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gH-Symmetrically Derivative of Interval-Valued Functions and Applications in Interval-Valued Optimization

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Abstract: In this paper, we present the gH-symmetrical derivative of interval-valued functions and its properties. In application, we apply this new derivative to investigate the Karush–Kuhn–Tucker (KKT) conditions of interval-valued optimization problems. Meanwhile, some examples are worked out to illuminate the obtained results.

Keywords: interval-valued functions; gH-symmetrically derivative; KKT optimality conditions

1. Introduction

In modern times, the optimization problems with uncertainty have received considerable attention and have great value in economic and control fields (e.g., [1–4]). From this point of view, Ishibuchi and Tanaka [5] derived the interval-valued optimization as an attempt to handle the problems with imprecise parameters. Since then, a collection of papers written by Chanas, Kuchta and Bitran et al. (e.g., [6–8]) offered many different approaches on this subject. For more profound results and applications, please see [9–15]. In addition, the importance of derivatives in nonlinear interval-valued optimization problems can not be ignored. Toward this end, Wu [16–18] discussed interval-valued nonlinear programming problems and gave a utilization of the H-derivative in interval-valued Karush–Kuhn–Tucker (KKT) optimization problems. Also, according to the results given by Chalco-Cano [19], the gH-differentiability was extended to learn interval-valued KKT optimality conditions. As for details of above mentioned derivatives, we refer the interested readers to [20,21].

Motivated by Wu [17] and Chalco-Cano [19], we introduce the gH-symmetrical derivative which is more general than the gH-derivative. Based on this derivative and its properties, we give KKT optimality conditions for interval-valued optimization problems.

The paper is discussed as follows. In Section 2, we recall some preliminaries. In Section 3, we put forward some concepts and theorems of the gH-symmetrical derivative. In Section 4, new KKT type optimality conditions are derived and some interesting examples are given. Finally, Section 5 contains some conclusions.

2. Preliminaries

Firstly, let $\mathbb{R}$ denote the space of real numbers and $\mathbb{Q}$ denote the set of rational numbers. We denote the set of real intervals by

$$\mathbb{I} = \{c = [\underline{c}, \overline{c}] | \underline{c}, \overline{c} \in \mathbb{R} \text{ and } \underline{c} \leq \overline{c}\},$$

the Hausdorff–Pompeiu distance between interval $[\underline{c}, \overline{c}]$ and $[\underline{d}, \overline{d}] \in \mathbb{I}$ is defined by

$$D([\underline{c}, \overline{c}], [\underline{d}, \overline{d}]) = \max\{|\underline{c} - \underline{d}|, |\overline{c} - \overline{d}|\}.$$

$(\mathbb{I}, D)$ is a complete metric space. The relation "$\preceq_{LU}$" of $\mathbb{I}$ is determined by

$$[\underline{c}, \overline{c}] \preceq_{LU} [\underline{d}, \overline{d}] \Leftrightarrow \underline{c} \leq \underline{d}, \overline{c} \leq \overline{d}.$$

Definition 1 ([21]). *The gH-difference of $c, d \in \mathbb{I}$ is defined as below*

$$c \ominus_g d = e \Leftrightarrow \begin{cases} (a) c = d + e; \\ or\ (b) d = c + (-1)e. \end{cases}$$

This gH-difference of two intervals always exists and it is equal to

$$c \ominus_g d = [\min\{\underline{c} - \underline{d}, \overline{c} - \overline{d}\}, \max\{\underline{c} - \underline{d}, \overline{c} - \overline{d}\}]. \tag{1}$$

Proposition 1 ([22]). *We recall some properties of intervals c, d and e.*
(1) Assume the length of interval c is defined by $l(c) = \overline{c} - \underline{c}$. Then

$$c \ominus_g d = \begin{cases} [\underline{c} - \underline{d}, \overline{c} - \overline{d}], & if\ l(c) \geq l(d); \\ [\overline{c} - \overline{d}, \underline{c} - \underline{d}], & if\ l(c) < l(d). \end{cases} \tag{2}$$

(2) If $(l(c) - l(d))(l(d) - l(e)) \geq 0$, then

$$c \ominus_g e = (c \ominus_g d) + (d \ominus_g e).$$

Let $f : (a, b) \to \mathbb{I}$ be an interval-valued function, and $f(t) = [\underline{f}(t), \overline{f}(t)]$ so that $\underline{f}(t) \leq \overline{f}(t)$ for all $t \in (a, b)$. The functions $\underline{f}, \overline{f}$ are called endpoint functions of f. In [21] Stefannini and Bede introduced the gH-derivative as follows.

Definition 2 ([21]). *Let $f : (a, b) \to \mathbb{I}$. Then f is gH-differentiable at $t_0 \in (a, b)$ if there exists $f'(t_0) \in \mathbb{I}$ such that*

$$\lim_{h \to 0} \frac{[f(t_0 + h) \ominus_g f(t_0)]}{h} = f'(t_0). \tag{3}$$

For more basic notations with interval analysis, see [21–24].

Definition 3 ([25]). *Let $f : (a, b) \to \mathbb{R}$. Then f is symmetrically differentiable at $t_0 \in (a, b)$ if there exists $A \in \mathbb{R}$ and*

$$\lim_{h \to 0} \frac{[f(t_0 + h) - f(t_0 - h)]}{2h} = A.$$

3. Main Results

Now, we introduce the gH-symmetrical derivative and some corresponding properties.

Definition 4. *Let $f : (a, b) \to \mathbb{I}$. Then f is symmetrically continuous at $t_0 \in (a, b)$ if*

$$\lim_{h \to 0} (f(t_0 + h) \ominus_g f(t_0 - h)) = 0.$$

Definition 5. *Let* $f : (a, b) \to \mathbb{I}$. *Then* f *is gH-symmetrically differentiable at* t_0 *if there exists* $f^s(t_0)$ *such that*

$$\lim_{h \to 0} \frac{f(t_0 + h) \ominus_g f(t_0 - h)}{2h} = f^s(t_0). \tag{4}$$

For convenience, let $\mathcal{D}_{\mathcal{I}}((a, b), \mathbb{I})$, $\mathcal{SD}_{\mathcal{I}}((a, b), \mathbb{I}))$ be the collection of gH-differentiable and gH-symmetrically differentiable interval functions on (a, b).

Lemma 1. *Let* c, d *and* $e \in \mathbb{I}$. *If* $(l(c) - l(d))(l(d) - l(e)) < 0$, *then we have*

$$c \ominus_g e = (c \ominus_g d) \ominus_g (-1)(d \ominus_g e).$$

Proof. If $(l(c) - l(d)) < 0$ and $(l(d) - l(e)) > 0$, by (2) we have

$$\begin{aligned}
(c \ominus_g d) &\ominus_g (-1)(d \ominus_g e) \\
&= [\bar{c} - \bar{d}, \underline{c} - \underline{d}] \ominus_g (-1)[\underline{d} - \underline{e}, \bar{d} - \bar{e}] \\
&= [\bar{c} - \bar{d}, \underline{c} - \underline{d}] \ominus_g [\bar{e} - \bar{d}, \underline{e} - \underline{d}] \\
&= [\min\{\bar{c} - \bar{e}, \underline{c} - \underline{e}\}, \max\{\bar{c} - \bar{e}, \underline{c} - \underline{e}\}] \\
&= c \ominus_g e.
\end{aligned}$$

If $(l(c) - l(d)) > 0$ and $(l(d) - l(e)) < 0$, the proof is similar to above. □

The following Theorem 1 shows the relation between $\mathcal{D}_{\mathcal{I}}((a, b), \mathbb{I})$ and $\mathcal{SD}_{\mathcal{I}}((a, b), \mathbb{I}))$.

Theorem 1. *Let* $f : (a, b) \to \mathbb{I}$ *be an interval-valued function. If* f *is gH-differentiable at* $t \in (a, b)$ *then* f *is gH-symmetrically differentiable at* t. *However, the converse is not true.*

Proof. Fix $t \in (a, b)$ and assume $f'(t)$ exists. Put

$$K = \big(l(f(t + h)) - l(f(t))\big)\big(l(f(t)) - l(f(t - h))\big).$$

Applying Proposition 1 and Lemma 1, we obtain

$$f(t + h) \ominus_g f(t - h) = \begin{cases} (f(t + h) \ominus_g f(t)) + (f(t) \ominus_g f(t - h)), & \text{if } K \geq 0; \\ (f(t + h) \ominus_g f(t)) \ominus_g (-1)(f(t) \ominus_g f(t - h)), & \text{if } K < 0. \end{cases} \tag{5}$$

Hence,
a. If $K \geq 0$, by (5) we have

$$\begin{aligned}
&\lim_{h \to 0} \frac{f(t + h) \ominus_g f(t - h)}{2h} \\
&= \lim_{h \to 0} \frac{(f(t + h) \ominus_g f(t)) + (f(t) \ominus_g f(t - h))}{2h} \\
&= f'(t).
\end{aligned}$$

According to Definition 5, $f^s(t)$ exists and

$$f^s(t) = f'(t). \tag{6}$$

b. If $K < 0$, by (5) we have

$$\lim_{h \to 0} \frac{f(t+h) \ominus_g f(t-h)}{2h}$$

$$= \lim_{h \to 0} \frac{(f(t+h) \ominus_g f(t)) \ominus_g (-1)(f(t) \ominus_g f(t-h))}{2h}$$

$$= \frac{f'(t)}{2} \ominus_g (-1) \frac{f'(t)}{2}.$$

Thus, $f^s(t)$ exists and

$$f^s(t) = \frac{f'(t)}{2} \ominus_g (-1) \frac{f'(t)}{2}. \tag{7}$$

Therefore, f is gH-symmetrically differentiable in view of (6) and (7).

Conversely, we now give a counter example as follows.

Let $f_1(t) = \begin{cases} [-|2t|, |3t|], & \text{if } t \neq 1; \\ 2, & \text{if } t = 1. \end{cases}$ Since

$$\lim_{h \to 0} \frac{[-|2(1+h)|, |3(1+h)|] \ominus_g [-|2(1-h)|, |3(1-h)|]}{2h} = [-2, 3],$$

f_1 is gH-symmetrically differentiable at $t = 1$. However,

$$\lim_{h \to 0} \frac{[-|2(1+h)|, |3(1+h)|] \ominus_g [2, 2]}{h}$$

does not exist. Then f_1 is not gH-differentiable at $t = 1$. $\square$

Remark 1. *Clearly the gH-symmetrically derivative is more general than gH-derivative reflected by Theorem 1. Moreover, $f'(t)$ and $f^s(t)$ are not necessarily equal according to (6) and (7). For example, consider interval-valued function $f_2(t) = [-|t|, |t|]$. We have*

$$f_2'(0) = \lim_{h \to 0} \frac{f(h) \ominus_g f(0)}{h} = \lim_{h \to 0} \frac{[-|h|, |h|] \ominus_g [0, 0]}{h} = [-1, 1].$$

However,

$$f_2^s(0) = \lim_{h \to 0} \frac{f(h) \ominus_g f(-h)}{2h} = \lim_{h \to 0} \frac{[-|h|, |h|] \ominus_g [-|h|, |h|]}{2h} = 0,$$

which implies $f_2'(0) \neq f_2^s(0)$.

Theorem 2. *Let $f : (a, b) \to \mathbb{I}$. Then f is gH-symmetrically differentiable at $t_0 \in (a, b)$ iff $\overline{f}$ and $\underline{f}$ are symmetrically differentiable at t_0. Moreover*

$$f^s(t_0) = [\min\{\underline{f}^s(t_0), \overline{f}^s(t_0)\}, \max\{\underline{f}^s(t_0), \overline{f}^s(t_0)\}]. \tag{8}$$

Proof. Suppose f is gH-symmetrically differentiable at t_0, then $f^s(t_0) = [\underline{g}(t_0), \overline{g}(t_0)]$ exists. According to Definition 5 and (1),

$$\underline{g}(t_0) = \lim_{h \to 0} \min \left\{ \frac{\underline{f}(t_0+h) - \underline{f}(t_0-h)}{2h}, \frac{\overline{f}(t_0+h) - \overline{f}(t_0-h)}{2h} \right\},$$

$$\overline{g}(t_0) = \lim_{h \to 0} \max \left\{ \frac{\underline{f}(t_0+h) - \underline{f}(t_0-h)}{2h}, \frac{\overline{f}(t_0+h) - \overline{f}(t_0-h)}{2h} \right\}$$

exist. Then $\underline{f}^s(t_0)$, $\overline{f}^s(t_0)$ must exist and (8) is workable.

Conversely, suppose $\overline{f}$ and $\underline{f}$ are symmetrically derivative at t_0.

If $(\overline{f})^s(t_0) \geq (\underline{f})^s(t_0)$, then

$$
\begin{aligned}
&[(\underline{f})^s(t_0), (\overline{f})^s(t_0)] \\
&= \Big[\lim_{h \to 0} \frac{\underline{f}(t_0 + h) - \underline{f}(t_0 - h)}{2h}, \lim_{h \to 0} \frac{\overline{f}(t_0 + h) - \overline{f}(t_0 - h)}{2h} \Big] \\
&= \lim_{h \to 0} \frac{f(t_0 + h) \ominus_g f(t_0 - h)}{2h} \\
&= f^s(t_0).
\end{aligned}
$$

So f is gH-symmetrically differentiable. Similarly, if $(\overline{f})^s(t_0) \leq (\underline{f})^s(t_0)$, then $f^s(t_0) = [(\overline{f})^s(t_0), (\underline{f})^s(t_0)]$. □

Next, we study the gH-symmetrically derivative of $f : M \subset \mathbb{R}^n \to \mathbb{I}$ where M is an open set.

Definition 6. *Let $f : M \to \mathbb{I}$, $t^0 = (t_1^0, t_2^0, \ldots, t_n^0) \in M$. If there exist $A_1, A_2, \ldots, A_n \in \mathbb{I}$ such that*

$$
\lim_{h \to 0} \frac{D(f(t^0 + h) \ominus_g f(t^0 - h), 2\sum_{i=1}^n h_i A_i)}{\sum_{i=1}^n |h_i|} = 0, \tag{9}
$$

$h = (h_1, \ldots, h_n)$. Then we call f gH-symmetrically differentiable at t^0, and define $(A_1, A_2, \ldots, A_n)$ (denote $\nabla_g^s f(t^0) = (A_1, A_2, \ldots, A_n)$) the symmetric gradient of f at t^0.

Theorem 3. *The function $f : M \to \mathbb{I}$ is gH-symmetrically differentiable iff $\underline{f}$ and $\overline{f}$ are symmetrically differentiable.*

Proof. The proof is similar to Theorem 2, so we omit it. □

Definition 7. *Let $f : M \to \mathbb{I}$ and $t^0 \in M$. If the interval-valued function $\varphi(t_i) = f(t_1^0, \ldots, t_{i-1}^0, t_i, t_{i+1}^0, \ldots, t_n^0)$ is gH-symmetrically differentiable at t_i^0, then f has the ith partial gH-symmetrical derivative $(\frac{\partial^s f}{\partial t_i})_g(t^0)$ at t^0, i.e.,*

$$
(\frac{\partial^s f}{\partial t_i})_g(t^0) = (\varphi)^s(t_i^0).
$$

The following Theorem illustrates the relation between symmetric gradients and partial gH-symmetrical derivatives.

Theorem 4. *Let $f : M \to \mathbb{I}$, $t^0 = (t_1^0, t_2^0, \ldots, t_n^0) \in M$. If f is gH-symmetrically differentiable at t^0, then $(\frac{\partial^s f}{\partial t_i})_g(t^0)$ exists, and $(\frac{\partial^s f}{\partial t_i})_g(t^0) = A_i (i = 1, 2, \ldots, n)$, where $(A_1, A_2, \ldots, A_n) = \nabla_g^s f(t^0)$.*

Proof. By Definition 6, substituting $h_j = 0$ $(j \neq i)$ and taking $h_i \to 0$ in M, it follows at once $A_i = (\frac{\partial^s f}{\partial t_i})_g(t^0)$. □

Example 1. *Let $f(t_1, t_2) = \begin{cases} [-|2t_1| + t_2^2, |3t_1| + t_2^2], & \text{if } (t_1, t_2) \neq (0, 0); \\ [t_2^2, t_2^2 + 1], & \text{if } (t_1, t_2) = (0, 0). \end{cases}$*

We have

$$
(\frac{\partial^s f}{\partial t_1})_g(0, 0) = \lim_{h \to 0} \frac{f(h, 0) \ominus_g f(-h, 0)}{2h} = \lim_{h \to 0} \frac{[-|2h|, 3|h|] \ominus_g [-|2h|, 3|h|]}{2h} = 0
$$

and

$$(\frac{\partial^s f}{\partial t_2})_g(0,0) = \lim_{h \to 0} \frac{f(0,h) \ominus_g f(0,-h)}{2h} = \lim_{h \to 0} \frac{[h^2,h^2] \ominus_g [h^2,h^2]}{2h} = 0.$$

Therefore, the symmetric gradient of f at the point $(0,0)$ is $\nabla^s_g f(0,0) = (0,0)$.

Remark 2. *The gradient of f in [19] is more restrictive than the symmetric gradient. For instance, the partial derivative $(\frac{\partial f}{\partial t_1})_g(0,0)$ does not exist in Example 1. So we can not obtain the gradient at $(0,0)$ using the gH-derivative.*

4. Mathematical Programming Applications

Now, we discuss the following interval-valued optimization problem (IVOP):

$$\min \quad f(t) \tag{IVOP1}$$
$$\text{subject to} \quad g_i(t) \le 0, \quad i = 1, \ldots, m,$$

where $g_1, g_2, \ldots, g_m : M \subset \mathbb{R}^n \to \mathbb{R}$ are symmetrically differentiable and convex on M, M is an open and convex set and $f : M \to \mathbb{I}$ is LU-convex (see [19], Definition 8). Then we study the LU-solution (see ([17], Definition 5.1)) of the problem (IVOP1).

Theorem 5. *Suppose $f : M \to \mathbb{I}$ is LU-convex and gH-symmetrically differentiable at t^*. If there exist (Lagrange) multipliers $0 < \lambda_1, \lambda_2 \in \mathbb{R}$ and $0 \le \mu_i \in \mathbb{R}$, $i = 1, \ldots, m$ so that*
(1) $\lambda_1 \nabla^s \underline{f}(t^*) + \lambda_2 \nabla^s \overline{f}(t^*) + \sum_{i=1}^m \mu_i \nabla^s g_i(t^*) = 0$;
(2) $\sum_{i=1}^m \mu_i g_i(t^*) = 0$, where $\mu = (\mu_1, \ldots, \mu_m)^T$.
   Then t^* is an optimal LU-solution of problem (IVOP1).

Proof. We define $f_l(t) = \lambda_1 \underline{f}(t) + \lambda_2 \overline{f}(t)$. Since f is LU-convex and gH-symmetrically differentiable at t^*, then f_l is convex and symmetrically differentiable at t^*. And

$$\nabla^s f_l(t^*) = \lambda_1 \nabla^s \underline{f}(t^*) + \lambda_2 \nabla^s \overline{f}(t^*),$$

then we have following conditions
(1) $\nabla^s f_l(t^*) + \sum_{i=1}^m \mu_i \nabla^s g_i(t^*) = 0$;
(2) $\sum_{i=1}^m \mu_i g_i(t^*) = 0$ where $\mu = (\mu_1, \ldots, \mu_m)^T$.
   Based on Theorem 3.1 of [26], t^* is an optimal solution of the real-valued objective function f_l subject to the same constraints of problem (IVOP1), i.e.,

$$f_l(t^*) \le f_l(\bar{t})$$

for any $\bar{t}(\ne t^*) \in M$.

Next, we illustrate this theorem by contradiction. Assume t^* is not a solution of (IVOP1), then there exists an $\bar{t} \in M$ such that $f(\bar{t}) \prec_{LU} f(t^*)$, i.e.,

$$\underline{f}(\bar{t}) < \underline{f}(t^*) \text{ and } \overline{f}(\bar{t}) \le \overline{f}(t^*);$$
$$or, \underline{f}(\bar{t}) \le \underline{f}(t^*) \text{ and } \overline{f}(\bar{t}) < \overline{f}(t^*);$$
$$or, \underline{f}(\bar{t}) < \underline{f}(t^*) \text{ and } \overline{f}(\bar{t}) < \overline{f}(t^*).$$

Therefore, we obtain that $f_l(\bar{t}) < f_l(t^*)$ which leads to a contradiction. This completes the proof. $\square$

Example 2. *Suppose the objective function*

$$f(t) = \begin{cases} [3t^2 + t - 6, 2t^2 + 2t], & if \ t \in (-1,0); \\ [2t - 6, 2t], & if \ t \in [0,1), \end{cases}$$

and the optimization problem as below

$$\min \quad f(t) \qquad\qquad \text{(IVOP2)}$$
$$subject \ to \quad -t \le 0,$$
$$t - 1 \le 0.$$

We have

$$\underline{f}(t) = \begin{cases} 3t^2 + t - 6, & if \ t \in (-1,0); \\ 2t - 6, & if \ t \in [0,1), \end{cases}$$

$$\overline{f}(t) = \begin{cases} 2t^2 + 2t, & if \ t \in (-1,0); \\ 2t, & if \ t \in [0,1). \end{cases}$$

Both $\underline{f}, \overline{f}$ are convex and symmetrically differentiable. Furthermore, the condition (1) and (2) of Theorem 5 are satisfying at $t^* = 0$ when $\lambda_1 = \frac{1}{3}\mu_1, \lambda_2 = \frac{1}{4}\mu_1$ and $\mu_2 = 0$. Hence, $t^* = 0$ is a LU-solution of (IVOP2).

Remark 3. *Note Theorem 4 in [19] can not be used in problem (IVOP2) since $\underline{f}$ is not differentiable at 0. Hence, Theorem 5 generalizes Theorem 4 in [19].*

Applying Theorem 5 we have the following result.

Corollary 1. *Under the same assumption of Theorem 5, let k be any integer with $1 < k < m$. If there exist (Lagrange) multipliers $0 \le \mu_i \in \mathbb{R}, i = 1, \ldots, m$, such that*
(1) $\nabla^s \underline{f}(t^*) + \sum_{i=1}^{k} \mu_i \nabla^s g_i(t^*) = 0$;
(2) $\nabla^s \overline{f}(t^*) + \sum_{i=k+1}^{m} \mu_i \nabla^s g_i(t^*) = 0$;
(3) $\sum_{i=1}^{k} \mu_i g_i(t^*) = 0 = \sum_{i=k+1}^{m} \mu_i g_i(t^*)$, where $\mu = (\mu_1, \ldots, \mu_m)^T$.
Then t^* is an optimal LU-solution of problem (IVOP1).

Proof. Let $\nu_i = \lambda_1 \mu_i$ $(i = 1, \ldots, k)$ and $\omega_i = \lambda_2 \mu_i$ $(i = k+1, \ldots, m)$. The conditions in this corollary can be written as
(1) $\lambda_1 \nabla^s \underline{f}(t^*) + \lambda_2 \nabla^s \overline{f}(t^*) + \sum_{i=1}^{k} \nu_i \nabla^s g_i(t^*) + \sum_{i=k+1}^{m} \omega_i \nabla^s g_i(t^*) = 0$;
(2) $\sum_{i=1}^{k} \mu_i g_i(t^*) = 0 = \sum_{i=k+1}^{m} \mu_i g_i(t^*)$.
Then from Theorem 5 the result follows. □

As shown in Example 1, symmetric gradient is more general than the gradient of f using the gH-derivative, we derive new KKT conditions for (IVOP1) using the symmetric gradient of interval-valued function given in Definition 6.

Theorem 6. *Under the same assumption of Theorem 5, the following KKT conditions hold*
(1) $\nabla^s_g f(t^*) + \sum_{i=1}^{m} \mu_i \nabla^s g_i(t^*) = 0$;
(2) $\sum_{i=1}^{m} \mu_i g_i(t^*) = 0$, where $\mu = (\mu_1, \ldots, \mu_m)^T$.
Then t^* is an optimal LU-solution of problem (IVOP1).

Proof. By Theorem 3, the equation $\nabla^s_g f(t^*) + \sum_{i=1}^{m} \mu_i \nabla^s g_i(t^*) = 0$ can be interpreted as

$$\nabla^s \underline{f}(t^*) + \sum_{i=1}^{m} \mu_i \nabla^s g_i(t^*) = 0 = \nabla^s \overline{f}(t^*) + \sum_{i=1}^{m} \mu_i \nabla^s g_i(t^*), \qquad (10)$$

which implies

$$\nabla^s \underline{f}(t^*) + \nabla^s \overline{f}(t^*) + \sum_{i=1}^{m} \nu_i \nabla^s g_i(t^*) = 0, \tag{11}$$

where $\nu_i = 2\mu_i$ $(i = 1, \ldots, m)$. Then the result meets all conditions of Theorem 5. That is the end of proof. □

Example 3. *Suppose*

$$f(t) = \begin{cases} [\frac{1}{3}t - 1, t], & \text{if } t > 0; \\ [-\frac{1}{3}t - 1, -t], & \text{if } t \leq 0. \end{cases} \tag{12}$$

and the programming problem

$$\begin{align} \min \quad & f(t) \tag{IVOP3} \\ \text{subject to} \quad & -t \leq 2, \\ & t - 1 < 3. \end{align}$$

We can observe that $\nabla^s_g f(0) = 0$. The conditions (1) and (2) of Theorem 6 are satisfied for $\mu_1 = \mu_2 = 0$. Hence, 0 is an optimal LU-solution of (IVOP3).

Remark 4. *It is worth noting that Theorem 9 of [19] can not solve the problem (IVOP3) since f is not gH-differentiable at 0. So Theorem 6 is more general than Theorem 9 in [19].*

Remark 5. *Comparing Example 2 with Example 3, it is easy to see Theorem 5 is more generic than Theorem 6. Nonetheless, Theorem 6 can be very effective for obtaining the solution of (IVOP1) in some cases.*

5. Conclusions and Further Research

We defined the gH-symmetrical derivative of interval-valued functions, which is more general than the gH-derivative. In addition, we generalized some results of Wu [17] and Chalco-Cano [19] by establishing sufficient optimality conditions for optimality problems involving gH-symmetrically differentiable objective functions. The symmetric gradient of interval functions is more general and it is more robust for optimization problems. However, the equality constraints are not considered in our paper. We can try to handle equality constraints using a similar methodology to the one proposed in this paper. Moreover, the constraint functions in this paper are still real-valued. In future research, we may extend the constraint functions as the interval-valued functions. And we may study the symmetric integral and more interesting properties about interval-valued functions.

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