

Lie Symmetry Analysis of the Time Fractional Generalized KdV Equations with Variable Coefficients

Cheng Chen ^{1,*} , Yao-Lin Jiang ² and Xiao-Tian Wang ³

¹ School of Science, Xi'an University of Posts and Telecommunications, Xi'an 710121, China

² School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China; yljiaing@xjtu.edu.cn

³ School of Electronic Engineering, Xi'an University of Posts and Telecommunications, Xi'an 710121, China; wangxt@xupt.edu.cn

* Correspondence: chencheng3468@163.com; Tel.: +86-137-5989-2702

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Abstract: The group classification of a class of time fractional generalized KdV equations with variable coefficient is presented. The Lie symmetry analysis method is extended to the certain subclasses of time fractional generalized KdV equations with initial and boundary values. Under the corresponding similarity transformation with similarity invariants, KdV equations with initial and boundary values have been transformed into fractional ordinary differential equations with initial value. Then we use the power series method to obtain the exact solution of the reduced equation with the Erdélyi-Kober fractional differential operator.

Keywords: infinitesimal operator; Riemann-Liouville derivative; initial and boundary value; Erdélyi-Kober operator

1. Introduction

In recent years, the fractional partial differential equations (FPDEs) have been widely studied. They can represent numerous intricate phenomena in many fields, such as engineering, mechanics, mechatronics, physics, control theory, and other fields of science [1–5] well. Consequently, some methods have been proposed to find exact solutions to FPDEs, such as the variational method [6,7], the Homotopy analysis method [8], the sub-equation method [9], the Lie symmetry analysis method [10–13], and so on.

The Korteweg-de Vries (KdV) equation is a very important mathematical model that was firstly derived by Korteweg and de Vries in 1895 and it has been used to describe the evolution and interaction of nonlinear waves [14,15]. Many scholars are also devoted to the study of the time fractional KdV equations. In this article, we are concerned with the time fractional generalized KdV equations of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au^m u_x + g(t)u_{xxx} = 0, \quad x > 0, \quad t > 0, \quad (1)$$

where $0 < \alpha < 1$, a is a nonzero constant, $m(m \geq 1)$ is a positive integer, and $g(t)$ is an arbitrary smooth non-vanishing function. In Equation (1), $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the Riemann-Liouville fractional derivative [2,3] with respect to t , which is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(s,x)}{(t-s)^{\alpha+1-n}} ds, & (0 \leq n-1 < \alpha < n, n \in \mathbb{N}), \\ \frac{\partial^n u}{\partial t^n}, & (\alpha = n \in \mathbb{N}), \end{cases}$$

where $\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt$ is Gamma function.

When $\alpha = 1$, Equation (1) can be reduced to variable coefficient generalized KdV equations, which is studied by way of the Lie group analysis method in detail in Reference [16]. The exhaustive group classification was presented. O.O. Vaneeva et al. applied the Lie symmetry method, which was proposed by Bluman [17] and the generalized KdV equations with the initial and boundary value was reduced to nonlinear third-order ODEs with initial value.

This paper is organized as follows. In Section 2, Lie symmetry analysis method of the fractional partial differential equation is presented. In Section 3, the group classification of the time fractional generalized KdV equations with variable coefficients is carried out by using the classical Lie symmetry analysis method. Then in Section 4, we first try to extend the Lie symmetry analysis method to the time fractional generalized KdV equations with initial and boundary value and use the power series method to obtain the exact solution. Finally, the conclusions are given.

2. Lie Symmetry Analysis of the Time Fractional Partial Differential Equation

Consider the following time fractional partial differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - F(t, x, u, u_x, u_{xx}, \dots) = 0 \quad (2)$$

with two independent variables t, x and one dependent variable $u = u(t, x)$. The one-parameter(ε) Lie group of transformations is given by

$$\begin{aligned} t^* &= t + \varepsilon \cdot \tau(t, x, u) + o(\varepsilon^2), \\ x^* &= x + \varepsilon \cdot \zeta(t, x, u) + o(\varepsilon^2), \\ u^* &= u + \varepsilon \cdot \eta(t, x, u) + o(\varepsilon^2), \\ \frac{\partial^\alpha u^*}{\partial t^{*\alpha}} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon \cdot \eta_\alpha^0(t, x, u) + o(\varepsilon^2), \\ \frac{\partial u^*}{\partial x^*} &= u_x + \varepsilon \cdot \eta^x(t, x, u) + o(\varepsilon^2), \\ \frac{\partial^2 u^*}{\partial x^{*2}} &= u_{xx} + \varepsilon \cdot \eta^{xx}(t, x, u) + o(\varepsilon^2), \\ &\vdots \end{aligned} \quad (3)$$

where

$$\begin{aligned} \eta^x &= \eta_x + (\eta_u - \zeta_x)u_x - \tau_x u_t - \zeta_u (u_x)^2 - \tau_u u_x u_t, \\ \eta^{xx} &= \eta_{xx} + (2\eta_{xu} - \zeta_{xx})u_x - \tau_{xx} u_t - (\eta_{uu} - 2\zeta_{xu})(u_x)^2 - 2\tau_{xu} u_x u_t - \zeta_{uu} (u_x)^3 \\ &\quad - \tau_{uu} (u_x)^2 u_t + (\eta_u - 2\zeta_x)u_{xx} - 2\tau_x u_{xt} - 3\zeta_u u_{xx} u_x - \tau_u u_{xx} u_t - 2\tau_u u_{xt} u_x, \\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\zeta), \\ &\vdots \\ \eta_\alpha^0 &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + (\eta_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^n \eta_u}{\partial t^n} \right. \\ &\quad \left. - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\zeta) D_t^{\alpha-n}(u_x), \end{aligned}$$

here

$$\mu = \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} (-u)^r \frac{\partial^m}{\partial t^m} (u^{k-r}) \frac{\partial^{n-m+k} \eta}{\partial t^{n-m} \partial u^k}.$$

(see [12–15,18–22]).

Definition 1. Under transformations (3), Equation (2) is invariant (or Equation (2) admits transformations (3)) if Equation (2) has the same form with respect to new variables t^*, x^*, u^* , i.e.,

$$\frac{\partial^\alpha u^*}{\partial t^{*\alpha}} - F(t^*, x^*, u^*, u_{x^*}^*, u_{x^* x^*}^*, \dots) = 0.$$

On the basis of the Lie theory [17,23–25], we can obtain

Theorem 1. Equation (2) is invariant under a one-parameter (ϵ) Lie group of infinitesimal transformations if and only if

$$PrV^{(k)}\left(\frac{\partial^\alpha u}{\partial t^\alpha} - F(t, x, u, u_x, u_{xx}, \dots)\right) \Big|_{\frac{\partial^\alpha u}{\partial t^\alpha} - F(t, x, u, u_x, u_{xx}, \dots) = 0} = 0.$$

where $PrV^{(k)}$ is k order prolongation infinitesimal operator of

$$V = \varsigma(t, x, u) \frac{\partial}{\partial x} + \tau(t, x, u) \frac{\partial}{\partial t} + \eta(t, x, u) \frac{\partial}{\partial u}.$$

This theorem is called criterion of invariance.

3. Lie Symmetry Analysis of the Time Fractional Generalized KdV Equations with Variable Coefficients

In this section, we carry out the group classification of Equation (1). The third prolongation of the infinitesimal operator $PrV^{(3)}$ is as follows

$$PrV^{(3)} = V + \eta_\alpha^0 \frac{\partial}{\partial(\frac{\partial^\alpha u}{\partial t^\alpha})} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xxx} \frac{\partial}{\partial u_{xxx}}.$$

Applying $PrV^{(3)}$ to Equation (1), one can get

$$\tau(g(t))_t u_{xxx} + am\eta u^{m-1} u_x + a\eta^x u^m + \eta^{xxx} g(t) + \eta_\alpha^0 = 0. \quad (4)$$

Substituting the expressions of η_α^0 , η^x , η^{xxx} into Equation (4), we sort it out in the form of power of partial derivatives of u . Then equating the coefficients to zero, we get a system of linear PDEs and FPDEs, called the determining equations, as follows

$$\begin{aligned} \tau_x &= \tau_u = \varsigma_u = \varsigma_t = \eta_{uu} = 0, \\ 3g(t)\varsigma_x &= \alpha(\tau(t))_t g(t) + \tau(t)(g(t))_t, \\ g(t)(\eta_{xu} - \varsigma_{xx}) &= 0, \\ a(\alpha\tau_t - \varsigma_x)u^m + am\eta u^{m-1} + g(t)(3\eta_{xxu} - \varsigma_{xxx}) &= 0, \\ au^m \eta_x + g(t)\eta_{xxx} + \frac{\partial^\alpha \eta}{\partial t^\alpha} - u \frac{\partial^\alpha \eta_u}{\partial t^\alpha} &= 0, \\ \binom{\alpha}{n} \frac{\partial^n \eta_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{(n+1)}(\tau) &= 0, n = 1, 2, \dots \end{aligned}$$

From the first five equations of the determining equations, we can obtain

$$\tau = \tau(t), \quad \varsigma = \varsigma(x), \quad \eta = \eta_1(t, x)u + \eta_0(t, x).$$

By analyzing the second equation of the determining equations, we have

$$\varsigma = a_1 x + a_2,$$

where a_1, a_2 are arbitrary constants. Then the second equation is reduced to

$$3g(t)a_1 = \alpha(\tau(t))_t g(t) + \tau(t)(g(t))_t.$$

To solve the above differential equation, $\tau(t)$ is given as

$$\tau(t) = (g(t))^{-\frac{1}{\alpha}} \left[\frac{3a_1}{\alpha} \int (g(t))^{\frac{1}{\alpha}} dt + a_3 \right],$$

where a_3 is a constant.

By analyzing the third equation of the determining equations, we can get $\eta_x = 0$, then the fourth equation is reduced to

$$(\alpha\tau_t - \zeta_x)u + m\eta = 0.$$

Namely,

$$\eta = \frac{a_1 - \alpha(\tau(t))_t}{m} u.$$

According to $\binom{\alpha}{n} = \frac{(-1)^{n-1} \alpha \Gamma(n-\alpha)}{\Gamma(1-\alpha) \Gamma(n+1)}$, $\Gamma(n+1) = n\Gamma(n)$, the last equation of the determining equations is written as

$$(n+1) \frac{\partial^n \eta_u}{\partial t^n} + (n-\alpha) D_t^{(n+1)}(\tau) = 0, \quad n = 1, 2, \dots.$$

When $n = 1$, equation is

$$2 \frac{\partial \eta_u}{\partial t} + (1-\alpha) D_t^2(\tau) = 0,$$

which gives

$$\frac{2\alpha}{m} (\tau(t))_{tt} + (1-\alpha) (\tau(t))_{tt} = 0.$$

The invariance condition $\tau(t) |_{t=0} = 0$ under the Lie group transformations (3), because of the conservative property of the Riemann-Liouville fractional derivative operator.

Next, we will carry out group classification according to different forms of function $g(t)$ as follows.

(1) If $g(t)$ is arbitrary, we obtain the explicit form of infinitesimals

$$\tau = 0, \quad \zeta = a_1, \quad \eta = 0$$

where a_1 is arbitrary constant. Hence the infinitesimal operator becomes

$$V = a_1 \frac{\partial}{\partial x}.$$

(2) If $g(t)$ is a nonzero constant, the infinitesimals are

$$\tau = \frac{3a_1}{\alpha} t, \quad \zeta = a_1 x + a_2, \quad \eta = -\frac{2a_1}{m} u,$$

where a_1, a_2 are arbitrary constants. Hence, the general form of the infinitesimal operator is

$$V = \frac{3a_1}{\alpha} t \frac{\partial}{\partial t} + (a_1 x + a_2) \frac{\partial}{\partial x} - \frac{2a_1}{m} u \frac{\partial}{\partial u}.$$

A two-dimensional Lie algebra is spanned by

$$V_1 = \frac{3}{\alpha} t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - \frac{2}{m} u \frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial x}.$$

(3) If $g(t) = t^p$ ($p \neq 0, p \neq -\alpha$), the expressions of infinitesimals are given as

$$\tau = \frac{3a_1}{p+\alpha} t, \quad \zeta = a_1 x + a_2, \quad \eta = \frac{p-2\alpha}{m(p+\alpha)} a_1 u$$

where a_1 and a_2 are arbitrary constants. The infinitesimal operator takes the form

$$V = \frac{3a_1}{p+\alpha} t \frac{\partial}{\partial t} + (a_1 x + a_2) \frac{\partial}{\partial x} + \frac{p-2\alpha}{m(p+\alpha)} a_1 u \frac{\partial}{\partial u}$$

We obtain a two-dimensional Lie algebra, spanned by

$$V_1 = \frac{3t}{p+\alpha} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{p-2\alpha}{m(p+\alpha)} u \frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial x}.$$

When $p = 2\alpha$, the infinitesimals are given as

$$\tau = \frac{a_1}{\alpha} t, \quad \zeta = a_1 x + a_2, \quad \eta = 0$$

where a_1 and a_2 are arbitrary constants. The corresponding infinitesimal generators are

$$V_1 = \frac{t}{\alpha} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial x}.$$

When $p = -\alpha$, the infinitesimals are given as

$$\tau = a_1 t, \quad \zeta = a_2, \quad \eta = -\frac{\alpha a_1}{m} u$$

where a_1 and a_2 are arbitrary constants. The corresponding infinitesimal generators are

$$V_1 = t \frac{\partial}{\partial t} - \frac{\alpha u}{m} \frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial x}.$$

(4) If $g(t) = e^{pt}$ (p is a nonzero real number), we obtain the following expressions of infinitesimals

$$\tau = \frac{3a_1}{p}, \quad \zeta = a_1 x + a_2, \quad \eta = \frac{a_1}{m} u,$$

where a_1 and a_2 are arbitrary constants. Then we have

$$V = \frac{3a_1}{p} \frac{\partial}{\partial t} + (a_1 x + a_2) \frac{\partial}{\partial x} + \frac{a_1}{m} u \frac{\partial}{\partial u}.$$

We have a two-dimensional Lie algebra, which is spanned by

$$V_1 = \frac{3}{p} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{u}{m} \frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial x}.$$

4. Lie Symmetry Analysis of Time Fractional Generalized KdV Equations with Initial and Boundary Value

4.1. Lie Symmetry Analysis of Time Fractional Partial Differential Equations with Initial and Boundary Value

In applications, the partial differential equation with the initial and boundary value has been paid more attention. In Reference [17], Bluman successfully applied the Lie symmetry analysis method to deal with the partial differential equations with the initial and boundary value. In this paper we will attempt this method to solve the following time fractional partial differential equations with initial and boundary value

$$\frac{\partial^\alpha u}{\partial t^\alpha} - F(t, x, u, u_x, u_{xx}, \dots) = 0, \quad (5)$$

$$u|_S = h(x), \quad (6)$$

where S is the manifold.

Assuming that Equation (5) admits l one-parameter symmetries V_i , $i = 1, 2, \dots, l$, we say that Equations (5) and (6) are invariant under a symmetry of the form

$$V = \sum_{i=1}^l a_i V_i$$

for some constants a_i . The following two conditions also need to be satisfied

- (i) under symmetry V , the manifold S is invariant;
- (ii) under symmetry V restricted to manifold S , the initial (boundary) condition $u|_S = h(x)$ is invariant.

4.2. Reduction for Time Fractional Generalized KdV Equations with Initial and Boundary Value

We first apply this method to the time fractional generalized KdV equation with initial and boundary value as follows

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au^m u_x + t^p u_{xxx} = 0, \quad x > 0, \quad t > 0, \quad (7)$$

$$\begin{aligned} u(0, x) &= 0, \quad x > 0, \\ u(t, 0) &= q(t), \quad t > 0, \\ u_x(t, 0) &= 0, \quad t > 0, \\ u_{xx}(t, 0) &= 0, \quad t > 0, \end{aligned} \quad (8)$$

where $0 < \alpha < 1$, a and $p \neq -\alpha$ are arbitrary nonzero constants, $m (m \geq 1)$ is a positive integer, $q(t)$ is a non-vanishing smooth function with respect to t .

Then we search for the infinitesimal generators, which leave initial and boundary conditions (8) of Equation (7) invariant.

Theorem 2. *The infinitesimal generator*

$$V = \frac{3t}{p + \alpha} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{p - 2\alpha}{m(p + \alpha)} u \frac{\partial}{\partial u}$$

leaves Equations (7) and (8) invariant, and boundary condition $u(t, 0)$ should satisfy

$$q(t) = \lambda t^{\frac{p-2\alpha}{3m}}, \quad t > 0,$$

where $\lambda > 0$ is a constant.

Proof. In order to find infinitesimal generator admitted by Equation (7) with initial and boundary conditions (8), we construct a special linear combination,

$$V = a_1 V_1 + a_2 V_2,$$

where a_1 and a_2 are constants to be determined, which cannot be zero at the same time. Namely,

$$V = a_1 \left(\frac{3t}{p+\alpha} \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{p-2\alpha}{m(p+\alpha)} u \frac{\partial}{\partial u} \right) + a_2 \frac{\partial}{\partial x}. \quad (9)$$

Applying (9) to first boundary condition $x = 0$, $u(t, 0) = q(t)$, and initial condition $u(0, x) = 0$, we have

$$V(x)|_{x=0} = 0, \quad (10)$$

$$V(u - q(t))|_{x=0, u(t,0)=q(t)} = 0, \quad (11)$$

$$V(t)|_{t=0} = 0, \quad (12)$$

$$V(u)|_{t=0, u(0,x)=0} = 0. \quad (13)$$

Equation (10) gives $a_1 \times 0 + a_2 \times 1 = 0$, namely $a_2 = 0$.

From Equation (11), we obtain

$$a_1 \times \left(\frac{3t}{p+\alpha} \frac{\partial q}{\partial t} + \frac{p-2\alpha}{m(p+\alpha)} q \right) + a_2 \times 0 = 0.$$

For nonzero a_1 , we have

$$\frac{3t}{p+\alpha} \frac{\partial q}{\partial t} + \frac{p-2\alpha}{m(p+\alpha)} q = 0,$$

which gives

$$q(t) = \lambda t^{\frac{p-2\alpha}{3m}},$$

where $\lambda > 0$ is an integral constant.

It is obvious that the initial conditions (12) and (13) are invariant under V with $a_2 = 0$.

Using the second extension of V with $a_2 = 0$, that is,

$$PrV^{(2)} = V_1 + \frac{(1-m)p - (m+2)\alpha}{3m} u_x \frac{\partial}{\partial u_x} + \frac{(1-2m)p - 2(m+1)\alpha}{3m} u_{xx} \frac{\partial}{\partial u_{xx}},$$

it can be shown that $PrV^{(2)}$ leaves the remaining boundary conditions $u_x(t, 0) = 0$, $u_{xx}(t, 0) = 0$ invariant.

Namely,

$$Pr^{(2)}(u_x)|_{u_x(t,0)=0} = 0,$$

$$Pr^{(2)}(u_{xx})|_{u_{xx}(t,0)=0} = 0,$$

which are obviously established.

Hence the V_1 is the general form of infinitesimal generator admitted by Equation (7) with Equation (8). \square

Remark 1. If boundary condition $q(t) \neq \lambda t^{\frac{p-2\alpha}{3m}}$, then Equation (7) with (8) is not invariant under V_1 . Hence, $q(t) = \lambda t^{\frac{p-2\alpha}{3m}}$ is an indispensable condition.

Remark 2. To obtain one-parameter (ϵ) transformation by V_1 , one solves the system of first order ODEs,

$$\begin{aligned}\frac{dt^*}{d\varepsilon} &= \frac{3t^*}{p+\alpha}, \\ \frac{dx^*}{d\varepsilon} &= x^*, \\ \frac{du^*}{d\varepsilon} &= \frac{p-2\alpha}{m(p+\alpha)}u^*,\end{aligned}$$

with $t^* = t$, $x^* = x$, $u^* = u$ at $\varepsilon = 0$. Then the corresponding one-parameter Lie group of transformations is

$$\begin{aligned}t^* &= te^{\frac{3\varepsilon}{p+\alpha}}, \\ x^* &= xe^{\varepsilon}, \\ u^* &= ue^{\frac{p-2\alpha}{m(p+\alpha)}\varepsilon}.\end{aligned}\tag{14}$$

Under transformations (14), the equations are obtained as following

$$\frac{\partial^\alpha u^*}{\partial (t^*)^\alpha} + a(u^*)^m u_{x^*}^* + (t^*)^p u_{x^* x^*}^* = 0, \quad x^* > 0, \quad t^* > 0,$$

$$\begin{aligned}u^*(0, x^*) &= 0, \quad x^* > 0, \\ u^*(t^*, 0) &= \lambda (t^*)^{\frac{p-2\alpha}{3m}}, \quad t^* > 0, \\ u_{x^*}^*(t^*, 0) &= 0, \quad t^* > 0, \\ u_{x^* x^*}^*(t^*, 0) &= 0, \quad t^* > 0,\end{aligned}$$

which we have checked.

Hence the admitted infinitesimal generator can be used to reduce Equation (7) with the initial and boundary conditions (8) to an ordinary differential equation. The characteristic equations corresponding to infinitesimal generator $V = V_1$ are

$$\frac{dt}{\frac{3t}{p+\alpha}} = \frac{dx}{x} = \frac{du}{\frac{p-2\alpha}{m(p+\alpha)}u}.$$

The similarity variable and similarity transformation take the following form

$$z = xt^{-\frac{p+\alpha}{3}}, \quad u = \phi(z)t^{\frac{p-2\alpha}{3m}},\tag{15}$$

respectively.

According to the similarity transformation $u = \phi(z)t^{\frac{p-2\alpha}{3m}}$, $z = xt^{-\frac{p+\alpha}{3}}$, the initial and boundary values are reduced to

$$\begin{aligned}\phi(0) &= \lambda, \\ \phi'(0) &= 0, \\ \phi''(0) &= 0.\end{aligned}$$

Theorem 3. Under the similarity transformation $u = \phi(z)t^{\frac{p-2\alpha}{3m}}$ with $z = xt^{-\frac{p+\alpha}{3}}$, time fractional generalized KdV equations(7) with initial and boundary value(8) is reduced into fractional ODEs of the form

$$\begin{aligned}\left(P^{\frac{1+\frac{p-2\alpha}{3m}-\alpha,\alpha}{p+\alpha}}\phi\right)(z) + a(\phi(z))^m \phi'(z) + \phi'''(z) &= 0, \quad z \in [0, +\infty), \\ \phi(0) &= \lambda, \\ \phi'(0) &= 0, \\ \phi''(0) &= 0,\end{aligned}$$

with the Erdélyi-Kober fractional differentia operator [10,12]

$$(P_{\beta}^{\tau, \alpha} \phi)(z) := \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\beta} z \frac{d}{dz} \right) (K_{\beta}^{\tau + \alpha, n - \alpha} \phi)(z), z > 0, \beta > 0, \alpha > 0,$$

$$n = \begin{cases} [\alpha] + 1, & (\alpha \neq \mathbb{N}), \\ \alpha, & (\alpha = \mathbb{N}), \end{cases}$$

where

$$(K_{\beta}^{\tau, \alpha} \phi)(z) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} [(u-1)^{\alpha-1} u^{-(\tau+\alpha)} \phi(zu^{\frac{1}{\beta}})] du, & (\alpha > 0), \\ \phi(z), & (\alpha = 0), \end{cases}$$

is the Erdélyi-Kober fractional integral operator and $\lambda > 0$ is a constant, the prime denotes differentiation with respect to z .

Proof. Suppose $n - 1 < \alpha < n, n = 1, 2, 3, \dots$. Under the similarity transformation (15), Riemann-Liouville fractional derivative becomes

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n - \alpha - 1} s^{\frac{p - 2\alpha}{3m}} \phi(x s^{-\frac{p + \alpha}{3}}) ds \right].$$

Letting $v = \frac{t}{s}$, we have $ds = -\frac{t}{v^2} dv$. Then the above equation can be written as

$$\begin{aligned} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} &= \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n - \alpha)} \int_0^t s^{n - \alpha - 1} \left(\frac{t}{s} - 1 \right)^{n - \alpha - 1} s^{\frac{p - 2\alpha}{3m}} \phi \left(x \frac{t^{\frac{\alpha + p}{3}}}{s^{\frac{\alpha + p}{3}}} t^{-\frac{(\alpha + p)}{3}} \right) ds \right] \\ &= \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{s^{n - \alpha - 1 + \frac{p - 2\alpha}{3m}}}{t^{n - \alpha - 1 + \frac{p - 2\alpha}{3m}}} t^{n - \alpha - 1 + \frac{p - 2\alpha}{3m}} \left(\frac{t}{s} - 1 \right)^{n - \alpha - 1} \phi \left(x t^{-\frac{\alpha + p}{3}} \left(\frac{t}{s} \right)^{\frac{\alpha + p}{3}} \right) ds \right] \\ &= \frac{\partial^n}{\partial t^n} \left[\frac{t^{n - \alpha - 1 + \frac{p - 2\alpha}{3m}}}{\Gamma(n - \alpha)} \int_0^t \left(\frac{s}{t} \right)^{n - \alpha - 1 + \frac{p - 2\alpha}{3m}} \left(\frac{t}{s} - 1 \right)^{n - \alpha - 1} \phi \left(x t^{-\frac{\alpha + p}{3}} \left(\frac{t}{s} \right)^{\frac{\alpha + p}{3}} \right) ds \right] \\ &= \frac{\partial^n}{\partial t^n} \left[\frac{t^{n - \alpha - 1 + \frac{p - 2\alpha}{3m}}}{\Gamma(n - \alpha)} \int_{+\infty}^1 \left(\frac{1}{v} \right)^{n - \alpha - 1 + \frac{p - 2\alpha}{3m}} (v - 1)^{n - \alpha - 1} \phi \left(x v^{\frac{\alpha + p}{3}} \right) \left(-\frac{t}{v^2} \right) dv \right] \\ &= \frac{\partial^n}{\partial t^n} \left[\frac{t^{n - \alpha + \frac{p - 2\alpha}{3m}}}{\Gamma(n - \alpha)} \int_1^{\infty} v^{-(n - \alpha + 1 + \frac{p - 2\alpha}{3m})} (v - 1)^{n - \alpha - 1} \phi \left(z v^{\frac{p + \alpha}{3}} \right) dv \right] \\ &= \frac{\partial^n}{\partial t^n} \left[t^{n - \alpha + \frac{p - 2\alpha}{3m}} \left(K_{\frac{3}{p + \alpha}}^{1 + \frac{p - 2\alpha}{3m}, n - \alpha} \phi \right)(z) \right]. \end{aligned} \tag{16}$$

Using the relation $(z = x t^{-\frac{p + \alpha}{3}}, \phi \in C^1(0, +\infty))$, we have

$$t \frac{\partial}{\partial t} \phi(z) = -\frac{p + \alpha}{3} z \frac{d}{dz} \phi(z).$$

Then, Equation (16) is simplified as follows

$$\begin{aligned} & \frac{\partial^n}{\partial t^n} \left[t^{n-\alpha+\frac{p-2\alpha}{3m}} \left(K_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3m}, n-\alpha} \phi \right) (z) \right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[\frac{\partial}{\partial t} \left(t^{n-\alpha+\frac{p-2\alpha}{3m}} \left(K_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3m}, n-\alpha} \phi \right) (z) \right) \right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[t^{n-\alpha-1+\frac{p-2\alpha}{3m}} \left(n-\alpha + \frac{p-2\alpha}{3m} - \frac{p+\alpha}{3} z \frac{d}{dz} \right) \left(K_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3m}, n-\alpha} \phi \right) (z) \right]. \end{aligned}$$

Repeating the similar procedure for $n-1$ times, we obtain

$$\begin{aligned} & \frac{\partial^n}{\partial t^n} \left[t^{n-\alpha+\frac{p-2\alpha}{3m}} \left(K_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3m}, n-\alpha} \phi \right) (z) \right] \\ &= \dots \\ &= t^{-\alpha+\frac{p-2\alpha}{3m}} \prod_{j=0}^{n-1} \left(1-\alpha + \frac{p-2\alpha}{3m} + j - \frac{p+\alpha}{3} z \frac{d}{dz} \right) \left(K_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3m}, n-\alpha} \phi \right) (z). \end{aligned}$$

By the definition of the Erdélyi-Kober fractional differential operator given in Theorem 3, the above equation can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha+\frac{p-2\alpha}{3m}} \left(P_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3m}+\alpha, \alpha} \phi \right) (z). \quad (17)$$

According to $u = \phi(z) t^{\frac{p-2\alpha}{3m}}$, $z = xt^{-\frac{p+\alpha}{3}}$, we obtain

$$u_x = t^{-\frac{p+\alpha}{3}+\frac{p-2\alpha}{3m}} \phi'(z), \quad u_{xx} = t^{-\frac{2p+2\alpha}{3}+\frac{p-2\alpha}{3m}} \phi''(z), \quad u_{xxx} = t^{-(p+\alpha)+\frac{p-2\alpha}{3m}} \phi'''(z). \quad (18)$$

Substituting (17) and (18) into (7), the time fractional generalized KdV equations (7) with initial and boundary value (8) are reduced into the fractional ordinary differential equation with the form

$$\left(P_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3m}-\alpha, \alpha} \phi \right) (z) + a(\phi(z))^m \phi'(z) + \phi'''(z) = 0, \quad z \in [0, +\infty), \quad (19)$$

$$\begin{aligned} \phi(0) &= \lambda, \\ \phi'(0) &= 0, \\ \phi''(0) &= 0, \end{aligned} \quad (20)$$

where the prime denotes differentiation with respect to z . \square

4.3. Explicit Power Series Solutions of Time Fractional Generalized KdV Equations with Initial and Boundary Value

Then, we investigate explicit power series solutions of Equation (19) with conditions (20) applying the power series method in Reference [12,20,22].

Suppose that the power series solutions for Equation (19) is of the form

$$\phi(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (21)$$

From Equation (21), we can obtain

$$\begin{aligned}\phi'(z) &= \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n, \\ \phi''(z) &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}z^n, \\ \phi'''(z) &= \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)a_{n+3}z^n.\end{aligned}\quad (22)$$

Here, we only consider the following cases:

Case 1. When $m = 1$ in Equation (19), Equation (19) is written as

$$\left(P_{\frac{3}{p+\alpha}}^{1+\frac{p-2\alpha}{3}-\alpha,\alpha}\phi\right)(z) + a(\phi(z))\phi'(z) + \phi'''(z) = 0. \quad (23)$$

After substituting (21) and (22) into Equation (23), we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})}{\Gamma(1+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})} a_n z^n + a \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n + \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)a_{n+3}z^n = 0. \quad (24)$$

When $n = 0$ in Equation (24), we have

$$a_3 = -\frac{1}{6} \left(\frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3})}{\Gamma(1+\frac{p-2\alpha}{3})} a_0 + a a_0 a_1 \right). \quad (25)$$

When $n \geq 1$ in Equation (24), comparing coefficient of z^n we have

$$a_{n+3} = -\frac{(n+3)!}{n!} \left(\frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})}{\Gamma(1+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n (n+1-k)a_k a_{n+1-k} \right). \quad (26)$$

Therefore, according to (25) and (26) the power series solution for Equation (23) is of form as:

$$\begin{aligned}\phi(z) &= a_0 + a_1 z + a_2 z^2 - \frac{1}{6} \left(\frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3})}{\Gamma(1+\frac{p-2\alpha}{3})} a_0 + a a_0 a_1 \right) z^3 \\ &\quad - \sum_{n=1}^{\infty} \frac{(n+3)!}{n!} \left(\frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})}{\Gamma(1+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n (n+1-k)a_k a_{n+1-k} \right) z^{n+3}.\end{aligned}\quad (27)$$

Then substituting the initial conditions (20) into Equation (27), we obtain

$$a_0 = \lambda, \quad a_1 = a_2 = 0, \quad a_3 = -\frac{1}{6} \frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3})}{\Gamma(1+\frac{p-2\alpha}{3})} \lambda.$$

Hence, when $m = 1$, applying the transformation of the Lie group, the explicit power series solutions of the time fractional generalized KdV equations with initial and boundary (Equations (7) and (8)) can be obtained as follows

$$\begin{aligned}u(x, t) &= \left(1 - \frac{1}{6} \frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3})}{\Gamma(1+\frac{p-2\alpha}{3})} x^3 t^{-(p+\alpha)} \right) \lambda t^{\frac{p-2\alpha}{3}} \\ &\quad - \sum_{n=1}^{\infty} \frac{(n+3)!}{n!} \left(\frac{\Gamma(1-\alpha+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})}{\Gamma(1+\frac{p-2\alpha}{3}-\frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n (n+1-k)a_k a_{n+1-k} \right) x^{n+3} t^{\frac{p-2\alpha-(p+\alpha)(n+3)}{3}}.\end{aligned}\quad (28)$$

Case 2. When $m = 2$ in Equation (19), Equation (19) is written as

$$\left(P^{\frac{1}{3} + \frac{p-2\alpha}{6} - \alpha, \alpha}\phi\right)(z) + a(\phi(z))^2\phi'(z) + \phi'''(z) = 0. \quad (29)$$

After substituting (21) and (22) into equation Equation (29), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})} a_n z^n + a \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n \\ + \sum_{n=0}^{\infty} (n+1)(n+2)(n+3) a_{n+3} z^n = 0. \end{aligned} \quad (30)$$

When $n = 0$ in Equation (30), we have

$$a_3 = -\frac{1}{6} \left(\frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6})}{\Gamma(1 + \frac{p-2\alpha}{6})} a_0 + a a_0^2 a_1 \right). \quad (31)$$

When $n \geq 1$ in Equation (30), comparing coefficient of z^n we have

$$a_{n+3} = -\frac{(n+3)!}{n!} \left(\frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n \sum_{j=0}^k (n+1-k) a_j a_{k-j} a_{n+1-k} \right). \quad (32)$$

Therefore, according to (31) and (32), the power series solution for Equation (29) is of the form:

$$\begin{aligned} \phi(z) = a_0 + a_1 z + a_2 z^2 - \frac{1}{6} \left(\frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6})}{\Gamma(1 + \frac{p-2\alpha}{6})} a_0 + a a_0^2 a_1 \right) z^3 \\ - \sum_{n=1}^{\infty} \frac{(n+3)!}{n!} \left(\frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n \sum_{j=0}^k (n+1-k) a_j a_{k-j} a_{n+1-k} \right) z^{n+3}. \end{aligned} \quad (33)$$

Then, substituting the initial conditions (20) into Equation (33), we obtain

$$a_0 = \lambda, \quad a_1 = a_2 = 0, \quad a_3 = -\frac{1}{6} \frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6})}{\Gamma(1 + \frac{p-2\alpha}{6})} \lambda.$$

Hence, when $m = 2$, applying the transformation of the Lie group, the explicit power series solutions of the time fractional generalized KdV equations with initial and boundary (Equations (7) and (8)) can be obtained as follows

$$\begin{aligned} u(x, t) = \left(1 - \frac{1}{6} \frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6})}{\Gamma(1 + \frac{p-2\alpha}{6})} x^3 t^{-(p+\alpha)} \right) \lambda t^{\frac{p-2\alpha}{3}} \\ - \sum_{n=1}^{\infty} \frac{(n+3)!}{n!} \left(\frac{\Gamma(1 - \alpha + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{6} - \frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n \sum_{j=0}^k (n+1-k) a_j a_{k-j} a_{n+1-k} \right) x^{n+3} t^{\frac{p-2\alpha - (p+\alpha)(n+3)}{3}}. \end{aligned} \quad (34)$$

Case 3. When $m = 3$ in Equation (19), Equation (19) is written as

$$\left(P^{\frac{1}{3} + \frac{p-2\alpha}{9} - \alpha, \alpha}\phi\right)(z) + a(\phi(z))^3\phi'(z) + \phi'''(z) = 0. \quad (35)$$

After substituting (21) and (22) into equation Equation (35), we obtain

$$\sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})} a_n z^n + a \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=0}^{\infty} (n+1)(n+2)(n+3) a_{n+3} z^n = 0. \quad (36)$$

When $n = 0$ in Equation (36), we have

$$a_3 = -\frac{1}{6} \left(\frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9})}{\Gamma(1 + \frac{p-2\alpha}{9})} a_0 + a a_0^3 a_1 \right). \quad (37)$$

When $n \geq 1$ in Equation (36), comparing coefficient of z^n , we have

$$a_{n+3} = -\frac{(n+3)!}{n!} \left(\frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n \sum_{j=0}^k \sum_{l=0}^j (n+1-k) a_l a_{j-l} k a_{k-j} a_{n+1-k} \right). \quad (38)$$

Therefore, according to (37) and (38), the power series solution for Equation (35) is of form as:

$$\phi(z) = a_0 + a_1 z + a_2 z^2 - \frac{1}{6} \left(\frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9})}{\Gamma(1 + \frac{p-2\alpha}{9})} a_0 + a a_0 a_1 \right) z^3 - \sum_{n=1}^{\infty} \frac{(n+3)!}{n!} \left(\frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n \sum_{j=0}^k \sum_{l=0}^j (n+1-k) a_l a_{j-l} k a_{k-j} a_{n+1-k} \right) z^{n+3}. \quad (39)$$

Then substituting the initial conditions (20) into Equation (39), we obtain

$$a_0 = \lambda, \quad a_1 = a_2 = 0, \quad a_3 = -\frac{1}{6} \frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9})}{\Gamma(1 + \frac{p-2\alpha}{9})} \lambda.$$

Hence, when $m = 3$, applying the transformation of the Lie group, the explicit power series solutions of the time fractional generalized KdV equations with initial and boundary (Equations (7) and (8)) can be obtained as follows

$$u(x, t) = \left(1 - \frac{1}{6} \frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9})}{\Gamma(1 + \frac{p-2\alpha}{9})} x^3 t^{-(p+\alpha)} \right) \lambda t^{\frac{p-2\alpha}{3}} - \sum_{n=1}^{\infty} \frac{(n+3)!}{n!} \left(\frac{\Gamma(1-\alpha + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})}{\Gamma(1 + \frac{p-2\alpha}{9} - \frac{n(p+\alpha)}{3})} a_n + a \sum_{k=0}^n \sum_{j=0}^k \sum_{l=0}^j (n+1-k) a_l a_{j-l} k a_{k-j} a_{n+1-k} \right) x^{n+3} t^{\frac{p-2\alpha-(p+\alpha)(n+3)}{3}}. \quad (40)$$

5. Conclusions

In this paper, the method of classical Lie symmetry analysis can be successfully extended to the time fractional partial differential equation with initial and boundary values. Then time fractional generalized KdV equations have been transformed into fractional ordinary differential equations with initial value. Then we utilize the power series method to find exact solutions of the reduced equation with the Erdélyi-Kober fractional differential operator. The exact solutions of the time fractional partial differential equation with initial and boundary value are given. Hence, it confirms that our work is feasible.

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