

it is easy to see that $\{x_n\}$ is a Cauchy sequence. Because C is complete, there exists $v \in C$ such that $\lim\limits_{n\to\infty} x_n = v$. By Inequalities (11) and the sandwich theorem, we get $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$ and then $\lim\limits_{n\to\infty} Tx_n = v$ and we have

$$d^2(Tx_n, Tv) \leq a \cdot d^2(x_n, Tx_n) + a \cdot d^2(v, Tv) + (1-2a) d^2(x_n, v).$$

Taking the limit as $n \to \infty$, we obtain

$$d^2(v, Tv) \leq ad^2(v, Tv).$$

This implies $d(v, Tv) = 0$, so $Tv = v$.

If u is another fixed point of T, then we have

$$d^2(Tu, Tv) \leq a \cdot d^2(u, Tu) + a \cdot d^2(v, Tv) + (1-2a) d^2(u, v),$$

hence

$$d^2(u, v) \leq (1-2a) d^2(u, v).$$

Therefore, $d(u, v) = 0$ and v is the unique fixed point of T. $\square$

Theorem 5. *Let X be a Banach space and C be a closed convex subset of X. Let $T : C \to C$ be a mapping satisfying the inequality:*

$$\|Tx - Ty\|^2 \leq a \cdot \|x - Tx\|^2 + a \cdot \|y - Ty\|^2 + b \cdot \|x - y\|^2 \tag{12}$$

for all $x, y \in C$, where $0 < a < \frac{1}{2}$, $b = 1 - 2a$. Then, T has a unique fixed point.

Proof. Taking $y = Tx$ in Inequality (12), we have

$$\left\|Tx - T^2x\right\|^2 \leq a \cdot \|x - Tx\|^2 + a \cdot \left\|Tx - T^2x\right\|^2 + b \cdot \|x - Tx\|^2 .$$

Then,

$$(1 - a) \cdot \left\|Tx - T^2x\right\|^2 \leq (a + b) \cdot \|x - Tx\|^2 = (1 - a) \cdot \|x - Tx\|^2 ,$$

so

$$\left\|Tx - T^2x\right\| \leq \|x - Tx\| \tag{13}$$

for all $x \in C$.

Let $x \in C$ fixed and $z = \frac{1}{2}T^2x + \frac{1}{2}T^3x$. Since C is convex, we have $z \in C$. Then, by Inequalities (12) and (13) , we get

$$\begin{aligned}
\left\|Tx - T^3x\right\|^2 &\leq a \cdot \|x - Tx\|^2 + a \cdot \left\|T^2x - T^3x\right\|^2 + b \cdot \left\|x - T^2x\right\|^2 \\
&\leq 2a \cdot \|x - Tx\|^2 + b \cdot \left(\|x - Tx\| + \left\|Tx - T^2x\right\|\right)^2 \\
&\leq (2a + 4b) \cdot \|x - Tx\|^2 \\
&= (1 + 3b) \cdot \|x - Tx\|^2 ,
\end{aligned}$$

so

$$\left\|Tx - T^3x\right\| \leq \sqrt{1 + 3b} \cdot \|x - Tx\| .$$

Therefore,

$$\begin{aligned}
\|Tx - z\| &= \left\|\frac{1}{2}\left(Tx - T^2x\right) + \frac{1}{2}\left(Tx - T^3x\right)\right\| \\
&\leq \frac{1}{2}\left\|Tx - T^2x\right\| + \frac{1}{2}\left\|Tx - T^3x\right\| \\
&\leq \frac{1}{2}\|x - Tx\| + \frac{1}{2}\sqrt{1 + 3b} \cdot \|x - Tx\| \\
&= \frac{1 + \sqrt{1 + 3b}}{2} \cdot \|x - Tx\| .
\end{aligned} \tag{14}$$

In addition,

$$\left\|T^2x - z\right\| = \frac{1}{2}\left\|T^2x - T^3x\right\| \leq \frac{1}{2}\|x - Tx\| . \tag{15}$$

Now, by Inequalities (12), (13) and (14), we obtain

$$\begin{aligned}
\left\|T^2x - Tz\right\|^2 &\le a \cdot \left\|Tx - T^2x\right\|^2 + a \cdot \|z - Tz\|^2 + b \cdot \|Tx - z\|^2 \\
&\le a \cdot \|x - Tx\|^2 + a \cdot \|z - Tz\|^2 \\
&\quad + b \cdot \left(\frac{1 + \sqrt{1 + 3b}}{2}\right)^2 \cdot \|x - Tx\|^2 \\
&= a \cdot \|z - Tz\|^2 \\
&\quad + \left[a + b \cdot \left(\frac{1 + \sqrt{1 + 3b}}{2}\right)^2\right] \cdot \|x - Tx\|^2 .
\end{aligned} \tag{16}$$

In addition, by Inequalities (12), (13) and (15), we have

$$\begin{aligned}
\left\|T^3x - Tz\right\| &\le a \cdot \left\|T^2x - T^3x\right\|^2 + a \cdot \|z - Tz\|^2 + b \cdot \left\|T^2x - z\right\|^2 \\
&\le a \cdot \|x - Tx\|^2 + a \cdot \|z - Tz\|^2 + \frac{b}{4} \cdot \|x - Tx\|^2 \\
&= \left(a + \frac{b}{4}\right) \cdot \|x - Tx\|^2 + a \cdot \|z - Tz\|^2 .
\end{aligned} \tag{17}$$

Since

$$\begin{aligned}
\|z - Tz\| &= \left\|\frac{1}{2}\left(T^2x - Tz\right) + \frac{1}{2}\left(T^3x - Tz\right)\right\| \\
&\le \frac{1}{2}\left\|T^2x - Tz\right\| + \frac{1}{2}\left\|T^3x - Tz\right\|,
\end{aligned}$$

by Inequalities (16) and (17), we obtain

$$\begin{aligned}
\|z - Tz\| &\le \frac{1}{2} \cdot \left\{a \cdot \|z - Tz\|^2 + \left[a + b \cdot \left(\frac{1 + \sqrt{1 + 3b}}{2}\right)^2\right] \cdot \|x - Tx\|^2\right\}^{\frac{1}{2}} \\
&\quad + \frac{1}{2} \cdot \left[a \cdot \|z - Tz\|^2 + \left(a + \frac{b}{4}\right) \cdot \|x - Tx\|^2\right]^{\frac{1}{2}} .
\end{aligned} \tag{18}$$

If $x = Tx$, then x is a fixed point of T.
Otherwise, dividing Inequality (18) by $\frac{1}{2} \cdot \|x - Tx\|$, we get

$$\begin{aligned}
2 \cdot \frac{\|z - Tz\|}{\|x - Tx\|} &\le \left[a \cdot \frac{\|z - Tz\|^2}{\|x - Tx\|^2} + a + b \cdot \left(\frac{1 + \sqrt{1 + 3b}}{2}\right)^2\right]^{\frac{1}{2}} \\
&\quad + \left(a \cdot \frac{\|z - Tz\|^2}{\|x - Tx\|^2} + a + \frac{b}{4}\right)^{\frac{1}{2}} .
\end{aligned}$$

Denoting $\frac{\|z - Tz\|^2}{\|x - Tx\|^2} = t$, we obtain

$$2\sqrt{t} \le \left[a \cdot t + a + \frac{b \cdot \left(1 + \sqrt{1 + 3b}\right)^2}{4}\right]^{\frac{1}{2}} + \left(a \cdot t + a + \frac{b}{4}\right)^{\frac{1}{2}},$$

where

$$2 \le \left[a + \frac{a}{t} + \frac{b \cdot \left(1 + \sqrt{1+3b}\right)^2}{4t}\right]^{\frac{1}{2}} + \left(a + \frac{a}{t} + \frac{b}{4t}\right)^{\frac{1}{2}}.$$

Let

$$f(t) = \left[a + \frac{a}{t} + \frac{b \cdot \left(1 + \sqrt{1+3b}\right)^2}{4t}\right]^{\frac{1}{2}} + \left(a + \frac{a}{t} + \frac{b}{4t}\right)^{\frac{1}{2}}$$

for all $t > 0$. Obviously, f is a decreasing function and

$$\begin{aligned} f(1) &= \left[2a + \frac{b \cdot \left(1 + \sqrt{1+3b}\right)^2}{4}\right]^{\frac{1}{2}} + \left(2a + \frac{b}{4}\right)^{\frac{1}{2}} \\ &= \left[1 - b + \frac{b \cdot \left(1 + \sqrt{1+3b}\right)^2}{4}\right]^{\frac{1}{2}} + \left(1 - \frac{3b}{4}\right)^{\frac{1}{2}}. \end{aligned}$$

We claim that $f(1) < 2$.
Let $\alpha = \sqrt{1+3b}$. Obviously, since $b \in (0,1)$, we have $\alpha \in (1,2)$ and

$$\begin{aligned} f(1) &= \left[1 - \frac{\alpha^2 - 1}{3} + \frac{\left(\alpha^2 - 1\right)}{12} \cdot (1+\alpha)^2\right]^{\frac{1}{2}} + \left(1 - \frac{\alpha^2 - 1}{4}\right)^{\frac{1}{2}} \\ &= \left(\frac{\alpha^4 + 2\alpha^3 - 4\alpha^2 - 2\alpha + 15}{12}\right)^{\frac{1}{2}} + \left(\frac{5 - \alpha^2}{4}\right)^{\frac{1}{2}}. \end{aligned}$$

Now,

$$\begin{aligned} f(1) < 2 \Leftrightarrow \left(\frac{\alpha^4 + 2\alpha^3 - 4\alpha^2 - 2\alpha + 15}{12}\right)^{\frac{1}{2}} &< 2 - \left(\frac{5 - \alpha^2}{4}\right)^{\frac{1}{2}} \\ \Longleftrightarrow \frac{\alpha^4 + 2\alpha^3 - 4\alpha^2 - 2\alpha + 15}{12} &< 4 + \frac{5 - \alpha^2}{4} - 2\sqrt{5 - \alpha^2} \\ \Longleftrightarrow \alpha^4 + 2\alpha^3 - 4\alpha^2 - 2\alpha &< 24\left(2 - \sqrt{5 - \alpha^2}\right) \\ \Longleftrightarrow \alpha(\alpha + 2)\left(\alpha^2 - 1\right) &< \frac{24\left(\alpha^2 - 1\right)}{2 + \sqrt{5 - \alpha^2}} \\ \Longleftrightarrow 2 + \sqrt{5 - \alpha^2} &< \frac{24}{\alpha(\alpha + 2)}. \end{aligned} \tag{19}$$

Let $h : [1,2] \to \mathbb{R}$, $h(\alpha) = 2 + \sqrt{5 - \alpha^2} - \frac{24}{\alpha(\alpha+2)}$.
To prove Inequality (19), we will show that h is an increasing function and $h(2) = 0$.
We have $h(\alpha) = 2 + \sqrt{5 - \alpha^2} - \frac{12}{\alpha} + \frac{12}{\alpha+2}$ and $h'(\alpha) = \frac{-\alpha}{\sqrt{5-\alpha^2}} + \frac{12}{\alpha^2} - \frac{12}{(\alpha+2)^2}$. However,

$$\begin{aligned} h'(\alpha) > 0 &\Longleftrightarrow 48(\alpha + 1)\sqrt{5 - \alpha^2} > \alpha^3(\alpha + 2)^2 \\ &\Longleftrightarrow 48\sqrt{5 - \alpha^2} > \frac{\alpha^3(\alpha + 2)^2}{\alpha + 1} = \frac{\alpha^5 + 4\alpha^4 + 4\alpha^3}{\alpha + 1}. \end{aligned} \tag{20}$$

Since $\varphi : [1,2] \to \mathbb{R}$, $\varphi(\alpha) = 48\sqrt{5-\alpha^2}$ is a decreasing function with $\varphi(2) = 48$, and $\psi : [1,2] \to \mathbb{R}$, $\psi(\alpha) = \frac{\alpha^5+4\alpha^4+4\alpha^3}{\alpha+1}$ is an increasing function with $\psi(2) = \frac{128}{3} < 48$, we obtain Inequality (20). This implies Inequality (19), so $f(1) < 2$. Since f is a decreasing function and $f(t) \geq 2$, there exists $c < 1$ such that $t \leq c$. Therefore, $\|z - Tz\| \leq \sqrt{c}\,\|x - Tx\|$.

Now, since

$$\begin{aligned} \|z-x\| &\leq \frac{1}{2}\left\|T^2x - x\right\| + \frac{1}{2}\left\|T^3x - x\right\| \\ &\leq \frac{1}{2}\left(\left\|T^2x - Tx\right\| + \|Tx - x\|\right) \\ &\quad + \frac{1}{2}\left(\left\|T^3x - T^2x\right\| + \left\|T^2x - Tx\right\| + \|Tx - x\|\right) \\ &\leq \frac{5}{2}\|x - Tx\|, \end{aligned}$$

applying Lemma 2, we get that T has a unique fixed point. □

Example 3. *Let $X = l^\infty(\mathbb{R})$ be the set of bounded sequences of real numbers and $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$, where $x = \{x_n\}_{n\in\mathbb{N}}$. It is known that $(X, \|\cdot\|)$ is a Banach space. Let $C = \{x \in X : \|x\| \leq 1\}$ and $T : C \to C$,*

$$Tx = \begin{cases} \mathbf{\frac{1}{2}}, & \text{if } x = -\mathbf{1}, \\ -\mathbf{1}, & \text{if } x_n \in \left[\frac{1}{2}, 1\right] \text{ for every } n \in \mathbb{N}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $x = \{x_n\}_{n\in\mathbb{N}}$, $\mathbf{c} = \{c, c, c, ...\}$. It is obvious that C is closed, convex and not compact. Since $T^n(-\mathbf{1}) = \mathbf{\frac{1}{2}}$ if n is odd and $T^n(-\mathbf{1}) = -\mathbf{1}$ if n is even, we note that T is not asymptotic regular.

If $x = -\mathbf{1}$ and $y = \{y_n\}_{n\in\mathbb{N}}$ where $y_n \in \left[\frac{1}{2}, 1\right]$ for every $n \in \mathbb{N}$, then

$$d(Tx, Ty) = \frac{3}{2} \text{ and}$$

$$\begin{aligned} E_1(x,y) &= \frac{3}{2}a + a \cdot \sup_{n\in\mathbb{N}}(1+y_n) + (1-2a)\cdot \sup_{n\in\mathbb{N}}(1+y_n) \\ &= \frac{3}{2}a + (1-a)\cdot \sup_{n\in\mathbb{N}}(1+y_n) \geq \frac{3}{2}a + \frac{3}{2}(1-a) = \frac{3}{2}, \end{aligned}$$

so $d(Tx, Ty) \leq E_1(x,y)$, and then $d^2(x,y) \leq E_2(x,y)$.

If $x = -\mathbf{1}$ and $y = \{y_n\}_{n\in\mathbb{N}}$ where there exists n_0 such that $y_{n_0} \notin \left[\frac{1}{2}, 1\right]$, then

$$d(Tx, Ty) = \frac{1}{2} \text{ and}$$

$$E_1(x,y) = \frac{3}{2}a + a \cdot \sup_{n\in\mathbb{N}}|y_n| + (1-2a)\cdot \sup_{n\in\mathbb{N}}(1+y_n) \geq \frac{3}{2}a.$$

Hence, for $a \geq \frac{1}{3}$, we have $d(Tx, Ty) \leq E_1(x,y)$, and then $d^2(x,y) \leq E_2(x,y)$.

If $x = \{x_n\}_{n\in\mathbb{N}}$ where $x_n \in \left[\frac{1}{2}, 1\right]$ for every $n \in \mathbb{N}$ and $y = \{y_n\}_{n\in\mathbb{N}}$ where there exists n_0 such that $y_{n_0} \notin \left[\frac{1}{2}, 1\right]$, then

$$d(Tx, Ty) = 1 \text{ and}$$

$$E_2(x,y) = a \cdot \sup_{n\in\mathbb{N}}(x_n+1)^2 + a \cdot \sup_{n\in\mathbb{N}} y_n^2 + (1-2a)\cdot \sup_{n\in\mathbb{N}}(x_n - y_n)^2 \geq \frac{9}{4}a.$$

Hence, for $a \geq \frac{4}{9}$*, we have* $d^2(Tx, Ty) \leq E_2(x, y)$*. We note that* $x = \frac{1}{2}$ *and* $y = \mathbf{0}$*, and we have* $E_1(x, y) = \frac{3}{2}a + \frac{1}{2}(1 - 2a) = \frac{1+a}{2} < 1 = d(Tx, Ty)$*. Therefore, T does not satisfy Theorem (3).*

In other cases $d^2(Tx, Ty) = 0 \leq E_2(x, y)$*.*

3. Conclusions

We have introduced the class of quadratic quasicontractive mapping and prove two generalizations of some classical fixed point theorems: Edelstein's theorem, Hardy-Rogers's theorem and Gregus's theorem. Furthermore, we have presented some examples to support our main results.

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