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Conserved Quantity and Adiabatic Invariant for Hamiltonian System with Variable Order

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Abstract: Hamiltonian mechanics plays an important role in the development of nonlinear science. This paper aims for a fractional Hamiltonian system of variable order. Several issues are discussed, including differential equation of motion, Noether symmetry, and perturbation to Noether symmetry. As a result, fractional Hamiltonian mechanics of variable order are established, and conserved quantity and adiabatic invariant are presented. Two applications, fractional isotropic harmonic oscillator model of variable order and fractional Lotka biochemical oscillator model of variable order are given to illustrate the Methods and Results.

Keywords: Hamiltonian system; fractional derivative of variable order; Noether symmetry; conserved quantity; adiabatic invariant

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1. Introduction

Fractional calculus was firstly proposed because the order of a function's integral or derivative is a non-integer constant. In fact, fractional calculus started more than 300 years ago when L'Hopital and Leibniz were discussing the meaning of $\mathrm{d}^{1/2}y/\mathrm{d}x^{1/2}$. The development of fractional calculus is much slower than that of integer calculus though many famous mathematicians, such as Fourier, Euler, Riemann, Liouville, Letnikov, Grunwald, etc., have contributed to it (see [1–3] for the history of fractional calculus). In recent decades, because of fractional calculus's many applications in various fields of science, engineering, biomechanics, economics, and so forth, it has gained more attention (see [4–7] for a review). For example, fractional calculus has been applied to physics [8], quantum mechanics [9–11], field theory [12,13], etc. Among several definitions of the fractional derivatives, which are generally nonlocal operators and are historically applied to study time-dependent or nonlocal processes, Riemann–Liouville fractional derivative and Caputo fractional derivative are the most famous ones.

Furthermore, Samko and Ross [14] introduced a generalization of fractional calculus. They considered the order of a function's integral or derivative as $\alpha(\cdot,\cdot)$, where $\alpha(\cdot,\cdot)$ is a function rather than a constant. Because of the nonlocality and memory of fractional order calculus, it is reasonable that the order of a function's integral or derivative may vary. Afterwards, several works were dedicated to the properties and applications of the fractional operators with variable order [15–20]. For instance, Coimbra [17] examined dynamical behaviors with frictional force using the variable order model. Diaz and Coimbra [18] utilized the differential equation with variable order to reveal some dynamical control behaviors of the nonlinear oscillator. Sun et al. [20] investigated a class of fractional models with variable order. Nowadays, more and more fractional models with variable order have been applied to mechanics, mathematics, and other related subjects (see [21–24] and references therein).

In 1996, Riewe started the theory of the fractional calculus of variations by considering that the first-order derivative terms should come from the half-order derivative terms [25,26]. From then on, scholars began to use fractional derivatives to describe the dissipative forces such as frictional forces for the nonconservative systems. Then, fractional calculus of variations was studied by several scholars such as Klimek [27], Agrawal [28,29], Baleanu et al. [30,31], Torres [32], etc. Moreover, the development of the fractional calculus of variations with variable order also made great progress. In [33–35], several results for the fractional calculus of variations with variable order were obtained. Particularly, motivated by references [36–38], a linear combination of the fractional derivative of variable order was introduced. Based on a combined Caputo fractional derivative of variable order (CCVO), the necessary optimality conditions for variational problems were established [39,40], the fractional variational problem of Herglotz type with variable order was studied [41], and two fractional isoperimetric problems and a new variational problem subject to a holonomic constraint were presented [42].

After differential equations of motion are established through the calculus of variations, the next task is to find the solutions to them in dynamics. In fact, the solution to the equations can be given if one can find all of the integrals of the equations. An integral is a conserved quantity; therefore, people try their best to find all of the conserved quantities of a mechanical system. By means of the analysis of forces, Newtonian mechanics gives three conservation laws, i.e., the conservation of momentum, the conservation of mechanical energy, and the conservation of moment of momentum. By means of the analysis of the form of Lagrangian, Lagrangian mechanics gives two conservation laws, i.e., the conservation of generalized energy and the conservation of generalized momentum. The conservation of generalized momentum may be a conservation of momentum, a conservation of moment of momentum, or neither. The physical meaning of the conservation law in Lagrangian mechanics is less clear than that in Newtonian mechanics, whose three conservation laws have very clear physical meaning. However, the conserved quantities deduced by Lagrangian mechanics are more than those deduced by Newtonian mechanics. Since German mathematician Emmy Noether published her famous paper [43], the Noether symmetry method has become a modern method for seeking the conservation law of mechanical systems (see [44] for a review). Similarly, although the physical meaning of the conservation law obtained from the Noether symmetry method is less than that obtained from Lagrangian mechanics, the numbers of the conserved quantities deduced by the Noether symmetry method are more than those deduced by Lagrangian mechanics.

Recently, fractional Noether symmetry and fractional conserved quantity are also under strong research. There are two different definitions of fractional conserved quantity. One was introduced by Frederico and Torres [45] by means of a bilinear fractional operator D $(D(C) = 0)$, and the other was introduced by Atanacković [46] by means of the classical definition $(\mathrm{d}C/\mathrm{d}t = 0)$, where C means a fractional conserved quantity. Fractional Noether symmetry and fractional conserved quantity have been investigated through both definitions. For example, works [47,48] were done based on the former definition, and results [49–53] were obtained on the basis of the latter one. Furthermore, the study of the fractional Noether symmetry and fractional conserved quantity with variable order has also begun. For instance, Odzijewicz, Malinowska, and Torres [54,55] investigated Noether theorem and the second Noether theorem for the fractional Lagrangian system with variable order. Yan and Zhang [56,57] studied Noether symmetry and conserved quantity for the fractional Birkhoffian system in terms of Caputo fractional derivative of variable order (CVO), Riemann–Liouville fractional derivative of variable order (RLVO), Riesz–Caputo fractional derivative of variable order (RCVO), and Riesz–Riemann–Liouville fractional derivative of variable order (RVO).

When a dynamical system is disturbed by small forces, the conserved quantity, which refers to the integrability of the system, may also change. Therefore, the research on the perturbation to Noether symmetry and adiabatic invariants is also of great significance for a dynamical system [58,59]. A classical adiabatic invariant means a certain physical quantity that changes more slowly than the parameter that varies very slowly. That is, when a dynamical system is disturbed by small forces and varies very slowly, a slower physical quantity, i.e., adiabatic invariant, will be found. Some important results on the study of perturbation to Noether symmetry and adiabatic invariants for constrained mechanical systems have been obtained [60,61]. Very recently, we studied the perturbation to fractional

Noether symmetry for the Hamiltonian system [53] as well as the Birkhoffian system [62]. Besides, the results on perturbation to Noether symmetry with RLVO for the fractional generalized Birkhoffian system [63] were also presented.

The purpose of this paper is to generalize the problem of the calculus of variations, Noether theory, and perturbation to Noether symmetry to the fractional Hamiltonian systems in terms of combined Riemann–Liouville fractional derivatives of variable order (CRLVO) and combined Caputo fractional derivatives of variable order (CCVO), respectively. Here, the CRLVO and the CCVO are the first time to be introduced, and many results obtained before are the special cases of this paper. As the main results, Hamilton equations with the CRLVO and the CCVO are established. Then, Noether symmetry and conserved quantities for the fractional Hamiltonian systems with the CRLVO and the CCVO are presented. Perturbation to Noether symmetry and adiabatic invariants for the disturbed fractional Hamiltonian systems with the CRLVO and the CCVO are also investigated, and two applications are given to illustrate the Methods and Results of this paper.

In a word, there are mainly three factors for presenting this paper. Firstly, analytical mechanics. Lagrange was the first man to publish the famous book *Analytical Mechanics*. Lagrange felt proud when he successfully described mechanics problems mathematically without any diagrams because within the framework of Newtonian mechanics, mechanical phenomena are treated mostly schematically. In fact, the representation by means of analytical mechanics is more general as well as convenient. Secondly, Noether theory. Noether theory not only has mathematical importance but can also be helpful in understanding some inherent physical properties of a dynamical system. Generally, conserved quantity helps reduce the degrees of freedom and make it easier to find the solutions to the differential equations of motion. Thirdly, fractional calculus. Fractional calculus is a useful tool to deal with the dissipative systems, and, as early as 1993, Miller and Ross [64] once said that fractional calculus has been involved in almost every field of science and engineering.

This paper is constructed as follows: Combined fractional derivatives of variable order and their properties are given in Section 2. Differential equations of motion for the Hamiltonian system of variable order are established in Section 3. In Sections 4 and 5, Noether symmetry and conserved quantity, and perturbation to Noether symmetry and adiabatic invariants are investigated. The Results and Methods are illustrated in Section 6 with two applications and we finish with Section 7 for Conclusions.

2. Combined Fractional Derivative of Variable Order

Following the references [54,57], we list the left and right Riemann–Liouville fractional integrals of variable order, the left and right RLVO, and the left and right CVO by

$$ {}_{t_1}I_t^{\alpha(\cdot,\cdot)} f(t) = \int_{t_1}^{t} \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\alpha(t,\tau)-1} f(\tau)\mathrm{d}\tau \tag{1} $$

$$ {}_{t}I_{t_2}^{\alpha(\cdot,\cdot)} f(t) = \int_{t}^{t_2} \frac{1}{\Gamma(\alpha(\tau,t))} (\tau-t)^{\alpha(\tau,t)-1} f(\tau)\mathrm{d}\tau \tag{2} $$

$$ {}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} {}_{t_1}I_t^{1-\alpha(\cdot,\cdot)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_1}^{t} \frac{1}{\Gamma(1-\alpha(t,\tau))} (t-\tau)^{-\alpha(t,\tau)} f(\tau)\mathrm{d}\tau \tag{3} $$

$$ {}_{t}^{RL}D_{t_2}^{\alpha(\cdot,\cdot)} f(t) = \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\Big) {}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} f(t) = \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\Big) \int_{t}^{t_2} \frac{1}{\Gamma(1-\alpha(\tau,t))} (\tau-t)^{-\alpha(\tau,t)} f(\tau)\mathrm{d}\tau \tag{4} $$

$$ {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} f(t) = \int_{t_1}^{t} \frac{1}{\Gamma(1-\alpha(t,\tau))} (t-\tau)^{-\alpha(t,\tau)} \frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\mathrm{d}\tau \tag{5} $$

$$ {}_{t}^{C}D_{t_2}^{\alpha(\cdot,\cdot)} f(t) = \int_{t}^{t_2} \frac{1}{\Gamma(1-\alpha(\tau,t))} (\tau-t)^{-\alpha(\tau,t)} \Big(-\frac{\mathrm{d}}{\mathrm{d}\tau}\Big) f(\tau)\mathrm{d}\tau \tag{6} $$

where $0 < \alpha(\cdot,\cdot) < 1$, $\Gamma(\cdot)$ is the Gamma function, $f(t) \in L_1[t_1, t_2]$, ${}_{t_1}I_t^{1-\alpha(\cdot,\cdot)} f(t) \in AC[t_1, t_2]$, ${}_tI_{t_2}^{1-\alpha(\cdot,\cdot)} f(t) \in AC[t_1, t_2]$. In this paper, we assume that $\alpha(t,\tau) = \alpha(t-\tau)$, $\alpha(\tau,t) = \alpha(\tau-t)$.

Based on the definitions of the RLVO and the CVO, the RVO and the RCVO are defined in [57] as

$$ {}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} f(t) = \frac{1}{2}\Big[{}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} f(t) - {}_t^{RL}D_{t_2}^{\alpha(\cdot,\cdot)} f(t)\Big] \tag{7} $$

$$ {}_{t_1}^{RC}D_{t_2}^{\alpha(\cdot,\cdot)} f(t) = \frac{1}{2}\Big[{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} f(t) - {}_t^{C}D_{t_2}^{\alpha(\cdot,\cdot)} f(t)\Big] \tag{8} $$

In this paper, CRLVO and CCVO are constructed as

$$ {}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} f(t) = \gamma\,{}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} f(t) - (1-\gamma)\,{}_t^{RL}D_{t_2}^{\beta(\cdot,\cdot)} f(t) \tag{9} $$

$$ {}^{C}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} f(t) = \gamma\,{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} f(t) - (1-\gamma)\,{}_t^{C}D_{t_2}^{\beta(\cdot,\cdot)} f(t) \tag{10} $$

where $0 < \alpha(\cdot,\cdot) < 1$, $0 < \beta(\cdot,\cdot) < 1$, ${}_{t_1}D_t^{\alpha(\cdot,\cdot)}$ and ${}_tD_{t_2}^{\beta(\cdot,\cdot)}$ show the arrow of time, and γ is a parameter, $0 \le \gamma \le 1$, which determines the different quantities of information from the past and the future.

The CRLVO and the CCVO are the most general operators with variable order to some extent. From Equations (9) and (10), the RVO, the RCVO, the left and right Riemann–Liouville fractional derivatives (RL), the left and right Caputo fractional derivatives (C), the Riesz–Riemann–Liouville fractional derivative (R), the Riesz–Caputo fractional derivative (RC), the combined Riemann–Liouville fractional derivative (CRL), the combined Caputo fractional derivative (CC), and the classical integer derivative can be deduced as special cases. We list them as follows:

Let $\beta = \alpha$, $\gamma = \frac{1}{2}$ in Equations (9) and (10), so the RVO and the RCVO can be achieved.

When $\alpha(\cdot,\cdot) \to \alpha$, $\beta(\cdot,\cdot) \to \beta$, with two constants $0 < \alpha, \beta < 1$, the left and right RL, the left and right C, the R, the RC, the CRL, and the CC can be obtained as

$$ {}_{t_1}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_1}^{t} (t-\tau)^{-\alpha} f(\tau)\mathrm{d}\tau \tag{11} $$

$$ {}_t^{RL}D_{t_2}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \Big(-\frac{\mathrm{d}}{\mathrm{d}t}\Big) \int_{t}^{t_2} (\tau-t)^{-\alpha} f(\tau)\mathrm{d}\tau \tag{12} $$

$$ {}_{t_1}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_1}^{t} (t-\tau)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\mathrm{d}\tau \tag{13} $$

$$ {}_t^{C}D_{t_2}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t}^{t_2} (\tau-t)^{-\alpha} \Big(-\frac{\mathrm{d}}{\mathrm{d}\tau}\Big) f(\tau)\mathrm{d}\tau \tag{14} $$

$$ {}_{t_1}^{R}D_{t_2}^{\alpha} f(t) = \frac{1}{2}\Big[{}_{t_1}^{RL}D_t^{\alpha} f(t) - {}_t^{RL}D_{t_2}^{\alpha} f(t)\Big] \tag{15} $$

$$ {}_{t_1}^{RC}D_{t_2}^{\alpha} f(t) = \frac{1}{2}\Big[{}_{t_1}^{C}D_t^{\alpha} f(t) - {}_t^{C}D_{t_2}^{\alpha} f(t)\Big] \tag{16} $$

$$ {}^{RL}D_\gamma^{\alpha,\beta} f(t) = \gamma\,{}_{t_1}^{RL}D_t^{\alpha} f(t) - (1-\gamma)\,{}_t^{RL}D_{t_2}^{\beta} f(t) \tag{17} $$

$$ {}^{C}D_\gamma^{\alpha,\beta} f(t) = \gamma\,{}_{t_1}^{C}D_t^{\alpha} f(t) - (1-\gamma)\,{}_t^{C}D_{t_2}^{\beta} f(t) \tag{18} $$

Equations (11) to (18) can be found in [65].

When $\alpha(\cdot,\cdot) \to 1$, we have

$$ {}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} = {}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} = \mathrm{d}/\mathrm{d}t,\ {}_t^{RL}D_{t_2}^{\alpha(\cdot,\cdot)} = {}_t^{C}D_{t_2}^{\alpha(\cdot,\cdot)} = -\mathrm{d}/\mathrm{d}t \tag{19} $$

The formulae of integration by parts for the CVO are

$$\int_{t_1}^{t_2} g(t)\Big({}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)} f(t)\Big)\mathrm{d}t = \int_{t_1}^{t_2} f(t)\Big({}_{t}^{RL}D_{t_2}^{\alpha(\cdot,\cdot)} g(t)\Big)\mathrm{d}t + \Big(f(t)\Big({}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} g(t)\Big)\Big)\Big|_{t_1}^{t_2} \tag{20}$$

$$\int_{t_1}^{t_2} g(t)\Big({}_{t}^{C}D_{t_2}^{\alpha(\cdot,\cdot)} f(t)\Big)\mathrm{d}t = \int_{t_1}^{t_2} f(t)\Big({}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} g(t)\Big)\mathrm{d}t - \Big(f(t)\Big({}_{t_1}I_t^{1-\alpha(\cdot,\cdot)} g(t)\Big)\Big)\Big|_{t_1}^{t_2} \tag{21}$$

The formulae of integration by parts for the RLVO, RVO, and RCVO are

$$\int_{t_1}^{t_2} g(t)\Big({}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} f(t)\Big)\mathrm{d}t = \int_{t_1}^{t_2} f(t)\Big({}_{t}^{RL}D_{t_2}^{\alpha(\cdot,\cdot)} g(t)\Big)\mathrm{d}t - \Big[\Big({}_{t_1}I_t^{1-\alpha(\cdot,\cdot)} f(t)\Big)g(t)\Big]\Big|_{t_1} \tag{22}$$

$$\int_{t_1}^{t_2} g(t)\Big({}_{t}^{RL}D_{t_2}^{\alpha(\cdot,\cdot)} f(t)\Big)\mathrm{d}t = \int_{t_1}^{t_2} f(t)\Big({}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} g(t)\Big)\mathrm{d}t - \Big[\Big({}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} f(t)\Big)g(t)\Big]\Big|_{t_2} \tag{23}$$

$$\begin{aligned}\int_{t_1}^{t_2} g(t)\Big({}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} f(t)\Big)\mathrm{d}t = & -\int_{t_1}^{t_2} f(t)\Big({}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} g(t)\Big)\mathrm{d}t + \tfrac{1}{2}\Big[\Big({}_{t}I_{t_2}^{1-\alpha(\cdot,\cdot)} f(t)\Big)g(t)\Big]\Big|_{t_2} \\ & -\tfrac{1}{2}\Big[\Big({}_{t_1}I_t^{1-\alpha(\cdot,\cdot)} f(t)\Big)g(t)\Big]\Big|_{t_1}\end{aligned} \tag{24}$$

$$\int_{t_1}^{t_2} g(t)\Big({}_{t_1}^{RC}D_{t_2}^{\alpha(\cdot,\cdot)} f(t)\Big)\mathrm{d}t = -\int_{t_1}^{t_2} f(t)\Big({}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} g(t)\Big)\mathrm{d}t + \Big[f(t)\Big({}_{t_1}^{R}I_{t_2}^{1-\alpha(\cdot,\cdot)} g(t)\Big)\Big]\Big|_{t_1}^{t_2} \tag{25}$$

For more details about the above Equations (19)–(25), please refer to [54,57,65].

3. Hamilton Equation of Variable Order

A unified Hamilton action and Hamilton principle of variable order will be given first, and then the Hamilton action and Hamilton principle in terms of the CRLVO and the CCVO will be discussed, respectively.

Assuming the Lagrangian of variable order is $L_U\Big(t, q_j, D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\Big)$, then the generalized momentum of variable order and the Hamiltonian of variable order can be listed as

$$\begin{gathered} p_{Ui} = \frac{\partial L_U\Big(t, q_j, D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\Big)}{\partial D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i}, \\ H_U = p_{Ui} \cdot D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - L_U\Big(t, q_j, D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\Big),\ i,j = 1,2,\cdots,n \end{gathered} \tag{26}$$

And the functional

$$S_{HU} = \int_{t_1}^{t_2} \Big[p_{Ui} \cdot D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_U\Big(t, q_j, p_{Uj}\Big)\Big]\mathrm{d}t \tag{27}$$

Represents a unified Hamilton action of variable order, where $q_j, p_{Uj} \in \mathrm{C}^1$, H_U is a C^1 function with respect to all its arguments.

The formula

$$\delta S_{HU} = \delta \int_{t_1}^{t_2} \Big[p_{Ui} \cdot D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_U\Big(t, q_j, p_{Uj}\Big)\Big]\mathrm{d}t = 0 \tag{28}$$

with [57]

$$\delta_{t_1} D_t^{\alpha(\cdot,\cdot)} q_i = {}_{t_1}D_t^{\alpha(\cdot,\cdot)} \delta q_i,\ \delta_t D_{t_2}^{\beta(\cdot,\cdot)} q_i = {}_tD_{t_2}^{\beta(\cdot,\cdot)} \delta q_i,\ \delta q_i\big|_{t=t_1} = \delta q_i\big|_{t=t_2} = 0 \tag{29}$$

is called a unified Hamilton principle of variable order.

3.1. Hamilton Equation with the CRLVO

When we consider the CRLVO, the Lagrangian of variable order can be denoted by $L_{RL}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)$. The generalized momentum of variable order and the Hamiltonian of variable order are

$$\begin{array}{c} p_{RLi} = \dfrac{\partial L_{RL}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)}{\partial {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i}, \\ H_{RL} = p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - L_{RL}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right) \end{array} \tag{30}$$

Under this condition, Hamilton action of variable order is

$$S_{HRL} = \int_{t_1}^{t_2} \left[p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_{RL}\left(t, q_j, p_{RLj}\right) \right] \mathrm{d}t \tag{31}$$

Hamilton principle of variable order is

$$\delta S_{HRL} = \delta \int_{t_1}^{t_2} \left[p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_{RL}\left(t, q_j, p_{RLj}\right) \right] \mathrm{d}t = 0 \tag{32}$$

with [57]

$$\delta {}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} q_i = {}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} \delta q_i,\ \delta {}_t^{RL}D_{t_2}^{\beta(\cdot,\cdot)} q_i = {}_t^{RL}D_{t_2}^{\beta(\cdot,\cdot)} \delta q_i,\ \delta q_i\big|_{t=t_1} = \delta q_i\big|_{t=t_2} = 0 \tag{33}$$

It follows from Equation (32) that

$$\begin{aligned} \delta S_{HRL} &= \delta \int_{t_1}^{t_2} \left[p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_{RL}\left(t, q_j, p_{RLj}\right) \right] \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left[\delta p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i + p_{RLi} \cdot \delta {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - \delta H_{RL}\left(t, q_j, p_{RLj}\right) \right] \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left(\delta p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - \delta q_i \cdot {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{RLi} - \tfrac{\partial H_{RL}}{\partial q_i} \delta q_i - \tfrac{\partial H_{RL}}{\partial p_{RLi}} \delta p_{RLi} \right) \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left[\delta p_{RLi} \cdot \left({}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - \tfrac{\partial H_{RL}}{\partial p_{RLi}} \right) - \delta q_i \cdot \left({}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{RLi} + \tfrac{\partial H_{RL}}{\partial q_i} \right) \right] \mathrm{d}t = 0 \end{aligned} \tag{34}$$

where

$$\begin{aligned} &\int_{t_1}^{t_2} p_{RLi} \cdot \delta {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i \mathrm{d}t = \int_{t_1}^{t_2} p_{RLi} \cdot \delta \left[\gamma {}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} q_i - (1-\gamma) {}_t^{RL}D_{t_2}^{\beta(\cdot,\cdot)} q_i \right] \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left[\gamma p_{RLi} \cdot {}_{t_1}^{RL}D_t^{\alpha(\cdot,\cdot)} \delta q_i - (1-\gamma) p_{RLi} \cdot {}_t^{RL}D_{t_2}^{\beta(\cdot,\cdot)} \delta q_i \right] \mathrm{d}t \\ &= \int_{t_1}^{t_2} \left[\delta q_i \cdot \gamma {}_t^{RL}D_{t_2}^{\alpha(\cdot,\cdot)} p_{RLi} - \delta q_i \cdot (1-\gamma) {}_{t_1}^{RL}D_t^{\beta(\cdot,\cdot)} p_{RLi} \right] \mathrm{d}t \\ &= -\int_{t_1}^{t_2} \left[\delta q_i \cdot {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{RLi} \right] \mathrm{d}t \end{aligned} \tag{35}$$

From Equations (30) and (34), we have

$${}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i = \frac{\partial H_{RL}}{\partial p_{RLi}},\ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{RLi} = -\frac{\partial H_{RL}}{\partial q_i} \tag{36}$$

Equation (36) is called the Hamilton equation with the CRLVO.

3.2. Hamilton Equation with the CCVO

When we consider the CCVO, the Lagrangian of variable order can be denoted by $L_C\left(t, q_j, {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)$. The generalized momentum of variable order and the Hamiltonian of variable order are

$$p_{Ci} = \frac{\partial L_C\left(t, q_j, {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)}{\partial {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i},\ H_C = p_{Ci} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - L_C\left(t, q_j, {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right) \tag{37}$$

Under this condition, the Hamilton action of variable order is

$$S_{HC} = \int_{t_1}^{t_2} \left[p_{Ci} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_C\left(t, q_j, p_{Cj}\right) \right] \mathrm{d}t \tag{38}$$

The Hamilton principle of variable order is

$$\delta S_{HC} = \delta \int_{t_1}^{t_2} \left[p_{Ci} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_C\left(t, q_j, p_{Cj}\right) \right] \mathrm{d}t = 0 \tag{39}$$

with [57]

$$\delta {}_{t_1}^{C}D_{t}^{\alpha(\cdot,\cdot)} q_i = {}_{t_1}^{C}D_{t}^{\alpha(\cdot,\cdot)} \delta q_i,\ \delta {}_{t}^{C}D_{t_2}^{\beta(\cdot,\cdot)} q_i = {}_{t}^{C}D_{t_2}^{\beta(\cdot,\cdot)} \delta q_i,\ \delta q_i\big|_{t=t_1} = \delta q_i\big|_{t=t_2} = 0 \tag{40}$$

Similarly, from Equations (37) and (39), we can get

$${}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i = \frac{\partial H_C}{\partial p_{Ci}},\ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{Ci} = -\frac{\partial H_C}{\partial q_i} \tag{41}$$

Equation (41) is called the Hamilton equation with the CCVO.

Remark 1. *When $\beta(\cdot,\cdot) = \alpha(\cdot,\cdot)$, $\gamma = \frac{1}{2}$, the Hamilton equation in terms of the RVO can be obtained from Equation (36) as*

$${}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} q_i = \frac{\partial H_R}{\partial p_{Ri}},\ {}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} p_{Ri} = -\frac{\partial H_R}{\partial q_i} \tag{42}$$

where $p_{Ri} = \frac{\partial L_R\left(t, q_j, {}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} q_j\right)}{\partial {}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} q_i}$, $H_R = p_{Ri} \cdot {}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} q_i - L_R\left(t, q_j, {}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} q_j\right)$.

Similarly, the Hamilton equation in terms of the RCVO can be obtained from Equation (41) as

$${}_{t_1}^{RC}D_{t_2}^{\alpha(\cdot,\cdot)} q_i = \frac{\partial H_{RC}}{\partial p_{RCi}},\ {}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)} p_{RCi} = -\frac{\partial H_{RC}}{\partial q_i} \tag{43}$$

where $p_{RCi} = \frac{\partial L_{RC}\left(t, q_j, {}_{t_1}^{RC}D_{t_2}^{\alpha(\cdot,\cdot)} q_j\right)}{\partial {}_{t_1}^{RC}D_{t_2}^{\alpha(\cdot,\cdot)} q_i}$, $H_{RC} = p_{RCi} \cdot {}_{t_1}^{RC}D_{t_2}^{\alpha(\cdot,\cdot)} q_i - L_{RC}\left(t, q_j, {}_{t_1}^{RC}D_{t_2}^{\alpha(\cdot,\cdot)} q_j\right)$.

Moreover, Equations (42) and (43) can also be obtained through their respective Hamilton principles.

Remark 2. *From Equations (30) and (36), the Lagrange equation with the CRLVO can be obtained as*

$$\frac{\partial L_{RL}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)}{\partial q_i} - {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \frac{\partial L_{RL}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)}{\partial {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} = 0 \tag{44}$$

It is noted that Equation (44) can be obtained through

$$\begin{gathered} \delta S_{LRL} = \delta \int_{t_1}^{t_2} L_{RL}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right) \mathrm{d}t = 0, \\ \delta {}_{t_1}^{RL}D_{t}^{\alpha(\cdot,\cdot)} q_i = {}_{t_1}^{RL}D_{t}^{\alpha(\cdot,\cdot)} \delta q_i,\ \delta {}_{t}^{RL}D_{t_2}^{\beta(\cdot,\cdot)} q_i = {}_{t}^{RL}D_{t_2}^{\beta(\cdot,\cdot)} \delta q_i,\ \delta q_i\big|_{t=t_1} = \delta q_i\big|_{t=t_2} = 0 \end{gathered} \tag{45}$$

Similarly, from Equations (37) and (41), the Lagrange equation with the CCVO can be obtained as

$$\frac{\partial L_C}{\partial q_i} - {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \frac{\partial L_C\left(t, q_j, {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)}{\partial {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} = 0 \tag{46}$$

And Equation (46) can also be obtained through

$$\begin{gathered}\delta S_{LC}=\delta\int_{t_1}^{t_2}L_C\left(t,q_j,{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_j\right)\mathrm{d}t=0,\ \delta{}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}q_i={}_{t_1}^{C}D_t^{\alpha(\cdot,\cdot)}\delta q_i,\\ \delta{}_{t}^{C}D_{t_2}^{\beta(\cdot,\cdot)}q_i={}_{t}^{C}D_{t_2}^{\beta(\cdot,\cdot)}\delta q_i,\ \delta q_i\big|_{t=t_1}=\delta q_i\big|_{t=t_2}=0\end{gathered} \tag{47}$$

Remark 3. *When $\alpha(\cdot,\cdot)\to\alpha$, $\beta(\cdot,\cdot)\to\beta$, Hamilton equations of variable order, i.e., Equations (36) and (41) reduce to*

$${}^{RL}D_{\gamma}^{\alpha,\beta}q_i=\frac{\partial H_{RLF}}{\partial p_{RLiF}},\ {}^{RL}D_{1-\gamma}^{\beta,\alpha}p_{RLiF}=-\frac{\partial H_{RLF}}{\partial q_i} \tag{48}$$

and

$${}^{C}D_{\gamma}^{\alpha,\beta}q_i=\frac{\partial H_{CF}}{\partial p_{CiF}},\ {}^{RL}D_{1-\gamma}^{\beta,\alpha}p_{CiF}=-\frac{\partial H_{CF}}{\partial q_i} \tag{49}$$

Equations (48) and (49) are the Hamilton equations with the CRL and the CC, respectively.

Lagrange equations of variable order, i.e., Equations (44) and (46) reduce to

$$\frac{\partial L_{RLF}\left(t,q_j,{}^{RL}D_{\gamma}^{\alpha,\beta}q_j\right)}{\partial q_i}-{}^{RL}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{RLF}\left(t,q_j,{}^{RL}D_{\gamma}^{\alpha,\beta}q_j\right)}{\partial^{RL}D_{\gamma}^{\alpha,\beta}q_i}=0 \tag{50}$$

and

$$\frac{\partial L_{CF}}{\partial q_i}-{}^{RL}D_{1-\gamma}^{\beta,\alpha}\frac{\partial L_{CF}\left(t,q_j,{}^{C}D_{\gamma}^{\alpha,\beta}q_j\right)}{\partial^{C}D_{\gamma}^{\alpha,\beta}q_i}=0 \tag{51}$$

Equations (50) and (51) are the Lagrange equations with the CRL and the CC, respectively. It is noted that Equations (48) to (51) are consistent with the results in [53].

Remark 4. *When $\alpha(\cdot,\cdot)$, $\beta(\cdot,\cdot)\to 1$, Hamilton equations of variable order, i.e., Equations (36) and (41) reduce to*

$$\dot{q}_i=\frac{\partial H\left(t,q_j,p_j\right)}{\partial p_i},\ \dot{p}_i=-\frac{\partial H\left(t,q_j,p_j\right)}{\partial q_i} \tag{52}$$

Lagrange equations of variable order, i.e., Equations (44) and (46) reduce to

$$\frac{\partial L\left(t,q_j,\dot{q}_j\right)}{\partial q_i}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L\left(t,q_j,\dot{q}_j\right)}{\partial \dot{q}_i}=0 \tag{53}$$

Equations (52) and (53) are the classical Hamilton equation and the classical Lagrange equation, respectively, which is consistent with the results in [66].

4. Noether Symmetry and Conserved Quantity

The classical Noether symmetry means the invariance of the classical Hamilton action under the infinitesimal transformations of time and coordinates. A Noether symmetry can lead to a conserved quantity. Here, we generalize the classical definition of Noether symmetry to fractional form with variable order. Then Noether symmetry and conserved quantity will be discussed in detail with the CRLVO and the CCVO, respectively.

Under the infinitesimal transformations

$$\bar{t}=t+\Delta t,\ \bar{q}_i\left(\bar{t}\right)=q_i(t)+\Delta q_i,\ \bar{p}_{Ui}\left(\bar{t}\right)=p_{Ui}(t)+\Delta p_{Ui} \tag{54}$$

Whose linear parts are

$$\begin{gathered}\bar{t}=t+\theta\xi_{U0}^{0}\big(t,q_j,p_{Uj}\big),\ \bar{q}_i\big(\bar{t}\big)=q_i(t)+\theta\xi_{Ui}^{0}\big(t,q_j,p_{Uj}\big),\\ \bar{p}_{Ui}\big(\bar{t}\big)=p_{Ui}(t)+\theta\eta_{Ui}^{0}\big(t,q_j,p_{Uj}\big)\end{gathered}\tag{55}$$

The unified Hamilton action of variable order, i.e., Equation (27) changes to be

$$\bar{S}_{HU}=\int_{\bar{t}_1}^{\bar{t}_2}\Big[\bar{p}_{Ui}\cdot D_{\gamma}^{\bar{\alpha}(\cdot,\cdot),\bar{\beta}(\cdot,\cdot)}\bar{q}_i-H_U\big(\bar{t},\bar{q}_j,\bar{p}_{Uj}\big)\Big]\mathrm{d}\bar{t}\tag{56}$$

where θ is an infinitesimal parameter, and ξ_{U0}^{0}, ξ_{Ui}^{0}, and η_{Ui}^{0} are infinitesimal generators.

Let ΔS_{HU} denote the linear part of $\bar{S}_{HU}-S_{HU}$, if there exists infinitesimal generators satisfying $\Delta S_{HU}=-\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\Delta G_U^0\big)\mathrm{d}t$, then fractional Noether symmetry with variable order can be determined by the corresponding infinitesimal transformations, where $\Delta G_U^0=\theta G_U^0\big(t,q_j,p_{Uj}\big)$, and G_U^0 is called a gauge function.

For a certain mechanical system, conserved quantity can be deduced from Noether symmetry. A conserved quantity can be seen as a solution to the differential equation of motion of the mechanical system, and help reduce the freedom of the mechanical system.

A quantity I_U is called a conserved quantity if and only if $\frac{\mathrm{d}}{\mathrm{d}t}I_U=0$.

4.1. Noether Symmetry and Conserved Quantity with the CRLVO

When we consider the CRLVO, the infinitesimal transformations can be expressed as

$$\begin{gathered}\bar{t}=t+\Delta t,\ \bar{q}_i\big(\bar{t}\big)=q_i(t)+\Delta q_i,\ \bar{p}_{RLi}\big(\bar{t}\big)=p_{RLi}(t)+\Delta p_{RLi}\\ \Big(\bar{t}=t+\theta_{RL}\xi_{RL0}^{0}\big(t,q_j,p_{RLj}\big),\ \bar{q}_i\big(\bar{t}\big)=q_i(t)+\theta_{RL}\xi_{RLi}^{0}\big(t,q_j,p_{RLj}\big),\\ \bar{p}_{RLi}\big(\bar{t}\big)=p_{RLi}(t)+\theta_{RL}\eta_{RLi}^{0}\big(t,q_j,p_{RLj}\big)\Big).\end{gathered}\tag{57}$$

Under the action of Equation (57), the Hamilton action of variable order, i.e., Equation (31) becomes

$$\bar{S}_{HRL}=\int_{\bar{t}_1}^{\bar{t}_2}\Big[\bar{p}_{RLi}\cdot{}^{RL}D_{\gamma}^{\bar{\alpha}(\cdot,\cdot),\bar{\beta}(\cdot,\cdot)}\bar{q}_i-H_{RL}\big(\bar{t},\bar{q}_j,\bar{p}_{RLj}\big)\Big]\mathrm{d}\bar{t}\tag{58}$$

Then we get

$$\begin{gathered}\Delta S_{HRL}=\bar{S}_{HRL}-S_{HRL}\\ =\int_{t_1}^{t_2}\Big\{-H_{RL}\big(t+\Delta t,q_j+\Delta q_j,p_{RLj}+\Delta p_{RLj}\big)+\Big[{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i\\ -\gamma q_i(t_1)\Delta t_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))}+{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\delta q_i+(1-\gamma)q_i(t_2)\Delta t_2\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))}\Big]\\ \cdot(p_{RLi}+\Delta p_{RLi})\Big\}\cdot\Big(1+\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\Big)\mathrm{d}t-\int_{t_1}^{t_2}\Big[p_{RLi}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i-H_{RL}\big(t,q_j,p_{RLj}\big)\Big]\mathrm{d}t\\ =\int_{t_1}^{t_2}\Big[p_{RLi}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\delta q_i+\Big(p_{RLi}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i-\frac{\partial H_{RL}}{\partial t}\Big)\Delta t\\ -\gamma p_{RLi}q_i(t_1)\Delta t_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))}+\Big(p_{RLi}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i-H_{RL}\Big)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\\ +(1-\gamma)p_{RLi}q_i(t_2)\Delta t_2\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))}-\frac{\partial H_{RL}}{\partial q_i}\Delta q_i\Big]\mathrm{d}t\end{gathered}\tag{59}$$

where

$$\begin{gathered}{}^{RL}D_{\gamma}^{\bar{\alpha}(\cdot,\cdot),\bar{\beta}(\cdot,\cdot)}\bar{q}_i={}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i+{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\delta q_i+\Delta t\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i\\ -\gamma q_i(t_1)\Delta t_1\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))}+(1-\gamma)q_i(t_2)\Delta t_2\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))}\end{gathered}\tag{60}$$

Let $\Delta S_{HRL} = -\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta G_{RL}^0\right)\mathrm{d}t$, and we obtain

$$\begin{gathered} p_{RLi}\cdot{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{RLi}^0 - \dot{q}_i\xi_{RL0}^0\right) + \left(p_{RLi}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - \frac{\partial H_{RL}}{\partial t}\right)\xi_{RL0}^0 \\ +\left(p_{RLi}\cdot{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - H_{RL}\right)\cdot\dot{\xi}_{RL0}^0 - \frac{\partial H_{RL}}{\partial q_i}\xi_{RLi}^0 + \dot{G}_{RL}^0\left(t, q_j, p_{RLj}\right) \\ -\gamma p_{RLi}q_i(t_1)\xi_{RL0}^0\left(t_1, q_j(t_1), p_{RLj}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} \\ +(1-\gamma)p_{RLi}q_i(t_2)\xi_{RL0}^0\left(t_2, q_j(t_2), p_{RLj}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} = 0 \end{gathered} \tag{61}$$

Equation (61) is called Noether identity with the CRLVO.

Theorem 1. *For the Hamiltonian system with the CRLVO (Equation (36)), if there exists a gauge function G_{RL}^0 such that the infinitesimal generators ξ_{RL0}^0 and ξ_{RLi}^0 satisfy Equation (61), then there exists a conserved quantity.*

$$\begin{gathered} I_{HRL0} = \int_{t_1}^{t}\left[p_{RLi}\cdot{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{RLi}^0 - \dot{q}_i\xi_{RL0}^0\right) + \left(\xi_{RLi}^0 - \dot{q}_i\xi_{RL0}^0\right)\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RLi}\right]\mathrm{d}\tau + (p_{RLi}\cdot \\ {}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - H_{RL}\Big)\cdot\xi_{RL0}^0 - \int_{t_1}^{t} p_{RLi}\left[\gamma q_i(t_1)\xi_{RL0}^0\left(t_1, q_j(t_1), p_{RLj}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{(\tau-t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))}\right. \\ \left.-(1-\gamma)q_i(t_2)\xi_{RL0}^0\left(t_2, q_j(t_2), p_{RLj}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))}\right]\mathrm{d}\tau + G_{RL}^0 \end{gathered} \tag{62}$$

Proof. Using Equations (36) and (61), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}I_{HRL0} &= \left(\dot{p}_{RLi}\cdot{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i + p_{RLi}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - \frac{\partial H_{RL}}{\partial t} - \frac{\partial H_{RL}}{\partial q_i}\dot{q}_i - \frac{\partial H_{RL}}{\partial p_{RLi}}\dot{p}_{RLi}\right) \\ &\cdot\xi_{RL0}^0 + \left(p_{RLi}\cdot{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - H_{RL}\right)\cdot\dot{\xi}_{RL0}^0 + p_{RLi}\cdot{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{RLi}^0 - \dot{q}_i\xi_{RL0}^0\right) + \Big(\xi_{RLi}^0 \\ &-\dot{q}_i\xi_{RL0}^0\Big)\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RLi} - \gamma p_{RLi}q_i(t_1)\xi_{RL0}^0\left(t_1, q_j(t_1), p_{RLj}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} \\ &+ (1-\gamma)p_{RLi}q_i(t_2)\xi_{RL0}^0\left(t_2, q_j(t_2), p_{RLj}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} + \dot{G}_{RL}^0 \\ &= \left(\dot{p}_{RLi}\cdot{}^{RL}D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - \frac{\partial H_{RL}}{\partial q_i}\dot{q}_i - \frac{\partial H_{RL}}{\partial p_{RLi}}\dot{p}_{RLi}\right)\cdot\xi_{RL0}^0 + \left(\xi_{RLi}^0 - \dot{q}_i\xi_{RL0}^0\right)\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RLi} + \frac{\partial H_{RL}}{\partial q_i}\xi_{RLi}^0 \\ &= \left(\xi_{RLi}^0 - \dot{q}_i\xi_{RL0}^0\right)\cdot\left({}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RLi} + \frac{\partial H_{RL}}{\partial q_i}\right) = 0 \end{aligned}$$

□

4.2. *Noether Symmetry and Conserved Quantity with the CCVO*

When we consider the CCVO, the infinitesimal transformations can be expressed as

$$\begin{aligned} &\bar{t} = t + \Delta t,\ \bar{q}_i\left(\bar{t}\right) = q_i(t) + \Delta q_i,\ \bar{p}_{Ci}\left(\bar{t}\right) = p_{Ci}(t) + \Delta p_{Ci}, \\ &\left(\bar{t} = t + \theta_C\xi_{C0}^0\left(t, q_j, p_{Cj}\right),\ \bar{q}_i\left(\bar{t}\right) = q_i(t) + \theta_C\xi_{Ci}^0\left(t, q_j, p_{Cj}\right),\right. \\ &\left.\bar{p}_{Ci}\left(\bar{t}\right) = p_{Ci}(t) + \theta_C\eta_{Ci}^0\left(t, q_j, p_{Cj}\right).\right. \end{aligned} \tag{63}$$

Under the action of Equation (63), the Hamilton action of variable order, i.e., Equation (38) becomes

$$\bar{S}_{HC} = \int_{\bar{t}_1}^{\bar{t}_2}\left[\bar{p}_{Ci}\cdot{}^{C}D_\gamma^{\bar{\alpha}(\cdot,\cdot),\bar{\beta}(\cdot,\cdot)}\bar{q}_i - H_C\left(\bar{t}, \bar{q}_j, \bar{p}_{Cj}\right)\right]\mathrm{d}\bar{t} \tag{64}$$

Then let $\Delta S_{HC} = \overline{S}_{HC} - S_{HC} = -\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta G_C^0\right)\mathrm{d}t$, and we obtain

$$\begin{aligned}
&p_{Ci}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{Ci}^{0} - \dot{q}_i\xi_{C0}^{0}\right) + \left(p_{Ci}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - \frac{\partial H_C}{\partial t}\right)\xi_{C0}^{0} \\
&-\gamma p_{Ci}\xi_{C0}^{0}\left(t_1, q_j(t_1), p_{Cj}(t_1)\right)\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))}\dot{q}_i(t_1) + \left(p_{Ci}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - H_C\right)\cdot\dot{\xi}_{C0}^{0} \\
&+(1-\gamma)p_{Ci}\xi_{C0}^{0}\left(t_2, q_j(t_2), p_{Cj}(t_2)\right)\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))}\dot{q}_i(t_2) - \frac{\partial H_C}{\partial q_i}\xi_{Ci}^{0} + \dot{G}_C^{0}\left(t, q_j, p_{Cj}\right) = 0
\end{aligned} \tag{65}$$

Equation (65) is called Noether identity with the CCVO.

Theorem 2. *For the Hamiltonian system with the CCVO (Equation (41)), if there exists a gauge function* G_C^0 *such that the infinitesimal generators* ξ_{C0}^0 *and* ξ_{Ci}^0 *satisfy Equation (65), then there exists a conserved quantity.*

$$\begin{aligned}
I_{HC0} = &\int_{t_1}^{t}\left[p_{Ci}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{Ci}^{0} - \dot{q}_i\xi_{C0}^{0}\right) + \left(\xi_{Ci}^{0} - \dot{q}_i\xi_{C0}^{0}\right)\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{Ci}\right]\mathrm{d}\tau + (p_{Ci} \\
&\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - H_C\Big)\cdot\xi_{C0}^{0} - \int_{t_1}^{t} p_{Ci}\left[\gamma\xi_{C0}^{0}\left(t_1, q_j(t_1), p_{Cj}(t_1)\right)\dot{q}_i(t_1)\frac{(\tau-t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))}\right. \\
&\left.-(1-\gamma)\dot{q}_i(t_2)\xi_{C0}^{0}\left(t_2, q_j(t_2), p_{Cj}(t_2)\right)\frac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))}\right]\mathrm{d}\tau + G_C^0
\end{aligned} \tag{66}$$

Proof. The proof is similar to that in Theorem 1 and thus is omitted. □

4.3. Some Explanations

We will present some special cases in this section on the basis of Theorems 1 and 2, which are the main results of fractional Noether symmetry and fractional conserved quantity with variable order in this paper.

Remark 5. *Let* $\beta(\cdot,\cdot) = \alpha(\cdot,\cdot)$, $\gamma = \frac{1}{2}$, *and conserved quantities for the Hamiltonian systems with variable order (Equations (42) and (43)) can be achieved as follows:*

Theorem 3. *For the Hamiltonian system with the RVO (Equation (42)), if there exists a gauge function* $G_R^0\left(t, q_j, p_{Rj}\right)$ *such that the infinitesimal generators* $\xi_{R0}^0\left(t, q_j, p_{Rj}\right)$ *and* $\xi_{Ri}^0\left(t, q_j, p_{Rj}\right)$ *satisfy*

$$\begin{aligned}
&p_{Ri}\cdot{}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)}\left(\xi_{Ri}^{0} - \dot{q}_i\xi_{R0}^{0}\right) + \left(p_{Ri}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)}q_i - \frac{\partial H_R}{\partial t}\right)\xi_{R0}^{0} + \left(p_{Ri}\cdot{}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)}q_i - H_R\right)\cdot\dot{\xi}_{R0}^{0} \\
&-\gamma p_{Ri}q_i(t_1)\xi_{R0}^{0}\left(t_1, q_j(t_1), p_{Rj}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} - \frac{\partial H_R}{\partial q_i}\xi_{Ri}^{0} + \dot{G}_R^{0} \\
&+(1-\gamma)p_{Ri}q_i(t_2)\xi_{R0}^{0}\left(t_2, q_j(t_2), p_{Rj}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} = 0
\end{aligned} \tag{67}$$

then there exists a conserved quantity.

$$\begin{aligned}
I_{HR0} = &\int_{t_1}^{t}\left[p_{Ri}\cdot{}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)}\left(\xi_{Ri}^{0} - \dot{q}_i\xi_{R0}^{0}\right) + \left(\xi_{Ri}^{0} - \dot{q}_i\xi_{R0}^{0}\right)\cdot{}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)}p_{Ri}\right]\mathrm{d}\tau + \Big(p_{Ri}\cdot{}_{t_1}^{R}D_{t_2}^{\alpha(\cdot,\cdot)}q_i \\
&-H_R\Big)\cdot\xi_{R0}^{0} - \int_{t_1}^{t} p_{Ri}\left[\gamma q_i(t_1)\xi_{R0}^{0}\left(t_1, q_j(t_1), p_{Rj}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{(\tau-t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))}\right. \\
&\left.-(1-\gamma)q_i(t_2)\xi_{R0}^{0}\left(t_2, q_j(t_2), p_{Rj}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))}\right]\mathrm{d}\tau + G_R^0
\end{aligned} \tag{68}$$

Theorem 4. *For the Hamiltonian system with the RCVO (Equation (43)), if there exists a gauge function* $G^0_{RC}\left(t, q_j, p_{RCj}\right)$ *such that the infinitesimal generators* $\xi^0_{RC0}\left(t, q_j, p_{RCj}\right)$ *and* $\xi^0_{RCi}\left(t, q_j, p_{RCj}\right)$ *satisfy*

$$\begin{aligned} &p_{RCi} \cdot {}^{RC}_{t_1}D^{\alpha(\cdot,\cdot)}_{t_2}\left(\xi^0_{RCi} - \dot{q}_i\xi^0_{RC0}\right) + \left(p_{RCi} \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}{}^{RC}_{t_1}D^{\alpha(\cdot,\cdot)}_{t_2} q_i - \tfrac{\partial H_{RC}}{\partial t}\right)\xi^0_{RC0} \\ &-\gamma p_{RCi}\xi^0_{RC0}\left(t_1, q_j(t_1), p_{RCj}(t_1)\right)\tfrac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))}\dot{q}_i(t_1) + \left(p_{RCi} \cdot {}^{RC}_{t_1}D^{\alpha(\cdot,\cdot)}_{t_2} q_i - H_{RC}\right) \cdot \dot{\xi}^0_{RC0} \\ &+(1-\gamma)p_{RCi}\xi^0_{RC0}\left(t_2, q_j(t_2), p_{RCj}(t_2)\right)\tfrac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))}\dot{q}_i(t_2) - \tfrac{\partial H_{RC}}{\partial q_i}\xi^0_{RCi} + \dot{G}^0_{RC} = 0 \end{aligned} \tag{69}$$

then there exists a conserved quantity.

$$\begin{aligned} I_{HRC0} = &\int_{t_1}^{t}\left[p_{RCi} \cdot {}^{RC}_{t_1}D^{\alpha(\cdot,\cdot)}_{t_2}\left(\xi^0_{RCi} - \dot{q}_i\xi^0_{RC0}\right) + \left(\xi^0_{RCi} - \dot{q}_i\xi^0_{RC0}\right) \cdot {}^{R}_{t_1}D^{\alpha(\cdot,\cdot)}_{t_2} p_{RCi}\right]\mathrm{d}\tau + (p_{RCi} \\ &\cdot {}^{RC}_{t_1}D^{\alpha(\cdot,\cdot)}_{t_2} q_i - H_{RC}\Big) \cdot \xi^0_{RC0} - \int_{t_1}^{t} p_{RCi}\Big[\gamma\xi^0_{RC0}\left(t_1, q_j(t_1), p_{RCj}(t_1)\right)\dot{q}_i(t_1)\tfrac{(\tau-t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))} \\ &-(1-\gamma)\dot{q}_i(t_2)\xi^0_{RC0}\left(t_2, q_j(t_2), p_{RCj}(t_2)\right)\tfrac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))}\Big]\mathrm{d}\tau + G^0_{RC} \end{aligned} \tag{70}$$

Remark 6. *Conserved quantities for the Lagrangian systems with variable order (i.e. Equations (44) and (46)) can be achieved as follows:*

Theorem 5. *For the Lagrangian system with the CRLVO (Equation (44)), if there exists a gauge function* $G^0_{LRL}\left(t, q_j, {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_j\right)$ *such that the infinitesimal generators* $\xi^0_{LRL0}\left(t, q_j\right)$ *and* $\xi^0_{LRLi}\left(t, q_j\right)$ *satisfy*

$$\begin{aligned} &\tfrac{\partial L_{RL}\left(t, q_j, {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_j\right)}{\partial {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i} \cdot {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma}\left(\xi^0_{LRLi} - \dot{q}_i\xi^0_{LRL0}\right) + L_{RL} \cdot \dot{\xi}^0_{RL0} + \dot{G}^0_{LRL} + \tfrac{\partial L_{RL}}{\partial q_i}\xi^0_{LRLi} + \Big(\tfrac{\partial L_{RL}}{\partial t} \\ &+ \tfrac{\partial L_{RL}}{\partial {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i} \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i\Big)\xi^0_{LRL0} - \tfrac{\gamma q_i(t_1)\partial L_{RL}}{\partial {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i}\xi^0_{LRL0}\left(t_1, q_j(t_1)\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} \\ &+ \tfrac{(1-\gamma)q_i(t_2)\partial L_{RL}}{\partial {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i}\xi^0_{LRL0}\left(t_2, q_j(t_2)\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} = 0 \end{aligned} \tag{71}$$

then there exists a conserved quantity.

$$\begin{aligned} I_{LRL0} = &\int_{t_1}^{t}\Big[\left(\xi^0_{LRLi} - \dot{q}_i\xi^0_{LRL0}\right){}^{RL}D^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}_{1-\gamma}\tfrac{\partial L_{RL}}{\partial {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i} + \tfrac{\partial L_{RL}}{\partial {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i} \\ &\cdot {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma}\left(\xi^0_{LRLi} - \dot{q}_i\xi^0_{LRL0}\right)\Big]\mathrm{d}\tau + L_{RL} \cdot \xi^0_{LRL0} - \int_{t_1}^{t}\tfrac{\partial L_{RL}}{\partial {}^{RL}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i}\Big[\gamma q_i(t_1)\xi^0_{LRL0}\left(t_1, q_j(t_1)\right) \\ &\cdot \tfrac{\mathrm{d}}{\mathrm{d}\tau}\tfrac{(\tau-t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))} - (1-\gamma)q_i(t_2)\xi^0_{LRL0}\left(t_2, q_j(t_2)\right)\tfrac{\mathrm{d}}{\mathrm{d}\tau}\tfrac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))}\Big]\mathrm{d}\tau + G^0_{LRL} \end{aligned} \tag{72}$$

Theorem 6. *For the Lagrangian system with the CCVO (Equation (46)), if there exists a gauge function* $G^0_{LC}\left(t, q_j, {}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_j\right)$ *such that the infinitesimal generators* $\xi^0_{LC0}\left(t, q_j\right)$ *and* $\xi^0_{LCi}\left(t, q_j\right)$ *satisfy*

$$\begin{aligned} &\tfrac{\partial L_C\left(t, q_j, {}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_j\right)}{\partial {}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i} \cdot {}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma}\left(\xi^0_{LCi} - \dot{q}_i\xi^0_{LC0}\right) + \Big(\tfrac{\partial L_C}{\partial {}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i} \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}{}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i \\ &+ \tfrac{\partial L_C}{\partial t}\Big)\xi^0_{LC0} + (1-\gamma)\tfrac{\partial L_C}{\partial {}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i}\xi^0_{LC0}\left(t_2, q_j(t_2)\right)\tfrac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))}\dot{q}_i(t_2) + \tfrac{\partial L_C}{\partial q_i}\xi^0_{LCi} \\ &-\gamma\tfrac{\partial L_C}{\partial {}^{C}D^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}_{\gamma} q_i}\xi^0_{LC0}\left(t_1, q_j(t_1)\right)\tfrac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))}\dot{q}_i(t_1) + L_C \cdot \dot{\xi}^0_{LC0} + \dot{G}^0_{LC} = 0 \end{aligned} \tag{73}$$

then there exists a conserved quantity.

$$\begin{gathered}
I_{LC0} = \int_{t_1}^{t}\left[\frac{\partial L_C}{\partial^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i}\cdot {}^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi^0_{LCi} - \dot{q}_i \xi^0_{LC0}\right) + \left(\xi^0_{LCi} - \dot{q}_i \xi^0_{LC0}\right)\right. \\
\left.\cdot {}^{RL} D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \frac{\partial L_C}{\partial^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i}\right]\mathrm{d}\tau - \int_{t_1}^{t} \frac{\partial L_C}{\partial^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i}\left[\gamma \xi^0_{LC0}\left(t_1, q_j(t_1)\right)\dot{q}_i(t_1)\frac{(\tau - t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))}\right. \\
\left. -(1-\gamma)\dot{q}_i(t_2)\xi^0_{LC0}\left(t_2, q_j(t_2)\right)\frac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))}\right]\mathrm{d}\tau + L_C\cdot \xi^0_{LC0} + G^0_{LC}
\end{gathered} \tag{74}$$

Remark 7. *When $\alpha(\cdot,\cdot) \to \alpha$, $\beta(\cdot,\cdot) \to \beta$, conserved quantities for the Hamiltonian systems with the CRL and the CC (Equations (48) and (49)) and the Lagrangian systems with the CRL and the CC (Equations (50) and (51)) can be deduced as follows:*

Theorem 7. *For the Hamiltonian system with the CRL (Equation (48)), if there exists a gauge function $G^0_{RLF}\left(t, q_j, p_{RLjF}\right)$ such that the infinitesimal generators $\xi^0_{RL0F}\left(t, q_j, p_{RLjF}\right)$ and $\xi^0_{RLiF}\left(t, q_j, p_{RLjF}\right)$ satisfy*

$$\begin{gathered}
p_{RLiF}\cdot {}^{RL}D_\gamma^{\alpha,\beta}\left(\xi^0_{RLiF} - \dot{q}_i\xi^0_{RL0F}\right) + \left(p_{RLiF}\cdot \frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_\gamma^{\alpha,\beta} q_i - \frac{\partial H_{RLF}}{\partial t}\right)\xi^0_{RL0F} + \left(p_{RLiF}\cdot {}^{RL}D_\gamma^{\alpha,\beta} q_i\right. \\
\left. -H_{RLF}\right)\cdot \dot{\xi}^0_{RL0F} - \frac{\partial H_{RLF}}{\partial q_i}\xi^0_{RLiF} + \dot{G}^0_{RLF} - \frac{\gamma p_{RLiF}}{\Gamma(1-\alpha)} q_i(t_1)\xi^0_{RL0F}\left(t_1, q_j(t_1), p_{RLjF}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha} \\
+\frac{(1-\gamma)p_{RLiF}}{\Gamma(1-\beta)} q_i(t_2)\xi^0_{RL0F}\left(t_2, q_j(t_2), p_{RLjF}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}t}(t_2 - t)^{-\beta} = 0
\end{gathered} \tag{75}$$

then there exists a conserved quantity.

$$\begin{gathered}
I_{HRL0F} = \int_{t_1}^{t}\left[p_{RLiF}\cdot {}^{RL}D_\gamma^{\alpha,\beta}\left(\xi^0_{RLiF} - \dot{q}_i\xi^0_{RL0F}\right) + \left(\xi^0_{RLiF} - \dot{q}_i\xi^0_{RL0F}\right)\cdot {}^{RL}D_{1-\gamma}^{\beta,\alpha} p_{RLiF}\right]\mathrm{d}\tau + (p_{RLiF} \\
\cdot {}^{RL}D_\gamma^{\alpha,\beta} q_i - H_{RLF})\cdot \xi^0_{RL0F} - \int_{t_1}^{t} p_{RLiF}\left[\frac{\gamma q_i(t_1)}{\Gamma(1-\alpha)}\xi^0_{RL0F}\left(t_1, q_j(t_1), p_{RLjF}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}(\tau - t_1)^{-\alpha}\right. \\
\left. -\frac{(1-\gamma)q_i(t_2)}{\Gamma(1-\beta)}\xi^0_{RL0F}\left(t_2, q_j(t_2), p_{RLjF}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}\tau}(t_2-\tau)^{-\beta}\right]\mathrm{d}\tau + G^0_{RLF}
\end{gathered} \tag{76}$$

Theorem 8. *For the Hamiltonian system with the CC (Equation (49)), if there exists a gauge function $G^0_{CF}\left(t, q_j, p_{CjF}\right)$ such that the infinitesimal generators $\xi^0_{C0F}\left(t, q_j, p_{CjF}\right)$ and $\xi^0_{CiF}\left(t, q_j, p_{CjF}\right)$ satisfy*

$$\begin{gathered}
p_{CiF}\cdot {}^{C}D_\gamma^{\alpha,\beta}\left(\xi^0_{CiF} - \dot{q}_i\xi^0_{C0F}\right) + \left(p_{CiF}\cdot \frac{\mathrm{d}}{\mathrm{d}t}{}^{C}D_\gamma^{\alpha,\beta} q_i - \frac{\partial H_{CF}}{\partial t}\right)\xi^0_{C0F} \\
-\gamma p_{CiF}\xi^0_{C0F}\left(t_1, q_j(t_1), p_{CjF}(t_1)\right)\frac{(t-t_1)^{-\alpha}}{\Gamma(1-\alpha)}\dot{q}_i(t_1) + \left(p_{CiF}\cdot {}^{C}D_\gamma^{\alpha,\beta} q_i - H_{CF}\right)\cdot \dot{\xi}^0_{C0F} \\
+(1-\gamma)p_{CiF}\xi^0_{C0F}\left(t_2, q_j(t_2), p_{CjF}(t_2)\right)\frac{(t_2-t)^{-\beta}}{\Gamma(1-\beta)}\dot{q}_i(t_2) - \frac{\partial H_{CF}}{\partial q_i}\xi^0_{CiF} + \dot{G}^0_{CF} = 0
\end{gathered} \tag{77}$$

then there exists a conserved quantity.

$$\begin{gathered}
I_{HC0F} = \int_{t_1}^{t}\left[p_{CiF}\cdot {}^{C}D_\gamma^{\alpha,\beta}\left(\xi^0_{CiF} - \dot{q}_i\xi^0_{C0F}\right) + \left(\xi^0_{CiF} - \dot{q}_i\xi^0_{C0F}\right)\cdot {}^{RL}D_{1-\gamma}^{\beta,\alpha} p_{CiF}\right]\mathrm{d}\tau + (p_{CiF} \\
\cdot {}^{C}D_\gamma^{\alpha,\beta} q_i - H_{CF})\cdot \xi^0_{C0F} - \int_{t_1}^{t} p_{CiF}\left[\gamma \xi^0_{C0F}\left(t_1, q_j(t_1), p_{CjF}(t_1)\right)\dot{q}_i(t_1)\frac{(\tau-t_1)^{-\alpha}}{\Gamma(1-\alpha)}\right. \\
\left. -(1-\gamma)\dot{q}_i(t_2)\xi^0_{C0F}\left(t_2, q_j(t_2), p_{CjF}(t_2)\right)\frac{(t_2-\tau)^{-\beta}}{\Gamma(1-\beta)}\right]\mathrm{d}\tau + G^0_{CF}
\end{gathered} \tag{78}$$

Theorem 9. *For the Lagrangian system with the CRL (Equation (50)), if there exists a gauge function* $G^0_{LRLF}\left(t,q_j,{}^{RL}D^{\alpha,\beta}_{\gamma}q_j\right)$ *such that the infinitesimal generators* $\xi^0_{LRL0F}\left(t,q_j\right)$ *and* $\xi^0_{LRLiF}\left(t,q_j\right)$ *satisfy*

$$\begin{aligned}&\tfrac{\partial L_{RLF}\left(t,q_j,{}^{RL}D^{\alpha,\beta}_{\gamma}q_j\right)}{\partial {}^{RL}D^{\alpha,\beta}_{\gamma}q_i}\cdot {}^{RL}D^{\alpha,\beta}_{\gamma}\left(\xi^0_{LRLiF}-\dot q_i\xi^0_{LRL0F}\right)+L_{RLF}\cdot\dot\xi^0_{RL0F}+\tfrac{\partial L_{RLF}}{\partial q_i}\xi^0_{LRLiF}+\Big(\tfrac{\partial L_{RLF}}{\partial t}\\&+\tfrac{\partial L_{RLF}}{\partial {}^{RL}D^{\alpha,\beta}_{\gamma}q_i}\cdot\tfrac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D^{\alpha,\beta}_{\gamma}q_i\Big)\xi^0_{LRL0F}-\tfrac{\gamma q_i(t_1)}{\Gamma(1-\alpha)}\tfrac{\partial L_{RLF}}{\partial {}^{RL}D^{\alpha,\beta}_{\gamma}q_i}\xi^0_{LRL0F}\left(t_1,q_j(t_1)\right)\tfrac{\mathrm{d}}{\mathrm{d}t}(t-t_1)^{-\alpha}\\&+\tfrac{(1-\gamma)q_i(t_2)}{\Gamma(1-\beta)}\tfrac{\partial L_{RLF}}{\partial {}^{RL}D^{\alpha,\beta}_{\gamma}q_i}\xi^0_{LRL0F}\left(t_2,q_j(t_2)\right)\tfrac{\mathrm{d}}{\mathrm{d}t}(t_2-t)^{-\beta}+\dot G^0_{LRLF}=0\end{aligned}\tag{79}$$

then there exists a conserved quantity.

$$\begin{aligned}I_{LRL0F}=&\int_{t_1}^{t}\Big[\left(\xi^0_{LRLiF}-\dot q_i\xi^0_{LRL0F}\right){}^{RL}D^{\beta,\alpha}_{1-\gamma}\tfrac{\partial L_{RLF}}{\partial {}^{RL}D^{\alpha,\beta}_{\gamma}q_i}+\tfrac{\partial L_{RLF}}{\partial {}^{RL}D^{\alpha,\beta}_{\gamma}q_i}\\&\cdot{}^{RL}D^{\alpha,\beta}_{\gamma}\left(\xi^0_{LRLiF}-\dot q_i\xi^0_{LRL0F}\right)\Big]\mathrm{d}\tau+L_{RLF}\cdot\xi^0_{LRL0F}-\int_{t_1}^{t}\tfrac{\partial L_{RLF}}{\partial {}^{RL}D^{\alpha,\beta}_{\gamma}q_i}\Big[\gamma q_i(t_1)\xi^0_{LRL0F}\left(t_1,q_j(t_1)\right)\\&\cdot\tfrac{\mathrm{d}}{\mathrm{d}\tau}\tfrac{(\tau-t_1)^{-\alpha}}{\Gamma(1-\alpha)}-(1-\gamma)q_i(t_2)\xi^0_{LRL0F}\left(t_2,q_j(t_2)\right)\tfrac{\mathrm{d}}{\mathrm{d}\tau}\tfrac{(t_2-\tau)^{-\beta}}{\Gamma(1-\beta)}\Big]\mathrm{d}\tau+G^0_{LRLF}\end{aligned}\tag{80}$$

Theorem 10. *For the Lagrangian system with the CC (Equation (51)), if there exists a gauge function* $G^0_{LCF}\left(t,q_j,{}^{C}D^{\alpha,\beta}_{\gamma}q_j\right)$ *such that the infinitesimal generators* $\xi^0_{LC0F}\left(t,q_j\right)$ *and* $\xi^0_{LCiF}\left(t,q_j\right)$ *satisfy*

$$\begin{aligned}&\tfrac{\partial L_{CF}\left(t,q_j,{}^{C}D^{\alpha,\beta}_{\gamma}q_j\right)}{\partial {}^{C}D^{\alpha,\beta}_{\gamma}q_i}\cdot{}^{C}D^{\alpha,\beta}_{\gamma}\left(\xi^0_{LCiF}-\dot q_i\xi^0_{LC0F}\right)+\Big(\tfrac{\partial L_{CF}}{\partial {}^{C}D^{\alpha,\beta}_{\gamma}q_i}\cdot\tfrac{\mathrm{d}}{\mathrm{d}t}{}^{C}D^{\alpha,\beta}_{\gamma}q_i\\&+\tfrac{\partial L_{CF}}{\partial t}\Big)\xi^0_{LC0F}+(1-\gamma)\tfrac{\partial L_{CF}}{\partial {}^{C}D^{\alpha,\beta}_{\gamma}q_i}\xi^0_{LC0F}\left(t_2,q_j(t_2)\right)\tfrac{(t_2-t)^{-\beta}}{\Gamma(1-\beta)}\dot q_i(t_2)+\tfrac{\partial L_{CF}}{\partial q_i}\xi^0_{LCiF}\\&-\gamma\tfrac{\partial L_{CF}}{\partial {}^{C}D^{\alpha,\beta}_{\gamma}q_i}\xi^0_{LC0F}\left(t_1,q_j(t_1)\right)\tfrac{(t-t_1)^{-\alpha}}{\Gamma(1-\alpha)}\dot q_i(t_1)+L_{CF}\cdot\dot\xi^0_{LC0F}+\dot G^0_{LCF}=0\end{aligned}\tag{81}$$

then there exists a conserved quantity.

$$\begin{aligned}I_{LC0F}=&\int_{t_1}^{t}\Big[\tfrac{\partial L_{CF}}{\partial {}^{C}D^{\alpha,\beta}_{\gamma}q_i}\cdot{}^{C}D^{\alpha,\beta}_{\gamma}\left(\xi^0_{LCiF}-\dot q_i\xi^0_{LC0F}\right)+\left(\xi^0_{LCiF}-\dot q_i\xi^0_{LC0F}\right)\\&\cdot{}^{RL}D^{\beta,\alpha}_{1-\gamma}\tfrac{\partial L_{CF}}{\partial {}^{C}D^{\alpha,\beta}_{\gamma}q_i}\Big]\mathrm{d}\tau-\int_{t_1}^{t}\tfrac{\partial L_{CF}}{\partial {}^{C}D^{\alpha,\beta}_{\gamma}q_i}\Big[\gamma\xi^0_{LC0F}\left(t_1,q_j(t_1)\right)\dot q_i(t_1)\tfrac{(\tau-t_1)^{-\alpha}}{\Gamma(1-\alpha)}\\&-(1-\gamma)\dot q_i(t_2)\xi^0_{LC0F}\left(t_2,q_j(t_2)\right)\tfrac{(t_2-\tau)^{-\beta}}{\Gamma(1-\beta)}\Big]\mathrm{d}\tau+L_{CF}\cdot\xi^0_{LC0F}+G^0_{LCF}\end{aligned}\tag{82}$$

Remark 8. *The results of Theorems 7–10 are consistent with those in [53].*

Remark 9. *When* $\alpha(\cdot,\cdot),\beta(\cdot,\cdot)\to 1$, *conserved quantities for the Hamiltonian system (52) and the Lagrangian system (53) can be obtained as follows:*

Theorem 11. *For the classical Hamiltonian system (Equation (52)), if there exists a gauge function* $G^0\left(t,q_j,p_j\right)$ *such that the infinitesimal generators* $\xi^0_0\left(t,q_j,p_j\right)$ *and* $\xi^0_i\left(t,q_j,p_j\right)$ *satisfy*

$$p_i\dot\xi^0_i-\frac{\partial H}{\partial t}\xi^0_0-H\dot\xi^0_0-\frac{\partial H}{\partial q_i}\xi^0_i+\dot G^0=0\tag{83}$$

then there exists a conserved quantity.

$$I_0=p_i\xi^0_i-H\xi^0_0+G^0\tag{84}$$

Theorem 12. *For the classical Lagrangian system (Equation (53)), if there exists a gauge function $G_L^0\left(t, q_j, \dot{q}_j\right)$ such that the infinitesimal generators $\xi_{L0}^0\left(t, q_j\right)$ and $\xi_{Li}^0\left(t, q_j\right)$ satisfy*

$$\frac{\partial L\left(t, q_j, \dot{q}_j\right)}{\partial \dot{q}_i}\cdot\left(\dot{\xi}_{Li}^0 - \dot{q}_i\dot{\xi}_{L0}^0\right) + L\cdot\dot{\xi}_{L0}^0 + \dot{G}_L^0 + \frac{\partial L}{\partial q_i}\xi_{Li}^0 + \frac{\partial L}{\partial t}\xi_{L0}^0 = 0 \tag{85}$$

then there exists a conserved quantity.

$$I_{L0} = \frac{\partial L}{\partial \dot{q}_i}\left(\xi_{Li}^0 - \dot{q}_i\xi_{L0}^0\right) + L\cdot\xi_{L0}^0 + G_L^0 \tag{86}$$

Remark 10. *The results of Theorems 11 and 12 are consistent with those in [66].*

5. Perturbation to Noether Symmetry and Adiabatic Invariants

Noether symmetry is one of the methods of finding the solutions to the differential equations of motion. However, if the mechanical system is disturbed by small forces, the original conserved quantity may also change. Perturbation to Noether symmetry and adiabatic invariants will be studied in this section because they also have a close relationship with the integration of the mechanical system.

Supposing the mechanical system is disturbed by εW_{Ui}, the way of disturbing the infinitesimal generators and the gauge function is set as

$$\begin{array}{c} G_U = G_U^0 + \varepsilon G_U^1 + \varepsilon^2 G_U^2 + \cdots,\ \xi_{U0} = \xi_{U0}^0 + \varepsilon\xi_{U0}^1 + \varepsilon^2\xi_{U0}^2 + \cdots \\ \xi_{Ui} = \xi_{Ui}^0 + \varepsilon\xi_{Ui}^1 + \varepsilon^2\xi_{Ui}^2 + \cdots,\ \eta_{Ui} = \eta_{Ui}^0 + \varepsilon\eta_{Ui}^1 + \varepsilon^2\eta_{Ui}^2 + \cdots \end{array} \tag{87}$$

where G_U is the gauge function of the disturbed system, and ξ_{U0}, ξ_{Ui}, and η_{Ui} are the infinitesimal generators of the disturbed system.

If a quantity I_z, which has a parameter ε and the highest power of ε as z, satisfies the rule that $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to ε^{z+1}, then the quantity I_z is called an adiabatic invariant.

Then we have:

Theorem 13. *For the disturbed Hamiltonian system in terms of the CRLVO*

$${}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i = \frac{\partial H_{RL}}{\partial p_{RLi}},\ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RLi} = -\frac{\partial H_{RL}}{\partial q_i} - \varepsilon W_{RLi}\left(t, q_j, p_{RLj}\right) \tag{88}$$

If there exists a gauge function $G_{RL}^m\left(t, q_j, p_{RLj}\right)$ such that the infinitesimal generators $\xi_{RL0}^m\left(t, q_j, p_{RLj}\right)$ and $\xi_{RLi}^m\left(t, q_j, p_{RLj}\right)$ satisfy

$$\begin{array}{c} p_{RLi}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{RLi}^m - \dot{q}_i\xi_{RL0}^m\right) + \left(p_{RLi}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i - \frac{\partial H_{RL}}{\partial t}\right)\xi_{RL0}^m + \left(p_{RLi}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_i\right. \\ \left.-H_{RL}\right)\cdot\dot{\xi}_{RL0}^m - \frac{\partial H_{RL}}{\partial q_i}\xi_{RLi}^m + \dot{G}_{RL}^m - \gamma p_{RLi}q_i(t_1)\xi_{RL0}^m\left(t_1, q_j(t_1), p_{RLj}(t_1)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} \\ +(1-\gamma)p_{RLi}q_i(t_2)\xi_{RL0}^m\left(t_2, q_j(t_2), p_{RLj}(t_2)\right)\frac{\mathrm{d}}{\mathrm{d}t}\frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} - W_{RLi}\left(\xi_{RLi}^{m-1} - \dot{q}_i\xi_{RL0}^{m-1}\right) = 0 \end{array} \tag{89}$$

then there exists an adiabatic invariant.

$$
\begin{aligned}
I_{HRLz} = \sum_{m=0}^{z} \varepsilon^m \Big\{ \int_{t_1}^{t} \Big[p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} \big(\xi_{RLi}^m - \dot{q}_i \xi_{RL0}^m \big) + \big(\xi_{RLi}^m - \dot{q}_i \xi_{RL0}^m \big) \cdot {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{RLi} \Big] \mathrm{d}\tau + \big(p_{RLi} \\
\cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_{RL} \big) \cdot \xi_{RL0}^m - \int_{t_1}^{t} p_{RLi} \Big[\gamma q_i(t_1) \xi_{RL0}^m \big(t_1, q_j(t_1), p_{RLj}(t_1) \big) \tfrac{\mathrm{d}}{\mathrm{d}\tau} \tfrac{(\tau - t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))} \\
-(1-\gamma) q_i(t_2) \xi_{RL0}^m \big(t_2, q_j(t_2), p_{RLj}(t_2) \big) \tfrac{\mathrm{d}}{\mathrm{d}\tau} \tfrac{(t_2 - \tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))} \Big] \mathrm{d}\tau + G_{RL}^m \Big\}
\end{aligned}
\tag{90}
$$

where $\xi_{RLi}^{m-1} = \xi_{RL0}^{m-1} = 0$ *when* $m = 0$.

Proof. Using Equations (88) and (89), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I_{HRLz} = \sum_{m=0}^{z} \varepsilon^m \Bigg[\Bigg(\dot{p}_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i + p_{RLi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - \frac{\partial H_{RL}}{\partial q_i} \dot{q}_i - \frac{\partial H_{RL}}{\partial p_{RLi}} \dot{p}_{RLi} \\
- \frac{\partial H_{RL}}{\partial t} \Bigg) \cdot \xi_{RL0}^m + \Big(p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_{RL} \Big) \cdot \dot{\xi}_{RL0}^m + p_{RLi} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} \big(\xi_{RLi}^m - \dot{q}_i \xi_{RL0}^m \big) \\
+ \big(\xi_{RLi}^m - \dot{q}_i \xi_{RL0}^m \big) \cdot {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{RLi} - \gamma p_{RLi} q_i(t_1) \xi_{RL0}^m \big(t_1, q_j(t_1), p_{RLj}(t_1) \big) \frac{\mathrm{d}}{\mathrm{d}t} \frac{(t - t_1)^{-\alpha(t,t_1)}}{\Gamma(1 - \alpha(t,t_1))} \\
+ (1-\gamma) p_{RLi} q_i(t_2) \xi_{RL0}^m \big(t_2, q_j(t_2), p_{RLj}(t_2) \big) \frac{\mathrm{d}}{\mathrm{d}t} \frac{(t_2 - t)^{-\beta(t_2,t)}}{\Gamma(1 - \beta(t_2,t))} + \dot{G}_{RL}^m \Bigg] \\
= \sum_{m=0}^{z} \varepsilon^m \Bigg[\big(\xi_{RLi}^m - \dot{q}_i \xi_{RL0}^m \big) \cdot \Bigg({}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{RLi} + \frac{\partial H_{RL}}{\partial q_i} \Bigg) + W_{RLi} \big(\xi_{RLi}^{m-1} - \dot{q}_i \xi_{RL0}^{m-1} \big) \Bigg] \\
= \sum_{m=0}^{z} \varepsilon^m \Big[-\varepsilon W_{RLi} \big(\xi_{RLi}^m - \dot{q}_i \xi_{RL0}^m \big) + W_{RLi} \big(\xi_{RLi}^{m-1} - \dot{q}_i \xi_{RL0}^{m-1} \big) \Big] = -\varepsilon^{z+1} W_{RLi} \big(\xi_{RLi}^z - \dot{q}_i \xi_{RL0}^z \big)
\end{aligned}
$$

□

Theorem 14. *For the disturbed Hamiltonian system in terms of the CCVO*

$$
{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i = \frac{\partial H_C}{\partial p_{Ci}}, \; {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{Ci} = -\frac{\partial H_C}{\partial q_i} - \varepsilon W_{Ci} \big(t, q_j, p_{Cj} \big) \tag{91}
$$

If there exists a gauge function $G_C^m \big(t, q_j, p_{Cj} \big)$ *such that the infinitesimal generators* $\xi_{C0}^m \big(t, q_j, p_{Cj} \big)$ *and* $\xi_{Ci}^m \big(t, q_j, p_{Cj} \big)$ *satisfy*

$$
\begin{aligned}
p_{Ci} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} \big(\xi_{Ci}^m - \dot{q}_i \xi_{C0}^m \big) + \Big(p_{Ci} \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - \tfrac{\partial H_C}{\partial t} \Big) \xi_{C0}^m - W_{Ci} \big(\xi_{Ci}^{m-1} - \dot{q}_i \xi_{C0}^{m-1} \big) \\
-\gamma p_{Ci} \xi_{C0}^m \big(t_1, q_j(t_1), p_{Cj}(t_1) \big) \tfrac{(t - t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} \dot{q}_i(t_1) + \Big(p_{Ci} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_C \Big) \cdot \dot{\xi}_{C0}^m \\
+(1-\gamma) p_{Ci} \xi_{C0}^m \big(t_2, q_j(t_2), p_{Cj}(t_2) \big) \tfrac{(t_2 - t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} \dot{q}_i(t_2) - \tfrac{\partial H_C}{\partial q_i} \xi_{Ci}^m + \dot{G}_C^m = 0
\end{aligned}
\tag{92}
$$

Then there exists an adiabatic invariant.

$$
\begin{aligned}
I_{HCz} = \sum_{m=0}^{z} \varepsilon^m \Big\{ \int_{t_1}^{t} \Big[p_{Ci} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} \big(\xi_{Ci}^m - \dot{q}_i \xi_{C0}^m \big) + \big(\xi_{Ci}^m - \dot{q}_i \xi_{C0}^m \big) \cdot {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} p_{Ci} \Big] \mathrm{d}\tau + \big(p_{Ci} \\
\cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i - H_C \big) \cdot \xi_{C0}^m - \int_{t_1}^{t} p_{Ci} \Big[\gamma \xi_{C0}^m \big(t_1, q_j(t_1), p_{Cj}(t_1) \big) \dot{q}_i(t_1) \tfrac{(\tau - t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))} \\
-(1-\gamma) \dot{q}_i(t_2) \xi_{C0}^m \big(t_2, q_j(t_2), p_{Cj}(t_2) \big) \tfrac{(t_2 - \tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))} \Big] \mathrm{d}\tau + G_C^m \Big\}
\end{aligned}
\tag{93}
$$

where $\xi_{Ci}^{m-1} = \xi_{C0}^{m-1} = 0$ *when* $m = 0$.

Theorems 13 and 14 are the main results of this paper for the disturbed Hamiltonian systems with variable order. Based on the two theorems, adiabatic invariants for the disturbed Lagrangian systems with variable order can also been obtained.

Theorem 15. *For the disturbed Lagrangian system in terms of the CRLVO*

$$\frac{\partial L_{RL}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)}{\partial q_i} - {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \frac{\partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} = \varepsilon W_{LRLi}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right) \tag{94}$$

If there exists a gauge function $G_{LRL}^{m}\left(t, q_j, {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)$ *such that the infinitesimal generators* $\xi_{LRL0}^{m}\left(t, q_j\right)$ *and* $\xi_{LRLi}^{m}\left(t, q_j\right)$ *satisfy*

$$\begin{aligned} &\frac{\partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{LRLi}^{m} - \dot{q}_i \xi_{LRL0}^{m}\right) + L_{RL} \cdot \dot{\xi}_{RL0}^{m} + \dot{G}_{LRL}^{m} + \frac{\partial L_{RL}}{\partial q_i} \xi_{LRLi}^{m} + \left(\frac{\partial L_{RL}}{\partial t}\right. \\ &\left. + \frac{\partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \cdot \frac{\mathrm{d}}{\mathrm{d}t} {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i\right) \xi_{LRL0}^{m} - \frac{\gamma q_i(t_1) \partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \xi_{LRL0}^{m}\left(t_1, q_j(t_1)\right) \frac{\mathrm{d}}{\mathrm{d}t} \frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} \\ &+ \frac{(1-\gamma) q_i(t_2) \partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \xi_{LRL0}^{m}\left(t_2, q_j(t_2)\right) \frac{\mathrm{d}}{\mathrm{d}t} \frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} - W_{LRLi}\left(\xi_{LRLi}^{m-1} - \dot{q}_i \xi_{LRL0}^{m-1}\right) = 0 \end{aligned} \tag{95}$$

Then there exists an adiabatic invariant.

$$\begin{aligned} I_{LRLz} = &\sum_{m=0}^{z} \varepsilon^m \left\{ \int_{t_1}^{t} \left[\left(\xi_{LRLi}^{m} - \dot{q}_i \xi_{LRL0}^{m}\right) {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \frac{\partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} + \frac{\partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \right.\right. \\ &\cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{LRLi}^{m} - \dot{q}_i \xi_{LRL0}^{m}\right) \Big] \mathrm{d}\tau + L_{RL} \cdot \xi_{LRL0}^{m} - \int_{t_1}^{t} \frac{\partial L_{RL}}{\partial^{RL} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \Big[\gamma q_i(t_1) \xi_{LRL0}^{m}\left(t_1, q_j(t_1)\right) \\ &\left.\left. \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{(\tau-t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))} - (1-\gamma) q_i(t_2) \xi_{LRL0}^{m}\left(t_2, q_j(t_2)\right) \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))} \right] \mathrm{d}\tau + G_{LRL}^{m} \right\} \end{aligned} \tag{96}$$

where $\xi_{LRLi}^{m-1} = \xi_{LRL0}^{m-1} = 0$ *when* $m = 0$.

Theorem 16. *For the disturbed Lagrangian system in terms of the CCVO*

$$\frac{\partial L_C}{\partial q_i} - {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \frac{\partial L_C\left(t, q_j, {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)}{\partial^{C} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} = \varepsilon W_{LCi}\left(t, q_j, {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right) \tag{97}$$

If there exists a gauge function $G_{LC}^{m}\left(t, q_j, {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_j\right)$ *such that the infinitesimal generators* $\xi_{LC0}^{m}\left(t, q_j\right)$ *and* $\xi_{LCi}^{m}\left(t, q_j\right)$ *satisfy*

$$\begin{aligned} &\frac{\partial L_C}{\partial^{C} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}\left(\xi_{LCi}^{m} - \dot{q}_i \xi_{LC0}^{m}\right) + \left(\frac{\partial L_C}{\partial^{C} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \cdot \frac{\mathrm{d}}{\mathrm{d}t} {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i + \frac{\partial L_C}{\partial t}\right) \xi_{LC0}^{m} \\ &+ (1-\gamma) \frac{\partial L_C}{\partial^{C} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \xi_{LC0}^{m}\left(t_2, q_j(t_2)\right) \frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} \dot{q}_i(t_2) + \frac{\partial L_C}{\partial q_i} \xi_{LCi}^{m} + L_C \cdot \dot{\xi}_{LC0}^{m} + \dot{G}_{LC}^{m} \\ &- \gamma \frac{\partial L_C}{\partial^{C} D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \xi_{LC0}^{m}\left(t_1, q_j(t_1)\right) \frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} \dot{q}_i(t_1) - W_{LCi}\left(\xi_{LCi}^{m-1} - \dot{q}_i \xi_{LC0}^{m-1}\right) = 0 \end{aligned} \tag{98}$$

Then there exists an adiabatic invariant.

$$
\begin{aligned}
I_{LCz} = \sum_{m=0}^{z} \varepsilon^m \Bigg\{ \int_{t_1}^{t} \Bigg[\frac{\partial L_C}{\partial {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \cdot {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} \left(\xi_{LCi}^m - \dot{q}_i \xi_{LC0}^m \right) + \left(\xi_{LCi}^m - \dot{q}_i \xi_{LC0}^m \right) \\
\cdot {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} \frac{\partial L_C}{\partial {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \Bigg] \mathrm{d}\tau - \int_{t_1}^{t} \frac{\partial L_C}{\partial {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q_i} \Bigg[\gamma \xi_{LC0}^m \Big(t_1, q_j(t_1)\Big) \dot{q}_i(t_1) \frac{(\tau - t_1)^{-\alpha(\tau,t_1)}}{\Gamma(1-\alpha(\tau,t_1))} \\
-(1-\gamma)\dot{q}_i(t_2) \xi_{LC0}^m \Big(t_2, q_j(t_2)\Big) \frac{(t_2-\tau)^{-\beta(t_2,\tau)}}{\Gamma(1-\beta(t_2,\tau))} \Bigg] \mathrm{d}\tau + L_C \cdot \xi_{LC0}^m + G_{LC}^m \Bigg\}
\end{aligned} \tag{99}
$$

where $\xi_{LCi}^{m-1} = \xi_{LC0}^{m-1} = 0$ *when* $m = 0$.

Moreover, adiabatic invariants for the disturbed Hamiltonian systems with the RVO and the RCVO, the disturbed Hamiltonian systems with the CRL and the CC, the disturbed Lagrangian systems with the CRL and the CC, the disturbed classical Hamiltonian system, and the disturbed classical Lagrangian system can also be deduced. We only present them in the forms of remarks rather than describe them in detail.

Remark 11. *Let* $\beta(\cdot,\cdot) = \alpha(\cdot,\cdot)$, $\gamma = \frac{1}{2}$, *so adiabatic invariants for the disturbed Hamiltonian systems in terms of the RVO and the RCVO can be achieved.*

Remark 12. *Let* $\alpha(\cdot,\cdot) \to \alpha$, $\beta(\cdot,\cdot) \to \beta$, *so adiabatic invariants for the disturbed Hamiltonian systems with the CRL and the CC as well as the disturbed Lagrangian systems with the CRL and the CC can be deduced.*

Remark 13. *Let* $\alpha(\cdot,\cdot), \beta(\cdot,\cdot) \to 1$, *so adiabatic invariants for the disturbed classical Hamiltonian system and the disturbed classical Lagrangian system can be deduced.*

6. Applications

In this section, we present two applications to illustrate the Methods and Results.

Application 1. *The differential equation of the Lotka biochemical oscillator [67] of variable order is*

$$
{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} x_1 = \alpha_1 + \beta_1 \exp x_2, \ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)} x_2 = \alpha_2 + \beta_2 \exp x_1 \tag{100}
$$

where α_1, α_2, β_1, *and* β_2 are constants.

Let $x_1 = q$, $x_2 = p_{RL}$, so the Hamiltonian can be expressed as

$$
H_{RL} = \alpha_1 p_{RL} - \alpha_2 q + \beta_1 \exp p_{RL} - \beta_2 \exp q \tag{101}
$$

From Equation (61), we have

$$
\begin{aligned}
p_{RL} \cdot \xi_{RL0}^0 \cdot \tfrac{\mathrm{d}}{\mathrm{d}t} {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q + \Big(p_{RL} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q - \alpha_1 p_{RL} + \alpha_2 q - \beta_1 \exp p_{RL} + \beta_2 \exp q \Big) \cdot \dot{\xi}_{RL0}^0 \\
- p_{RL} \cdot \gamma q(t_1) \xi_{RL0}^0 \Big(t_1, q_j(t_1), p_{RLj}(t_1)\Big) \tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} + p_{RL} \cdot {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} \left(\xi_{RL}^0 - \dot{q} \xi_{RL0}^0 \right) + \dot{G}_{RL}^0 \\
+ (\alpha_2 + \beta_2 \exp q) \xi_{RL}^0 + p_{RL} \cdot (1-\gamma) q(t_2) \xi_{RL0}^0 \Big(t_2, q_j(t_2), p_{RLj}(t_2)\Big) \tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} = 0
\end{aligned} \tag{102}
$$

It is obvious that

$$
\xi_{RL0}^0 = -1, \ \xi_{RL}^0 = 0, \ G_{RL}^0 = 0 \tag{103}
$$

satisfies Equation (102), where [57]

$$
\frac{\mathrm{d}}{\mathrm{d}t} {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} q = {}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} \dot{q} + \gamma q(t_1) \frac{\mathrm{d}}{\mathrm{d}t} \frac{(t-t_1)^{-\alpha(t,t_1)}}{\Gamma(1-\alpha(t,t_1))} - (1-\gamma) q(t_2) \frac{\mathrm{d}}{\mathrm{d}t} \frac{(t_2-t)^{-\beta(t_2,t)}}{\Gamma(1-\beta(t_2,t))} \tag{104}
$$

Then, a conserved quantity can be obtained from Theorem 1 as

$$I_{HRL0}=\int_{t_1}^{t}\left(p_{RL}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q+\dot{q}{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RL}\right)\mathrm{d}\tau-\left(p_{RL}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q-H_{RL}\right) \tag{105}$$

When the Lotka biochemical oscillator model is disturbed as

$${}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q=\alpha_1+\beta_1\exp p_{RL},\ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RL}=\alpha_2+\beta_2\exp q-\varepsilon(2q+1) \tag{106}$$

We can get the following solution to Equation (89)

$$\xi_{RL0}^{1}=1,\ \xi_{RL}^{1}=0,\ G_{RL}^{1}=q^2+q \tag{107}$$

Then, the first order adiabatic invariant can be obtained from Theorem 13 as

$$\begin{aligned}I_{HRL1}=&\int_{t_1}^{t}\left(p_{RL}\cdot\tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q+\dot{q}{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RL}\right)\mathrm{d}\tau-\left(p_{RL}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q-\alpha_1p_{RL}+\alpha_2q\right.\\&\left.-\beta_1\exp p_{RL}+\beta_2\exp q\right)+\varepsilon\left[q^2+q+\left(p_{RL}\cdot{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q-\alpha_1p_{RL}+\alpha_2q-\beta_1\exp p_{RL}+\beta_2\exp q\right)\right.\\&\left.-\int_{t_1}^{t}\left(p_{RL}\cdot\tfrac{\mathrm{d}}{\mathrm{d}\tau}{}^{RL}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q+\dot{q}{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{RL}\right)\mathrm{d}\tau\right]\end{aligned} \tag{108}$$

When $\alpha,\ \beta\to 1$, the classical differential equation of the Lotka biochemical oscillator, the classical Hamiltonian, the classical conserved quantity, and the classical adiabatic invariant can be obtained, which are consistent with the results in [66].

Application 2. *The Lagrangian of the two-dimensional isotropic harmonic oscillator of variable order is*

$$L_C=\frac{1}{2}m\left[\left({}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1\right)^2+\left({}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2\right)^2\right]-\frac{1}{2}k\left[(q_1)^2+(q_2)^2\right] \tag{109}$$

where k and m are constants.

Firstly, the generalized momentum and the Hamiltonian of variable order can be obtained from Equation (37):

$$p_{C1}=\frac{\partial L_C\left(t,q_j,{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_j\right)}{\partial{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1}=m{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1$$

$$p_{C2}=\frac{\partial L_C\left(t,q_j,{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_j\right)}{\partial{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2}=m{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2$$

$$\begin{aligned}H_C&=p_{C1}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1+p_{C2}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2-L_C\left(t,q_j,{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_j\right)\\&=p_{C1}\cdot\tfrac{p_{C1}}{m}+p_{C2}\cdot\tfrac{p_{C2}}{m}-\tfrac{1}{2}m\left[\left(\tfrac{p_{C1}}{m}\right)^2+\left(\tfrac{p_{C2}}{m}\right)^2\right]+\tfrac{1}{2}k\left[(q_1)^2+(q_2)^2\right]\\&=\tfrac{(p_{C1})^2}{2m}+\tfrac{(p_{C2})^2}{2m}+\tfrac{1}{2}k\left[(q_1)^2+(q_2)^2\right]\end{aligned} \tag{110}$$

From Equation (41), we have

$$\begin{aligned}&{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1=\tfrac{p_{C1}}{m},\ {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2=\tfrac{p_{C2}}{m}\\&{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C1}=-kq_1,\ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C2}=-kq_2\end{aligned} \tag{111}$$

And from Equation (65), we find that

$$\xi_{C0}^{0}=-1,\ \xi_{C1}^{0}=\xi_{C2}^{0}=0,\ G_{C}^{0}=0 \tag{112}$$

is a proper solution. A conserved quantity can be obtained from Theorem 2 as

$$\begin{array}{c} I_{HC0}=-\left[p_{C1}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1+p_{C2}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2-\frac{(p_{C1})^2}{2m}-\frac{(p_{C2})^2}{2m}-\frac{1}{2}k\left((q_1)^2+(q_2)^2\right)\right] \\ +\int_{t_1}^{t}\left(p_{C1}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1+p_{C2}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2+\dot{q}_1\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C1}\right. \\ \left.+\dot{q}_2\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C2}\right)\mathrm{d}\tau \end{array} \tag{113}$$

When the isotropic harmonic oscillator model of variable order is disturbed as

$$\begin{array}{c} {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1=\frac{p_{C1}}{m},\ {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2=\frac{p_{C2}}{m},\ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C1}=-kq_1-\varepsilon q_2 \\ {}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C2}=-kq_2-\varepsilon q_1 \end{array} \tag{114}$$

we obtain

$$\xi_{C0}^{1}=1,\ \xi_{C1}^{1}=\xi_{C2}^{1}=0,\ G_{C}^{1}=q_1q_2 \tag{115}$$

which satisfies Equation (92). Then, the first order adiabatic invariant can be obtained from Theorem 14 as

$$\begin{array}{c} I_{HC1}=-\left[p_{C1}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1+p_{C2}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2-\frac{(p_{C1})^2}{2m}-\frac{(p_{C2})^2}{2m}-\frac{1}{2}k\left((q_1)^2+(q_2)^2\right)\right] \\ +\int_{t_1}^{t}\left(p_{C1}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1+p_{C2}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2+\dot{q}_1\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C1}+\dot{q}_2\right. \\ \left.\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C2}\right)\mathrm{d}\tau-\varepsilon\left\{\int_{t_1}^{t}\left(p_{C1}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1+p_{C2}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2+\dot{q}_1\right.\right. \\ \left.\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C1}+\dot{q}_2\cdot{}^{RL}D_{1-\gamma}^{\beta(\cdot,\cdot),\alpha(\cdot,\cdot)}p_{C2}\right)\mathrm{d}\tau-\left[p_{C1}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_1+p_{C2}\cdot{}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}q_2\right. \\ \left.\left.-\frac{(p_{C1})^2}{2m}-\frac{(p_{C2})^2}{2m}-\frac{1}{2}k\left((q_1)^2+(q_2)^2\right)\right]-q_1q_2\right\} \end{array} \tag{116}$$

When $\alpha,\ \beta\to 1$, the classical Lagrangian, the classical Hamiltonian of the two-dimensional isotropic harmonic oscillator, the classical conserved quantity, and the classical adiabatic invariant can be obtained, which is consistent with the results in [66].

7. Conclusions

Equations of motion, Noether symmetry and conserved quantities, and perturbation to Noether symmetry and adiabatic invariants are investigated here. Hamilton equations with the CRLVO (Equation (36)), CCVO (Equation (41)), RVO (Equation (42)), RCVO (Equation (43)), CRL (Equation (48)), and CC (Equation (49)) are established. Then, Noether symmetry and conserved quantities are studied for those six Hamiltonian systems (Theorem 1, Theorem 2, Theorem 3, Theorem 4, Theorem 7, and Theorem 8). Meanwhile, Lagrange equations with the CRLVO (Equation (44)), CCVO (Equation (46)), CRL (Equation (50)), and CC (Equation (51)) are presented. Then Noether symmetry and conserved quantities are investigated for those four Lagrangian systems (Theorem 5, Theorem 6, Theorem 9, and Theorem 10). As for perturbation to Noether symmetry and adiabatic invariants, only adiabatic invariants with the CRLVO and CCVO for the Hamiltonian systems and the Lagrangian systems are studied and described in detail (Theorem 13, Theorem 14, Theorem 15, and Theorem 16).

Among the results obtained in this paper, Equations (36), (41)–(44), and (46), Theorems 1–6, and Theorems 13–16 are new. Equations (48)–(53) and Theorems 7–12, which are deduced from the main results of this paper, are consistent with the existing results.

It is generally known that there are three kinds of symmetry in analytical mechanics, i.e., Noether symmetry, Lie symmetry, and Mei symmetry. Noether symmetry plays an important role in finding the solutions to the differential equations of motion for mechanical systems because conserved quantities can be deduced from it. In fact, Lie symmetry and Mei symmetry are also able to deduce conserved quantities under certain conditions. The Lie symmetry means the invariance of the differential equations of motion under the infinitesimal transformations of time and coordinates. The Mei symmetry means the invariance under which the transformed dynamical functions still satisfy the original differential equations of motion. The conserved quantities deduced directly by the Lie symmetry and the Mei symmetry are called the Hojman conserved quantity and the Mei conserved quantity, respectively. However, only Noether symmetry is considered here, so Lie symmetry, Mei symmetry, perturbation to Lie symmetry, and perturbation to Mei symmetry are important and valuable aspects waiting to be studied.

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References

1. Oldham, K.B.; Spanier, J. *The Fractional Calculus*; Academic Press: New York, NY, USA, 1974.
2. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simulat.* **2011**, *16*, 1140–1153. [CrossRef]
3. Machado, J.T.; Galhano, A.M.; Trujillo, J.J. Science metrics on fractional calculus development since 1966. *Fract. Calc. Appl. Anal.* **2013**, *16*, 479–500.
4. Herrmann, R. *Fractional Calculus: An Introduction for Physicists*; World Sci. Publishing: Singapore, 2011.
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
6. Magin, R.L. *Fractional Calculus in Bioengineering*; Begell House Publishers: Redding, CA, USA, 2006.
7. Sabatier, J.; Agrawal, O.P.; Machado, J.T. *Advances in Fractional Calculus*; Springer: Dordrecht, The Netherlands, 2007.
8. Metzler, R.; Klafter, J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **2000**, *339*, 1–77. [CrossRef]
9. Kusnezov, D.; Bulgac, A.; Dang, G.D. Quantum Lévy processes and fractional kinetics. *Phys. Rev. Lett.* **1999**, *82*, 1136–1139. [CrossRef]
10. Naber, M. Time fractional Schrdinger equation. *J. Math. Phys.* **2004**, *45*, 3339–3352. [CrossRef]
11. Muslih, S.I.; Agrawal, O.P.; Baleanu, D. A fractional Dirac equation and its solution. *J. Phys. A Math. Theor.* **2010**, *43*, 055203. [CrossRef]
12. Herrmann, R. Gauge invariance in fractional field theories. *Phys. Lett. A* **2008**, *372*, 5515–5522. [CrossRef]
13. Lazo, M.J. Gauge invariant fractional electromagnetic fields. *Phys. Lett. A* **2011**, *375*, 3541–3546. [CrossRef]
14. Samko, S.G.; Ross, B. Integration and differentiation to a variable fractional order. *Integral Transform. Spec. Funct.* **1993**, *1*, 277–300. [CrossRef]
15. Ross, B.; Samko, S.G. Fractional integration operator of a variable order in the Holder spaces. *Internat. J. Math. Math. Sci.* **1995**, *18*, 777–788. [CrossRef]
16. Samko, S.G. Fractional integration and differentiation of variable order. *Anal. Math.* **1995**, *21*, 213–236. [CrossRef]
17. Coimbra, C.F.M. Mechanics with variable-order differential operators. *Ann. Phys.* **2003**, *12*, 692–703. [CrossRef]
18. Diaz, G.; Coimbra, C.F.M. Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation. *Nonlinear Dynam.* **2009**, *56*, 145–157. [CrossRef]
19. Lorenzo, C.F.; Hartley, T.T. Variable order and distributed order fractional operators. *Nonlinear Dynam.* **2002**, *29*, 57–98. [CrossRef]
20. Sun, H.G.; Chen, W.; Chen, Y.Q. Variable-order fractional differential operators in anomalous diffusion modeling. *Phys. A Stat. Mech. Appl.* **2009**, *388*, 4586–4592. [CrossRef]

21. Ramirez, L.E.S.; Coimbra, C.F.M. On the selection and meaning of variable order operators for dynamic modeling. *Int. J. Differ. Equ.* **2010**, *2010*, 846107. [CrossRef]
22. Ramirez, L.E.S.; Coimbra, C.F.M. On the variable order dynamics of the nonlinear wake caused by a sedimenting particle. *Phys. D* **2011**, *240*, 1111–1118. [CrossRef]
23. Tavares, D.; Almeida, R.; Torres, D.F.M. Caputo derivatives of fractional variable order: Numerical approximations. *Commun. Nonlinear Sci. Numer. Simulat.* **2016**, *35*, 69–87. [CrossRef]
24. Ortigueira, M.D.; Valério, D.; Machado, J.T. Variable order fractional systems. *Commun. Nonlinear Sci. Numer. Simulat.* **2019**, *71*, 231–243. [CrossRef]
25. Riewe, F. Nonconservative Lagrangian and Hamiltonian mechanics. *Phys. Rev. E* **1996**, *53*, 1890–1899. [CrossRef]
26. Riewe, F. Mechanics with fractional derivatives. *Phys. Rev. E* **1997**, *55*, 3581–3592. [CrossRef]
27. Klimek, M. Lagrangian and Hamiltonian fractional sequential mechanics. *Czechoslov. J. Phys.* **2002**, *52*, 1247–1253. [CrossRef]
28. Agrawal, O.P. Formulation of Euler–Lagrange equations for fractional variational problems. *J. Math. Anal. Appl.* **2002**, *272*, 368–379. [CrossRef]
29. Agrawal, O.P. Fractional variational calculus in terms of Riesz fractional derivatives. *J. Phys. A Math. Theor.* **2007**, *40*, 6287–6303. [CrossRef]
30. Muslih, S.I.; Baleanu, D. Hamiltonian formulation of systems with linear velocities within Riemann–Liouville fractional derivatives. *J. Math. Anal. Appl.* **2005**, *304*, 599–606. [CrossRef]
31. Rabei, E.M.; Nawafleh, K.I.; Hijjawi, R.S.; Muslih, S.I.; Baleanu, D. The Hamilton formalism with fractional derivatives. *J. Math. Anal. Appl.* **2007**, *327*, 891–897. [CrossRef]
32. Malinowska, A.B.; Torres, D.F.M. *Introduction to the Fractional Calculus of Variations*; Imp. Coll. Press: London, UK, 2012.
33. Atanackovic, T.M.; Pilipovic, S. Hamilton's principle with variable order fractional derivatives. *Fract. Calc. Appl. Anal.* **2011**, *14*, 94–109. [CrossRef]
34. Odzijewicz, T.; Malinowska, A.B.; Torres, D.F.M. Variable order fractional variational calculus for double integrals. In Proceedings of the 51st IEEE Conference on Decision and Control, Maui, HI, USA, 10–13 December 2012; Article No. 6426489. pp. 6873–6878.
35. Odzijewicz, T.; Malinowska, A.B.; Torres, D.F.M. Fractional variational calculus of variable order. In *Advances in Harmonic Analysis and Operator Theory*; Almeida, A., Castro, L.P., Eds.; Birkhauser: Basel, Switzerland, 2013; Volume 229, pp. 291–301.
36. Malinowska, A.B.; Torres, D.F.M. Fractional calculus of variations for a combined Caputo derivative. *Fract. Calc. Appl. Anal.* **2011**, *14*, 523–537. [CrossRef]
37. Malinowska, A.B.; Torres, D.F.M. Multiobjective fractional variational calculus in terms of a combined Caputo derivative. *Appl. Math. Comput.* **2012**, *218*, 5099–5111. [CrossRef]
38. Odzijewicz, T.; Malinowska, A.B.; Torres, D.F.M. Fractional variational calculus with classical and combined Caputo derivatives. *Nonlinear Anal.* **2012**, *75*, 1507–1515. [CrossRef]
39. Tavares, D.; Almeida, R.; Torres, D.F.M. Optimality conditions for fractional variational problems with dependence on a combined Caputo derivative of variable order. *Optimization* **2015**, *64*, 1381–1391. [CrossRef]
40. Tavares, D.; Almeida, R.; Torres, D.F.M. Combined fractional variational problems of variable order and some computation aspects. *J. Comput. Appl. Math.* **2018**, *339*, 374–388. [CrossRef]
41. Tavares, D.; Almeida, R.; Torres, D.F.M. Fractional Herglotz variational problems of variable order. *Discret. Cont. Dyn. S* **2017**, *11*, 143–154. [CrossRef]
42. Tavares, D.; Almeida, R.; Torres, D.F.M. Constrained fractional variational problems of variable order. *IEEE/CAA J. Autom. Sin.* **2017**, *4*, 80–88. [CrossRef]
43. Noether, E. Invariante variations probleme. *Kgl. Ges. Wiss. Nachr. Göttingen. Math. Phys.* **1918**, *2*, 235–257.
44. Mei, F.X.; Wu, H.B. *Dynamics of Constrained Mechanical Systems*; Beijing Institute of Technology Press: Beijing, China, 2009.
45. Frederico, G.S.F.; Torres, D.F.M. A formulation of Noether's theorem for fractional problems of the calculus of variations. *J. Math. Anal. Appl.* **2007**, *334*, 834–846. [CrossRef]
46. Atanacković, T.M.; Konjik, S.; Pilipović, S.; Simić, S. Variational problems with fractional derivatives: Invariance conditions and Noether's theorem. *Nonlinear Anal.* **2009**, *71*, 1504–1517. [CrossRef]

47. Frederico, G.S.F.; Torres, D.F.M. Fractional Noether's theorem in the Riesz-Caputo sense. *Appl. Math. Comput.* **2010**, *217*, 1023–1033. [CrossRef]
48. Zhou, S.; Fu, H.; Fu, J.L. Symmetry theories of Hamiltonian systems with fractional derivatives. *Sci. Chin. Phys. Mech. Astron.* **2011**, *54*, 1847–1853. [CrossRef]
49. Frederico, G.S.F.; Lazo, M.J. Fractional Noether's theorem with classical and Caputo derivatives: Constants of motion for non-conservative systems. *Nonlinear Dyn.* **2016**, *85*, 839–851. [CrossRef]
50. Jin, S.X.; Zhang, Y. Noether theorem for non-conservative systems with time delay in phase space based on fractional model. *Nonlinear Dyn.* **2015**, *82*, 663–676. [CrossRef]
51. Zhai, X.H.; Zhang, Y. Noether symmetries and conserved quantities for fractional Birkhoffian systems with time delay. *Commun. Nonlinear Sci. Numer. Simulat.* **2016**, *36*, 81–97. [CrossRef]
52. Song, C.J.; Zhang, Y. Noether symmetry and conserved quantity for fractional Birkhoffian mechanics and its applications. *Fract. Calc. Appl. Anal.* **2018**, *21*, 509–526. [CrossRef]
53. Song, C.J. Noether symmetry for fractional Hamiltonian system. *Phys. Lett. A* **2019**, *383*, 125914. [CrossRef]
54. Odzijewicz, T.; Malinowska, A.B.; Torres, D.F.M. Noether's theorem for fractional variational problems of variable order. *Cent. Eur. J. Phys.* **2013**, *11*, 691–701. [CrossRef]
55. Malinowska, A.B.; Odzijewicz, T. Noether's second theorem for variable order fractional variational problems. *Lect. Notes Electr. Eng.* **2015**, *320*, 37–46.
56. Yan, B.; Zhang, Y. Noether's theorem for fractional Birkhoffian systems of variable order. *Acta Mech.* **2016**, *227*, 2439–2449. [CrossRef]
57. Yan, B. Research of the Fractional Pfaff-Birkhoff Variational Problem and Symmetry of Variable Order. Master's Thesis, Suzhou University of Science and Technology, Suzhou, China, 2016. (In Chinese).
58. Arnold, V.I.; Kozlov, V.V.; Neishtadt, A.I. Mathematical Aspects of Classical and Celestial Mechanics. In *Encyclopaedia of Mathematical Sciences*; Springer: Berlin, Germany, 2006.
59. Mei, F.X.; Liu, D.; Luo, Y. *Advanced Analytical Mechanics*; Beijing Institute of Technology Press: Beijing, China, 1991. (In Chinese)
60. Zhao, Y.Y.; Mei, F.X. Exact invariant and adiabatic invariant of a general dynamical system. *Acta Mech. Sin.* **1996**, *28*, 207–216. (In Chinese)
61. Chen, J.; Zhang, Y. Perturbation to Noether symmetries and adiabatic invariants for disturbed Hamiltonian systems based on El-Nabulsi nonconservative dynamics model. *Nonlinear Dyn.* **2014**, *77*, 353–360. [CrossRef]
62. Song, C.J. Adiabatic invariants for generalized fractional Birkhoffian mechanics and their applications. *Math. Probl. Eng.* **2018**, *2018*, 6414960. [CrossRef]
63. Song, C.J.; Zhang, Y. Perturbation to Noether symmetry for fractional dynamic systems of variable order. *Indian J. Phys.* **2018**, *93*, 1057–1067. [CrossRef]
64. Miller, K.S.; Ross, B. *An Introduction to the Fractional Integrals and Derivatives-Theory and Applications*; John Wiley and Sons Inc.: New York, NY, USA, 1993.
65. Luo, S.K.; Xu, Y.L. Fractional Birkhoffian mechanics. *Acta Mech.* **2015**, *226*, 829–844. [CrossRef]
66. Mei, F.X. *Analytical Mechanics (II)*; Beijing Institute of Technology Press: Beijing, China, 2013. (In Chinese)
67. Mei, F.X.; Shi, R.C.; Zhang, Y.F.; Wu, H.B. *Dynamics of Birkhoff Systems*; Beijing Institute of Technology Press: Beijing, China, 1996. (In Chinese)