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Some New Quantum Hermite–Hadamard-Type Estimates Within a Class of Generalized (s, m) -Preinvex Functions

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Abstract: In this work, we discover a new version of Hermite–Hadamard quantum integrals inequality via m -preinvex functions. Moreover, the authors present a quantum integrals identity and drive some new quantum integrals of Hermite–Hadamard-type inequalities involving generalized (s, m) -preinvex functions.

Keywords: quantum calculus; Hermite–Hadamard-type inequalities; generalized m -preinvex functions; generalized (s, m) -preinvex functions

1. Introduction

Quantum calculus or q -calculus is a methodology applicable to the typical study of calculus but it is mainly centered on the idea of derivation of q -analogous results excluding the use of limits. This concept was first introduced by Euler who started his study in the earlier years of the 18th century. It is the q -analogue of the ordinary derivative of a function, it is also known as Jackson or quantum derivative in some branches of mathematics, especially in combinatorics, see [1]. In recent years, the topic of q -calculus has attracted the attention of several scholars. That is why q -calculus is called a bridge between mathematics and physics. Having numerous applications in mathematics as well as in physics, q -calculus has emerged as an interesting and most fascinating field of research in recent years. Many researchers have written a number of papers on quantum integrals, for more details, see [2–9].

Inequality theory plays a key role in pure and applied sciences, and also has comprehensive applications in various areas of pure and applied mathematics.

A function $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ is called convex on J if the inequality

$$h(\tau\phi + (1 - \tau)\psi) \leq \tau h(\phi) + (1 - \tau)h(\psi)$$

holds for all $\phi, \psi \in I$ and $\tau \in [0, 1]$.

Motivated by the idea of convex function, Hermite and Hardamard [10] first introduced the following inequality that is called Hermite–Hadamard inequality:

$$h\left(\frac{\phi + \psi}{2}\right) \leq \frac{1}{\psi - \phi} \int_{\phi}^{\psi} h(x) dx \leq \frac{h(\phi) + h(\psi)}{2}. \quad (1)$$

Due to its geometrical interpretation and applications, the Hermite–Hadamard inequality is one of the finest inequalities among the inequalities of convex functions. This fundamental result of Hermite and Hadamard has attracted many mathematicians and consequently this inequality has been generalized and extended in different directions using novel and innovative ideas, see [11–14].

Next, Tariboon et al. [15,16] obtained some of the most important integral inequalities of analysis are extended to quantum calculus which is q -analogue of Hermite–Hadamard’s inequality on finite integral.

$$h\left(\frac{\phi + \psi}{2}\right) \leq \frac{1}{\psi - \phi} \int_{\phi}^{\psi} h(x) {}_{\phi}d_q x \leq \frac{qh(\phi) + h(\psi)}{1 + q}. \quad (2)$$

An important contribution to the subject was made by Alp et al. [17] who introduced corrected q -Hermite–Hadamard inequality, which can be written as:

$$h\left(\frac{q\phi + \psi}{2}\right) \leq \frac{1}{\psi - \phi} \int_{\phi}^{\psi} h(x) {}_{\phi}d_q x \leq \frac{qh(\phi) + h(\psi)}{2}. \quad (3)$$

Recently, Noor et al. [18] proposed some important results on quantum Hermite–Hadamard inequality for preinvex functions that can be written as follows:

A function $h : J_v = [\phi, \phi + v(\psi, \phi)] \subset \mathbb{R} \rightarrow \mathbb{R}$ be integrable and preinvex function with $v(\psi, \phi) > 0$. If the bifunction $v(.,.)$ satisfies the Condition C, then, we have

$$h\left(\frac{2\phi + v(\psi, \phi)}{2}\right) \leq \frac{1}{v(\psi, \phi)} \int_{\phi}^{\phi + v(\psi, \phi)} h(x) {}_{\phi}d_q x \leq \frac{qh(\phi) + h(\psi)}{1 + q}. \quad (4)$$

Proposition 1. Let $h : J_v = [\phi, \phi + v(\psi, \phi)] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a quantum differential mapping over J_v^o (interior of J_v) with $q \in (0, 1)$. If ${}_{\phi}D_q h$ is continuous and integrable over J_v . Then, the following identity holds:

$$\frac{qh(\phi) + h(\psi)}{1 + q} - \frac{1}{v(\psi, \phi)} \int_{\phi}^{\phi + v(\psi, \phi)} h(x) {}_{\phi}d_q x = \frac{qv(\psi, \phi)}{1 + q} \int_0^1 (1 - (1 + q)\tau) {}_{\phi}D_q h(\phi + \tau v(\psi, \phi)) {}_0d_q \tau.$$

Liu et al. [19] proposed the following results based on twice quantum integral identity and developed some trapezoid-type inequalities for convex function.

Proposition 2. Let $h : J = [\phi, \psi] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differential mapping over J^o (the interior of J) with ${}_{\phi}D_q^2 h$ being continuous and q -integrable over J , where $q \in (0, 1)$. Then, the following identity holds:

$$\frac{qh(\phi) + h(\psi)}{1 + q} - \frac{1}{\psi - \phi} \int_{\phi}^{\psi} h(x) {}_{\phi}d_q x = \frac{q^2(\psi - \phi)^2}{1 + q} \int_0^1 \tau(1 - q\tau) {}_{\phi}D_q^2 h((1 - \tau)\phi + \tau\psi) {}_0d_q \tau.$$

Theorem 3. Let $h : J = [\phi, \psi] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differential mapping over J^o (the interior of J) with ${}_{\phi}D_q^2 h$ being continuous and q -integrable over J , where $q \in (0, 1)$. If $|{}_{\phi}D_q^2 h|$ is convex on $[\phi, \psi]$, then

$$\left| \frac{qh(\phi) + h(\psi)}{1 + q} - \frac{1}{\psi - \phi} \int_{\phi}^{\psi} h(x) {}_{\phi}d_q x \right| \leq \frac{q^2(\psi - \phi)^2 (q^2 |{}_{\phi}D_q^2 h(\phi)| + |{}_{\phi}D_q^2 h(\psi)|)}{(1 + q)(1 + q + q^2)(1 + q + q^2 + q^3)}.$$

Theorem 4. Let $h : J = [\phi, \psi] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differential mapping over J^o (the interior of J) with ${}_{\phi}D_q^2 h$ being continuous and q -integrable over I , where $q \in (0, 1)$. If $|{}_{\phi}D_q^2 h|^r$ is convex on $[\phi, \psi]$, where $s, r > 1, \frac{1}{s} + \frac{1}{r} = 1$, then

$$\left| \frac{qh(\phi) + h(\psi)}{1 + q} - \frac{1}{\psi - \phi} \int_{\phi}^{\psi} h(x) {}_{\phi}d_q x \right| \leq \frac{q^2(\psi - \phi)^2}{1 + q} (\eta)^{\frac{1}{s}} \left(\frac{q |{}_{\phi}D_q^2 h(\phi)|^r + |{}_{\phi}D_q^2 h(\psi)|^r}{1 + q} \right)^{\frac{1}{r}},$$

where

$$\eta = (1 - q) \sum_{n=0}^{\infty} (q^n)^{s+1} (1 - q^{n+1})^s.$$

Theorem 5. Let $h : J = [\phi, \psi] \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differential mapping over I^o (the interior of J) with ${}_{\phi}D_q^2 h$ being continuous and q -integrable over I , where $q \in (0, 1)$. If $|{}_{\phi}D_q^2 h|^r$ is convex on $[\phi, \psi]$ for $r \geq 1$, then

$$\left| \frac{qh(\phi) + h(\psi)}{1 + q} - \frac{1}{\psi - \phi} \int_{\phi}^{\psi} h(x) {}_{\phi}d_q x \right| \leq \frac{q^2(\psi - \phi)^2}{(1 + q)^{2 - \frac{1}{r}}} \left(\frac{q^2 |{}_{\phi}D_q^2 h(\phi)|^r + |{}_{\phi}D_q^2 h(\psi)|^r}{1 + q + q^2 + q^3} \right)^{\frac{1}{r}}.$$

2. Preliminaries

In this section, Suppose that Δ is a nonempty bounded set in \mathbb{R}^n and Δ^o denotes the interior of Δ . The generic n -dimensional vector-space will be represented by \mathbb{R}^n and $\mathbb{R}_0 = [0, \infty)$.

Ben-Israel et al. [20] defined the concept of invex set as follows, which is a generalization of convex set:

Definition 1. Let $\Delta \subseteq \mathbb{R}^n$ be an invex set with respect to $v : \Delta \times \Delta \rightarrow \mathbb{R}^n$, if

$$\phi + \tau v(\psi, \phi) \in \Delta$$

for all $\phi, \psi \in \Delta$ and $\tau \in [0, 1]$.

Pini. R in [21] introduced the idea of invexity and generalized convexity

Definition 2. Let $\Delta \subseteq \mathbb{R}^n$ be an invex set with respect to $v : \Delta \times \Delta \rightarrow \mathbb{R}^n$. Let $h : \Delta \rightarrow \mathbb{R}$ be called a preinvex function if

$$h(\phi + \tau v(\psi, \phi)) \leq (1 - \tau)h(\phi) + \tau h(\psi)$$

holds for all $\phi, \psi \in \Delta$ and $\tau \in [0, 1]$.

The following definitions for generalized (s, m) -preinvex function, quantum derivative and integral of function h are stated as:

Author J. Y. Li [22] has introduced the concept of inequality for s -preinvex function

Definition 3. Let $\Delta \subseteq \mathbb{R}_0$ be an invex set with respect to $v : \Delta \times \Delta \rightarrow \mathbb{R}^n$. A function $h : \Delta \rightarrow \mathbb{R}$ is called s -preinvex function if

$$h(\phi + \tau v(\psi, \phi)) \leq (1 - \tau)^s h(\phi) + \tau^s h(\psi)$$

holds for all $\phi, \psi \in \Delta$, $\tau \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Ting-Song Du et al. [23] first established the idea of m -invex set and generalized (s, m) -preinvex functions as follows:

Definition 4. Let $\Delta \subseteq \mathbb{R}^n$ be m -invex set with respect to the function $v : \Delta \times \Delta \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if

$$m\phi + \tau v(\psi, \phi, m) \in \Delta$$

holds for each $\phi, \psi \in \Delta$ and any $\tau \in [0, 1]$.

Example 1. Consider $\Delta = [-3, -2] \cup [-1, 2]$ and

$$v(\psi, \phi, m) = \begin{cases} \psi - m\phi, & \text{if } 2 \geq \phi \geq -1, 2 \geq \psi \geq -1; \\ \psi - m\phi, & \text{if } -3 \leq \phi \leq -2, -3 \leq \psi \leq -2; \\ -1 - m\phi, & \text{if } -3 \leq \phi \leq -2, -1 \leq \psi \leq 2; \\ -3 - m\phi, & \text{if } -1 \leq \phi \leq 2, -3 \leq \psi \leq -2. \end{cases},$$

Clearly, Δ is an invex set with respect to v but not a convex set.

Remark 1. Definition 4 shows that the m -invex set $v(\psi, \phi, m)$ degenerates to an invex set $v(\psi, \phi)$, if we take $m = 1$.

We introduce the new concept of generalized m -preinvex and (s, m) -preinvex functions

Definition 5. Let $h : \Delta \rightarrow \mathbb{R}$, h is said to be a generalized m -preinvex with respect to function $v : \Delta \times \Delta \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if

$$h(m\phi + \tau v(\psi, \phi, m)) \leq (1 - \tau)h(m\phi) + \tau h(\psi)$$

holds for all $\phi, \psi \in \Delta$ and $\tau \in [0, 1]$.

Definition 6. Let $h : \Delta \rightarrow \mathbb{R}$, h is said to be a generalized (s, m) -preinvex with respect to function $v : \Delta \times \Delta \times (0, 1] \rightarrow \mathbb{R}^n$ for some fixed $s, m \in (0, 1]$ if

$$h(\phi m + \tau v(\psi, \phi, m)) \leq (1 - \tau)^s h(m\phi) + \tau^s h(\psi)$$

holds for all $\phi, \psi \in \Delta$, $\tau \in [0, 1]$.

Remark 2. If we take $v(\psi, \phi, m) = \psi - m\phi$ in Definition 6, then the generalized (s, m) -preinvex function could reduce to (s, m) -convex function.

Example 2. Let $h(\phi) = -|\phi|$, $s = 1$ and

$$v(\psi, \phi, m) = \begin{cases} \psi - m\phi, & \text{if } \phi \geq 0, \psi \geq 0; \\ \psi - m\phi, & \text{if } \phi \leq 0, \psi \leq 0; \\ m\phi - \psi, & \text{if } \phi \geq 0, \psi \leq 0; \\ m\phi - \psi, & \text{if } \phi \leq 0, \psi \geq 0. \end{cases}$$

Then, $h(\phi)$ is a generalized $(1, m)$ -preinvex function with respect to $v : \mathbb{R} \times \mathbb{R} \times (0, 1] \rightarrow \mathbb{R}$ and for some fixed $m \in (0, 1]$.

Note: If we take $m = 1$ in Example 1 and Example 2, then $v(\psi, \phi, 1)$ could reduce to $v(\psi, \phi)$.

In [24], Mohan et al. introduced the concept of well-known Condition C, rewritten as follows:

Definition 7. Let $\Delta \subset \mathbb{R}$ be an invex set with respect to bifunction $v(., .)$. Then, for any $\phi, \psi \in \Delta$ and $\tau \in [0, 1]$,

$$\begin{aligned} v(\psi, \psi + \tau v(\phi, \psi)) &= -\tau v(\phi, \psi), \\ v(\phi, \psi + \tau v(\phi, \psi)) &= (1 - \tau)v(\phi, \psi). \end{aligned}$$

The idea of preinvex function is more generalized than convex function because every convex function is preinvex with respect to the property $v(\psi, \phi) = \psi - \phi$, but converse is not true.

We recall some previously known concepts on q -calculus which will be used in this paper. Tariboon et al. [16] proposed the concept of quantum derivative and integration over finite interval $[\phi, \psi]$.

Definition 8. Consider a continuous function $h : [\phi, \psi] \rightarrow \mathbb{R}$, then, the quantum derivative of function h at $\tau \in [\phi, \psi]$ with $q \in (0, 1)$ is written as

$${}_{\phi}D_q h(\tau) = \frac{h(\tau) - h(q\tau + (1-q)\phi)}{(1-q)(\tau - \phi)}, \quad \tau \neq \phi. \quad (5)$$

Since $h : [\phi, \psi] \rightarrow \mathbb{R}$ is a continuous function, we thus have ${}_{\phi}D_q h(\phi) = \lim_{\tau \rightarrow \phi} ({}_{\phi}D_q h(\tau))$.

If we take $\phi = 0$ in (5), then ${}_0D_q h = D_q h$, where $D_q h$ is a familiar quantum derivative of $h(\tau)$ defined by

$${}_{\phi}D_q h(\tau) = \frac{h(\tau) - h(q\tau)}{(1-q)\tau}, \quad \tau \neq 0.$$

Definition 9. Consider a continuous function $h : [\phi, \psi] \rightarrow \mathbb{R}$. We introduce the concept of 2nd-order quantum derivative on interval $[\phi, \psi]$. In quantum calculus, the 2nd-order quantum derivative, denoted as ${}_{\phi}D_q^2 h$, is defined as ${}_{\phi}D_q^2 h = {}_{\phi}D_q ({}_{\phi}D_q h) : [\phi, \psi] \rightarrow \mathbb{R}$. Similarly, we define ${}_{\phi}D_q^n : [\phi, \psi] \rightarrow \mathbb{R}$, which is called a higher-order quantum derivative on $[\phi, \psi]$ with $q \in (0, 1)$.

Example 3. Define a function $h : [\phi, \psi] \rightarrow \mathbb{R}$ by $h(\tau) = \tau^2 + 1$ with $q \in (0, 1)$. Then, for $\tau \neq \phi$ we have

$$\begin{aligned} {}_{\phi}D_q^2(\tau^2 + 1) &= {}_{\phi}D_q \left(\frac{\tau^2 + 1 - ((q\tau + (1-q)\phi)^2 + 1)}{(1-q)(\tau - \phi)} \right) \\ &= {}_{\phi}D_q \left(\frac{(1+q)\tau^2 - 2\phi\tau q - (1+q)\phi^2}{(\tau - \phi)} \right) \\ &= {}_{\phi}D_q ((1+q)\tau + (1-q)\phi) \\ &= \frac{(1+q)\tau + (1-q)\phi - ((1+q)(q\tau + (1-q)\phi) + (1-q)\phi)}{(1-q)(\tau - \phi)} \\ &= 1 + q. \end{aligned}$$

Definition 10. Consider a continuous function $h : [\phi, \psi] \rightarrow \mathbb{R}$. The quantum integral on $[\phi, \psi]$ with $q \in (0, 1)$ is stated as

$$\int_{\phi}^{\tau} h(x) {}_{\phi}d_q x = (1-q)(\tau - \phi) \sum_{n=0}^{\infty} q^n h(q^n \tau + (1-q^n)\phi), \quad (6)$$

for $\tau \in [\phi, \psi]$.

Example 4. Define function $h : [\phi, \psi] \rightarrow \mathbb{R}$ by $h(x) = 4x + 1$ with $q \in (0, 1)$. Then, we have

$$\begin{aligned} \int_{\phi}^{\tau} (4x + 1) {}_{\phi}d_q x &= (1-q)(\tau - \phi) \left(4 \sum_{n=0}^{\infty} q^n (q^n \tau + (1-q^n)\phi) + \sum_{n=0}^{\infty} q^n \right) \\ &= \frac{(\tau - \phi) [4(\tau + q\phi) + (1+q)]}{1+q}. \end{aligned}$$

Note that if we take $\phi = 0$ in (6), then we obtain the concept of classical quantum integral as

$$\int_0^\tau h(x)_0 d_q x = (1-q)\tau \sum_{n=0}^{\infty} q^n h(q^n \tau),$$

for $\tau \in [0, \infty)$.

If $c \in (\phi, \tau)$, then the definite quantum integral on $[\phi, \psi]$ is expressed as

$$\int_c^\tau h(x)_\phi d_q x = \int_\phi^\tau h(x)_\phi d_q x - \int_\phi^c h(x)_\phi d_q x.$$

Lemma 1. Let $\delta \in \mathbb{R} \setminus \{-1\}$, then

$$\int_\phi^\tau (x - \phi)_\phi^\delta d_q x = \left(\frac{1-q}{1-q^{\delta+1}} \right) (\tau - \phi)^{\delta+1}.$$

Theorem 6. Consider continuous functions $h_1, h_2 : [\phi, \psi] \rightarrow \mathbb{R}$, where $\alpha \in \mathbb{R}$. Then, for $\tau \in [\phi, \psi]$, with $q \in (0, 1)$,

$$\begin{aligned} \int_\phi^\tau [h_1(x) + h_2(x)]_\phi d_q x &= \int_\phi^\tau h_1(x)_\phi d_q x + \int_\phi^\tau h_2(x)_\phi d_q x, \\ \int_\phi^\tau (\alpha h_1(x))_\phi d_q x &= \alpha \int_\phi^\tau h_1(x)_\phi d_q x. \end{aligned}$$

In addition, we introduce the quantum analogues of σ , $(x - \sigma)^n$ and the definition of the quantum Beta function, see [25].

Definition 11. For any $\sigma \in \mathbb{R}$,

$$[\sigma] = \frac{1 - q^\sigma}{1 - q}$$

is called the quantum analogues of σ .

In particular, for $n \in \mathbb{Z}^+$, we denote

$$[n] = \frac{1 - q^n}{1 - q} = q^{n-1} + \dots + q + 1.$$

Definition 12. If n is an integer, the quantum analogue of $(x - \sigma)^n$ is the polynomial

$$(x - \sigma)_q^n = \begin{cases} 1 & \text{if } n = 0 \\ (x - \sigma)(x - q\sigma) \dots (x - q^{n-1}\sigma), & \text{if } n \geq 1. \end{cases}$$

Definition 13. For any $r, \tau > 0$,

$$\beta_q(r, \tau) = \int_0^1 x^{r-1} (1 - qx)_q^{\tau-1} d_q x$$

is called the quantum Beta function.

Note that

$$\beta_q(r, 1) = \int_0^1 x^{r-1} d_q x = \frac{1}{[r]},$$

where $[r]$ is the quantum analogue of r .

3. Main Results

In this section, we introduce new quantum Hermite–Hadamard-type estimates within a class of generalized m -preinvex functions. Furthermore, we derive identity for twice q -differentiable function. By the help of this identity, we will prove our main results, these results are generalizations of the results proved by Liu et al. in [19]. Before that, for simplicity of the notations, we take $J_v = [m\phi, m\phi + v(\psi, \phi, m)] \subset \mathbb{R}$, $m\phi < m\phi + v(\psi, \phi, m)$ as the interval and J_v^0 as the interior of J_v .

Theorem 7. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be an integrable and generalized m -preinvex function with $v(\psi, \phi, m) > 0$, for some $m \in (0, 1]$ and $q \in (0, 1)$. If $v(\cdot, \cdot)$ satisfies condition C, we have

$$h\left(\frac{2m\phi + v(\psi, \phi, m)}{2}\right) \leq \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) {}_m\phi d_q z \leq \frac{qh(m\phi) + h(\psi)}{1 + q}.$$

Proof. Let h be a generalized m -preinvex function over $v(\cdot, \cdot)$ and let condition C hold, then

$$h\left(\frac{2m\phi + v(\psi, \phi, m)}{2}\right) \leq \frac{1}{2} (h(m\phi + \tau v(\psi, \phi, m)) + h(m\phi + (1 - \tau)v(\psi, \phi, m))), \quad (7)$$

applying quantum integral identity in Equation (7) over τ on $[0, 1]$, we get the following integral

$$\begin{aligned} h\left(\frac{2m\phi + v(\psi, \phi, m)}{2}\right) &\leq \frac{1}{2} \left(\frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) {}_m\phi d_q z \right. \\ &\quad \left. + \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) {}_m\phi d_q z \right) \\ &= \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) {}_m\phi d_q z. \end{aligned}$$

Since h is generalized m -preinvex function, then $\tau \in [0, 1]$. Therefore,

$$h(m\phi + \tau v(\psi, \phi, m)) \leq (1 - \tau)h(m\phi) + \tau h(\psi), \quad (8)$$

again applying quantum integral identity in (8) over τ on $[0, 1]$ and using Definition 10, we have

$$\frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) {}_m\phi d_q z \leq \frac{qh(m\phi) + h(\psi)}{1 + q}. \quad (9)$$

Thus, our required result can be obtained by combining Equations (7) and (9). \square

Remark 3. Under these conditions, the new inequalities recapture well-known previous inequalities.

1. If $m = 1$, $v(\psi, \phi, 1) = \psi - \phi$ and $q \rightarrow 1^-$, then Theorem 7 reduces to inequality (1).
2. If $m = 1$ and $v(\psi, \phi, 1) = \psi - \phi$, then Theorem 7 reduces to inequality (3).
3. If $q \rightarrow 1^-$ and $m = 1$, then Theorem 7 reduces to inequality (4).

Lemma 2. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differentiable function on J_v^0 with ${}_m\phi D_q^2 h$ being continuous and integrable on J_v with $q \in (0, 1)$ and for some $m \in (0, 1]$. Then, the following identity holds:

$$\begin{aligned} &\frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1 + q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) {}_m\phi d_q z \\ &= \frac{q^2 v^2(\psi, \phi, m)}{1 + q} \int_0^1 \tau(1 - q\tau) {}_m\phi D_q^2 h(m\phi + \tau v(\psi, \phi, m)) {}_0 d_q \tau. \end{aligned}$$

Proof. Utilizing Definition 8 and Definition 9, we get

$$\begin{aligned}
 & {}_{m\phi}D_q^2 h(m\phi + \tau v(\psi, \phi, m)) \\
 &= {}_{m\phi}D_q ({}_{m\phi}D_q h(m\phi + \tau v(\psi, \phi, m))) \\
 &= {}_{m\phi}D_q \left(\frac{h(m\phi + \tau v(\psi, \phi, m)) - h(m\phi + q\tau v(\psi, \phi, m))}{\tau(1-q)v(\psi, \phi, m)} \right) \\
 &= \frac{1}{\tau(1-q)v(\psi, \phi, m)} \left[\begin{aligned} & \frac{h(m\phi + \tau v(\psi, \phi, m)) - h(m\phi + q\tau v(\psi, \phi, m))}{\tau(1-q)v(\psi, \phi, m)} \\ & - \frac{h(m\phi + q\tau v(\psi, \phi, m)) - h(m\phi + q^2\tau v(\psi, \phi, m))}{\tau q(1-q)v(\psi, \phi, m)} \end{aligned} \right] \\
 &= \frac{1}{\tau^2 q(1-q)^2 v^2(\psi, \phi, m)} \left[\begin{aligned} & qh(m\phi + \tau v(\psi, \phi, m)) \\ & -(1+q)h(m\phi + q\tau v(\psi, \phi, m)) \\ & + h(m\phi + q^2\tau v(\psi, \phi, m)) \end{aligned} \right].
 \end{aligned}$$

Applying this expression and Definition 10, we have

$$\begin{aligned}
 & \int_0^1 \tau(1-q\tau) {}_{m\phi}D_q^2 h(m\phi + \tau v(\psi, \phi, m))_0 d_q \tau \\
 &= \frac{1}{q(1-q)^2 v^2(\psi, \phi, m)} \left[\begin{aligned} & q \int_0^1 \frac{(1-q\tau)}{\tau} h(m\phi + \tau v(\psi, \phi, m))_0 d_q \tau \\ & -(1+q) \int_0^1 \frac{(1-q\tau)}{\tau} h(m\phi + q\tau v(\psi, \phi, m))_0 d_q \tau \\ & + \int_0^1 \frac{(1-q\tau)}{\tau} h(m\phi + q^2\tau v(\psi, \phi, m))_0 d_q \tau \end{aligned} \right] \\
 &= \frac{1}{q(1-q)^2 v^2(\psi, \phi, m)} \left[\begin{aligned} & q(1-q) \sum_{n=0}^{\infty} h(m\phi + q^n v(\psi, \phi, m)) \\ & -(1+q)(1-q) \sum_{n=0}^{\infty} h(m\phi + q^{n+1} v(\psi, \phi, m)) \\ & + (1-q) \sum_{n=0}^{\infty} h(m\phi + q^{n+2} v(\psi, \phi, m)) \end{aligned} \right] \\
 &= \frac{q}{(1-q)^2 v^3(\psi, \phi, m)} \left[\begin{aligned} & \frac{q(1-q)v(\psi, \phi, m) \sum_{n=0}^{\infty} q^n h(m\phi + q^n v(\psi, \phi, m))}{q} \\ & - \frac{(1+q)(1-q)v(\psi, \phi, m) \sum_{n=0}^{\infty} q^{n+1} h(m\phi + q^{n+1} v(\psi, \phi, m))}{q^2} \\ & + \frac{(1-q)v(\psi, \phi, m) \sum_{n=0}^{\infty} q^{n+2} h(m\phi + q^{n+2} v(\psi, \phi, m))}{q^3} \end{aligned} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{q(1-q)v^2(\psi, \phi, m)} \left[\begin{array}{c} q \left[\begin{array}{c} \sum_{n=0}^{\infty} h(m\phi + q^n v(\psi, \phi, m)) \\ - \sum_{n=0}^{\infty} h(m\phi + q^{n+1} v(\psi, \phi, m)) \\ - \sum_{n=0}^{\infty} h(m\phi + q^{n+1} v(\psi, \phi, m)) \\ + \sum_{n=0}^{\infty} h(m\phi + q^{n+2} v(\psi, \phi, m)) \end{array} \right] \end{array} \right] \\
&- \frac{q}{(1-q)^2 v^3(\psi, \phi, m)} \left[\begin{array}{c} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \\ - \frac{1}{q^2} \left[\begin{array}{c} (1+q) \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \\ + (1+q)(1-q)v(\psi, \phi, m)h(m\phi + v(\psi, \phi, m)) \end{array} \right] \\ + \frac{1}{q^3} \left[\begin{array}{c} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \\ - (1-q)v(\psi, \phi, m)h(m\phi + v(\psi, \phi, m)) \\ - (1-q)v(\psi, \phi, m)qh(m\phi + qv(\psi, \phi, m)) \end{array} \right] \end{array} \right] \\
&= \left[\begin{array}{c} \frac{1}{q(1-q)v^2(\psi, \phi, m)} \left[\begin{array}{c} q(h(m\phi + v(\psi, \phi, m)) - h(m\phi)) \\ - h(m\phi + qv(\psi, \phi, m)) + h(m\phi) \end{array} \right] \\ - \frac{(1+q)}{q^2 v^3(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \\ - \frac{q^2+q-1}{q^2(1-q)v^2(\psi, \phi, m)} h(m\phi + v(\psi, \phi, m)) \\ + \frac{qh(m\phi + qv(\psi, \phi, m))}{q^2(1-q)v^2(\psi, \phi, m)} \end{array} \right] \\
&= \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{q^2 v^2(\psi, \phi, m)} - \frac{1+q}{q^2 v^3(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z.
\end{aligned}$$

Multiplying both sides with $\frac{q^2 v^2(\psi, \phi, m)}{1+q}$, we complete the proof. \square

Remark 4. If we substitute $m = 1$ and $v(\psi, \phi, 1) = \psi - \phi$ in Lemma 2, then it reduces to Proposition 2.

Theorem 8. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differentiable function on J_v^0 with ${}_{m\phi}D_q^2 h$ being continuous and q -integrable over J_v with $q \in (0, 1)$. If $|{}_{m\phi}D_q^2 h|$ is a generalized (s, m) -preinvex function on $[m\phi, m\phi + v(\psi, \phi, m)]$ for some fixed $s, m \in (0, 1]$, then the following inequalities hold:

$$\begin{aligned}
&\left| \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1+q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \right| \\
&\leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \left(\omega_1 \left| {}_{\phi}D_q^2 h(m\phi) \right| + \omega_2 \left| {}_{m\phi}D_q^2 h(\psi) \right| \right), \quad (10)
\end{aligned}$$

where

$$\begin{aligned}\omega_1 &= (1-q) \sum_{n=0}^{\infty} (q^{2n} - q^{3n+1})(1-q^n)^s, \\ \omega_2 &= (1-q) \sum_{n=0}^{\infty} q^{n(s+2)}(1-q^{n+1}).\end{aligned}$$

Proof. Using Lemma 2 and the generalized (s, m) -preinvex of $|{}_m\phi D_q^2 h|$, we have

$$\begin{aligned}& \left| \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1+q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_m\phi d_q z \right| \\& \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \int_0^1 \tau(1-q\tau) \left| {}_m\phi D_q^2 h(m\phi + \tau v(\psi, \phi, m)) \right| {}_0 d_q \tau \\& \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \left[\left| {}_m\phi D_q^2 h(m\phi) \right| \int_0^1 \tau(1-q\tau)(1-\tau)_0^s d_q \tau \right. \\& \quad \left. + \left| {}_m\phi D_q^2 h(m\phi + v(\psi, \phi, m)) \right| \int_0^1 \tau^{s+1}(1-q\tau)_0 d_q \tau \right].\end{aligned}$$

Now, we calculate the above quantum integral by applying Definition 10, then we get

$$\begin{aligned}\omega_1 &= \int_0^1 \tau(1-q\tau)(1-\tau)_0^s d_q \tau = (1-q) \sum_{n=0}^{\infty} (q^{2n} - q^{3n+1})(1-q^n)^s, \\ \omega_2 &= \int_0^1 \tau^{s+1}(1-q\tau)_0 d_q \tau = (1-q) \sum_{n=0}^{\infty} q^{n(s+2)}(1-q^{n+1}).\end{aligned}$$

Hence, the proof is complete. \square

Corollary 1. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differentiable function on J_v^0 with ${}_m\phi D^2 h$ being continuous and integrable over J_v . If $|{}_m\phi D^2 h|$ is a generalized (s, m) -preinvex function on $[m\phi, m\phi + v(\psi, \phi, m)]$ for some fixed $s, m \in (0, 1]$, then the following inequalities hold:

$$\begin{aligned}& \left| \frac{h(m\phi) + h(m\phi + v(\psi, \phi, m))}{2} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) dz \right| \\& \leq \frac{v^2(\psi, \phi, m) \left(|h''(m\phi)| + |h''(\psi)| \right)}{2(s+2)(s+3)}.\end{aligned}$$

Proof. We substitute $q \rightarrow 1^-$ in Theorem 8. Then, the quantum integral reduces to a classical integral and

$$\begin{aligned}\int_0^1 \tau(1-\tau)^{s+1} d\tau &= \frac{1}{(s+2)(s+3)}, \\ \int_0^1 \tau^{s+1}(1-\tau) d\tau &= \frac{1}{(s+2)(s+3)}.\end{aligned}$$

\square

Remark 5. Substituting $m = 1 = s$, $v(\psi, \phi, 1) = \psi - \phi$ and by using Definition 10 in Theorem 8, we obtain Theorem 3.

Theorem 9. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differentiable function on J_v^0 with ${}_m\phi D_q^2 h$ being continuous and q -integrable on J_v with $q \in (0, 1)$. If $\left| {}_m\phi D_q^2 h \right|^r$ is a generalized (s, m) -preinvex on $[m\phi, m\phi + v(\psi, \phi, m)]$ for some fixed $s, m \in (0, 1]$, where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then

$$\left| \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1+q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_m\phi d_q z \right| \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} (\theta_1)^{\frac{1}{p}} \left(\theta_2 \left| {}_m\phi D_q^2 h(m\phi) \right|^r + \theta_3 \left| {}_m\phi D_q^2 h(m\phi + v(\psi, \phi, m)) \right|^r \right)^{\frac{1}{r}}, \quad (11)$$

where

$$\begin{aligned} \theta_1 &= (1-q) \sum_{n=0}^{\infty} q^{n(p+1)} (1-q^{n+1})^p, \\ \theta_2 &= (1-q) \sum_{n=0}^{\infty} q^n (1-q^n)^s, \end{aligned}$$

and

$$\theta_3 = \frac{1}{[s+1]}.$$

$[s+1]$ is q -analogue of $s+1$.

Proof. Using Lemma 2, application of Hölder inequality, and the generalized (s, m) -preinvex of $\left| {}_m\phi D_q^2 h \right|^r$, we have

$$\begin{aligned} & \left| \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1+q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_m\phi d_q z \right| \\ & \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \left(\int_0^1 \tau^p (1-q\tau)_0^p d_q \tau \right)^{\frac{1}{p}} \left(\int_0^1 \left| {}_m\phi D_q^2 h(m\phi + \tau v(\psi, \phi, m)) \right|^r {}_0 d_q \tau \right)^{\frac{1}{r}} \\ & \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \left(\int_0^1 \tau^p (1-q\tau)_0^p d_q \tau \right)^{\frac{1}{p}} \left(\frac{\left| {}_m\phi D_q^2 h(m\phi) \right|^r \int_0^1 (1-\tau)_0^s d_q \tau}{+ \left| {}_m\phi D_q^2 h(m\phi + v(\psi, \phi, m)) \right|^r \int_0^1 \tau^s {}_0 d_q \tau} \right)^{\frac{1}{r}}. \end{aligned}$$

Applying Definition 10, we get

$$\begin{aligned} \theta_1 &= \int_0^1 \tau^p (1-q\tau)_0^p d_q \tau = (1-q) \sum_{n=0}^{\infty} q^{n(p+1)} (1-q^{n+1})^p, \\ \theta_2 &= \int_0^1 (1-\tau)_0^s d_q \tau = (1-q) \sum_{n=0}^{\infty} q^n (1-q^n)^s, \\ \theta_3 &= \int_0^1 \tau^s {}_0 d_q \tau = \frac{1-q}{1-q^{s+1}} = \frac{1}{[s+1]}. \end{aligned}$$

Hence, the proof is complete. \square

Corollary 2. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differentiable function on J_v^0 with ${}_m\phi D^2 h$ being continuous and integrable over J_v . If $|{}_m\phi D^2 h|$ is a generalized (s, m) -preinvex function on $[m\phi, m\phi + v(\psi, \phi, m)]$ for some fixed $s, m \in (0, 1]$, where $p, r > 1$, $\frac{1}{p} + \frac{1}{r} = 1$, then the following inequalities hold:

$$\begin{aligned} & \left| \frac{h(m\phi) + h(m\phi + v(\psi, \phi, m))}{2} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) dz \right| \\ & \leq \frac{v^2(\psi, \phi, m)}{2} (\beta(p+1, p+1))^{\frac{1}{p}} \left(\frac{|h''(m\phi)|^r + |h''(\psi)|^r}{s+1} \right)^{\frac{1}{r}}, \\ & \leq \frac{v^2(\psi, \phi, m)}{8} \left(\frac{\Gamma(p+1)}{\Gamma(\frac{3}{2} + p)} \right)^{\frac{1}{p}} \left(\frac{|h''(m\phi)|^r + |h''(\psi)|^r}{s+1} \right)^{\frac{1}{r}}. \end{aligned}$$

Proof. We substitute $q \rightarrow 1^-$ in Theorem 9. Then, the quantum integral reduces to a classical integral and

$$\begin{aligned} \int_0^1 \tau^p (1-\tau)^p d\tau &= \beta(p+1, p+1), \\ \int_0^1 (1-\tau)^s d\tau &= \frac{1}{s+1}, \\ \int_0^1 \tau^s d\tau &= \frac{1}{s+1}. \end{aligned}$$

Applying the properties of Beta function, that is,

$$\begin{aligned} \beta(\phi, \phi) &= 2^{1-2\phi} \beta\left(\frac{1}{2}, \phi\right), \\ \beta(\phi, \psi) &= \frac{\Gamma(\phi)\Gamma(\psi)}{\Gamma(\phi\psi)}, \end{aligned}$$

we obtain

$$\begin{aligned} \beta(p+1, p+1) &= 2^{1-2(p+1)} \beta\left(\frac{1}{2}, p+1\right) \\ &= 2^{-2p-1} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma(p+1)}{\Gamma\left(\frac{3}{2} + p\right)}, \end{aligned}$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\tau)$ is a Gamma function:

$$\Gamma(\tau) = \int_0^\infty x^{\tau-1} e^{-x} dx, \quad \tau > 0.$$

□

Corollary 3. If $p \in \mathbb{Z}^+$, $p > 1$, then

$$(1 - q\tau)^p \leq (1 - q\tau)_q^p.$$

Under the above condition, Theorem 9 reduces to

$$\begin{aligned} & \left| \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1+q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \right| \\ & \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \left[\left(\beta_q(p+1, p+1) \right)^{\frac{1}{p}} \times \left(\theta_2 \left| {}_{m\phi}D_q^2 h(m\phi) \right|^r + \theta_3 \left| {}_{m\phi}D_q^2 h(m\phi + v(\psi, \phi, m)) \right|^r \right)^{\frac{1}{r}} \right]. \end{aligned}$$

Remark 6. If we take $m = 1 = s$, $v(\psi, \phi, 1) = \psi - \phi$ and by using Definition 10 in Theorem 9, then we obtain Theorem 4.

Theorem 10. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differentiable function on J_v^0 with ${}_{m\phi}D_q^2 h$ being continuous and q -integrable on J_v with $q \in (0, 1)$. If $\left| {}_{m\phi}D_q^2 h \right|^r$ is generalized (s, m) -preinvex on $[m\phi, m\phi + v(\psi, \phi, m)]$ for some fixed $s, m \in (0, 1]$, where $r \geq 1$, then

$$\begin{aligned} & \left| \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1+q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \right| \\ & \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} (\varphi_1)^{1-\frac{1}{r}} \left(\varphi_2 \left| {}_{m\phi}D_q^2 h(m\phi) \right|^r + \varphi_3 \left| {}_{m\phi}D_q^2 h(m\phi + v(\psi, \phi, m)) \right|^r \right)^{\frac{1}{r}}, \quad (12) \end{aligned}$$

where

$$\begin{aligned} \varphi_1 &= \frac{1}{(1+q)(1+q+q^2)}, \\ \varphi_2 &= (1-q) \sum_{n=0}^{\infty} (1-q^n)^s (q^{2n} - q^{3n+1}), \\ \varphi_3 &= (1-q) \sum_{n=0}^{\infty} q^{n(s+2)} (1-q^{n+1}). \end{aligned}$$

Proof. Using Lemma 2, application of power mean inequality, and the generalized (s, m) -preinvex of $\left| {}_{m\phi}D_q^2 h \right|^r$, we have

$$\begin{aligned} & \left| \frac{qh(m\phi) + h(m\phi + v(\psi, \phi, m))}{1+q} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi+v(\psi, \phi, m)} h(z) {}_{m\phi}d_q z \right| \\ & \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \left(\int_0^1 \tau(1-q\tau) {}_0d_q \tau \right)^{1-\frac{1}{r}} \left(\int_0^1 \tau(1-q\tau) \left| {}_{m\phi}D_q^2 h(m\phi + \tau v(\psi, \phi, m)) \right|^r {}_0d_q \tau \right)^{\frac{1}{r}} \\ & \leq \frac{q^2 v^2(\psi, \phi, m)}{1+q} \left(\int_0^1 \tau(1-q\tau) {}_0d_q \tau \right)^{1-\frac{1}{r}} \left(\left| {}_{m\phi}D_q^2 h(m\phi) \right|^r \int_0^1 \tau(1-\tau)^s (1-q\tau) {}_0d_q \tau \right. \\ & \quad \left. + \left| {}_{m\phi}D_q^2 h(m\phi + v(\psi, \phi, m)) \right|^r \int_0^1 \tau^{s+1} (1-q\tau) {}_0d_q \tau \right)^{\frac{1}{r}}. \end{aligned}$$

Applying Definition 10, we can easily calculate as

$$\begin{aligned} \varphi_1 &= \int_0^1 \tau(1-q\tau) {}_0d_q \tau = \frac{1}{(1+q)(1+q+q^2)}, \\ \varphi_2 &= \int_0^1 \tau(1-\tau)^s (1-q\tau) {}_0d_q \tau = (1-q) \sum_{n=0}^{\infty} (1-q^n)^s (q^{2n} - q^{3n+1}), \\ \varphi_3 &= \int_0^1 \tau^{s+1} (1-q\tau) {}_0d_q \tau = (1-q) \sum_{n=0}^{\infty} q^{n(s+2)} (1-q^{n+1}). \end{aligned}$$

The proof is completed. \square

Corollary 4. Let $h : J_v \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice quantum differentiable function on J_v^0 with ${}_m\phi D^2 h$ being continuous and integrable over J_v . If $|{}_m\phi D^2 h|$ is generalized (s, m) -preinvex function on $[m\phi, m\phi + v(\psi, \phi, m)]$ for some fixed $s, m \in (0, 1]$, where $r \geq 1$, then the following inequalities hold:

$$\left| \frac{h(m\phi) + h(m\phi + v(\psi, \phi, m))}{2} - \frac{1}{v(\psi, \phi, m)} \int_{m\phi}^{m\phi + v(\psi, \phi, m)} h(z) dz \right| \leq \frac{v^2(\psi, \phi, m)}{2} \left(\frac{1}{6} \right)^{1-\frac{1}{r}} \left(\frac{|h''(m\phi)|^r + |h''(\psi)|^r}{(s+2)(s+3)} \right)^{\frac{1}{r}}.$$

Proof. We substitute $q \rightarrow 1^-$ in Theorem 10. Then, the quantum integral reduces to a classical integral and

$$\begin{aligned} \int_0^1 \tau(1-\tau) d\tau &= \frac{1}{6}, \\ \int_0^1 \tau(1-\tau)^{s+1} d\tau &= \frac{1}{(s+2)(s+3)}, \\ \int_0^1 \tau^{s+1}(1-\tau) d\tau &= \frac{1}{(s+2)(s+3)}. \end{aligned}$$

□

Remark 7. If we take $m = 1 = s$, $v(\psi, \phi, 1) = \psi - \phi$ and by using Definition 10 in Theorem 10, then we obtain Theorem 5.

4. Conclusions

Quantum calculus has large applications in many mathematical areas such as number theory, special functions, quantum mechanics, and mathematical inequalities. In this paper, we first establish a new quantum integral identity and then develop some quantum estimates of Hermite–Hadamard-type inequalities for generalized (s, m) -preinvex functions. These results in some special cases recapture the known results. We hope that our results may be helpful for further study.

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