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The Order of Strongly Starlikeness of the Generalized α-Convex Functions

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Abstract: We consider the order of the strongly-starlikeness of the generalized α-convex functions. Some sufficient conditions for functions to be p-valently strongly-starlike are given.

Keywords: analytic; α-convex function; starlike function; strongly-starlike function; subordination

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1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, real numbers and complex numbers, respectively.

Definition 1. *A function f is called p-valent in a domain $\mathbb{D} \subset \mathbb{C}$ if the equation $f(z) = w$ has at most p roots in $\mathbb{D}$ for every complex number w, and there is a complex number w_0 such that $f(z) = w_0$ has exactly p roots in $\mathbb{D}$.*

Let $\mathcal{A}(p)$ denote the class of analytic functions in $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N}). \tag{1}$$

For $p = 1$, we denote $\mathcal{A} := \mathcal{A}(1)$.

Definition 2. *A function $f \in \mathcal{A}(p)$ is said to be p-valently starlike in $\mathbb{U}$ if it satisfies:*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{2}$$

We denote by $\mathcal{S}_p^*$ the subclass of $\mathcal{A}(p)$ consisting of all p-valently starlike functions in $\mathbb{U}$.

Definition 3. *If $f \in \mathcal{A}(p)$ satisfies:*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}) \tag{3}$$

for some $\beta \in (0, 1]$, then the function f is called p-valently strongly-starlike of order β in $\mathbb{U}$. We denote this class by $\mathcal{SS}_p^*(\beta)$.

For $p = 1$, the class $\mathcal{SS}_1^*(\beta)$ was introduced by Brannan and Kirwan [1]. It is clear that:

$$\mathcal{SS}_p^*(\beta) \subset \mathcal{S}_p^* \quad \text{and} \quad \mathcal{SS}_p^*(1) = \mathcal{S}_p^*.$$

The strongly-starlike functions and related functions have been extensively studied by several authors (see, e.g., [1–16]).

We say that for functions f and g analytic in $\mathbb{U}$, g is subordinate to f, written $g \prec f$, if there exists a Schwarz function w such that $g(z) = f(w(z))$ for $z \in \mathbb{U}$. In particular, if f is univalent in $\mathbb{U}$, then:

$$g(z) \prec f(z) \quad (z \in \mathbb{U}) \Longleftrightarrow g(0) = f(0) \quad \text{and} \quad g(\mathbb{U}) \subset f(\mathbb{U}).$$

In [17], Mocanu first introduced the class:

$$\mathcal{M}(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{ \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)} \right\} > 0, \quad \alpha \in \mathbb{R}, \quad z \in \mathbb{U} \right\} \tag{4}$$

of α-convex functions, which give a continuous passage from convex to starlike functions. He proved that every α-convex function is starlike. Recently, Nunokawa, Sokól and Trabka-Wieclaw [8] considered the generalized α-convex function class:

$$\mathcal{M}(\alpha,\beta) = \left\{ f \in \mathcal{A} : \left| \arg\left\{ \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)} \right\} \right| < \frac{\beta\pi}{2}, \quad \alpha \in \mathbb{R}, \quad \beta \in (0,1],\ z \in \mathbb{U} \right\}.$$

In this paper, we shall further study the properties of the generalized α-convex functions. Several sufficient conditions for functions to be p-valently strongly starlike are obtained.

The following lemmas will be required in our investigation.

Lemma 1 (See [18])**.** *Let g be analytic and univalent in $\mathbb{U}$. Furthermore, let θ and φ be analytic in a domain $\mathbb{D} \supseteq g(\mathbb{U})$ with $\varphi(w) \neq 0$ for $w \in g(\mathbb{U})$. Put:*

$$Q(z) = zg'(z)\varphi(g(z)) \quad \text{and} \quad h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) Q is univalent starlike in $\mathbb{U}$ and

(ii) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{U})$.

If q is analytic in $\mathbb{U}$ with $q(0) = g(0)$, $q(\mathbb{U}) \subset \mathbb{D}$ and:

$$\theta(q(z)) + zq'(z)\varphi(q(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z) \quad (z \in \mathbb{U}), \tag{5}$$

then $q(z) \prec g(z)$ $(z \in \mathbb{U})$. The function g is the best dominant of (5).

Lemma 2 (See [19])**.** *Let $p(z)$ be an analytic function in $\mathbb{U}$ of the form*:

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n, \quad c_m \neq 0, \quad m \geq 1,$$

with $p(z) \neq 0$ in $\mathbb{U}$. If there exists a point z_0, $|z_0| < 1$, such that:

$$|\arg\{p(z)\}| < \frac{\pi}{2}$$

for $|z| < |z_0|$ and:

$$|\arg\{p(z_0)\}| = \frac{\pi}{2},$$

then:

$$\frac{z_0 p'(z_0)}{p(z_0)} = il,$$

where:

$$l \geq \frac{m}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg\{p(z_0)\} = \frac{\pi}{2}$$

and:

$$l \leq -\frac{m}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \quad \arg\{p(z_0)\} = -\frac{\pi}{2},$$

where $p(z_0) = \pm ia$, $a > 0$.

2. Main Results

Theorem 1. *Let* $\lambda_0, \lambda, \beta, a \in \mathbb{R}$ *satisfy* $\lambda \geq 0$, $\lambda_0 a \geq 0$, $0 < \beta \leq 1$ *and* $|a| \leq \frac{1}{\beta}$. *If* q *is analytic in* $\mathbb{U}$ *with* $q(0) = 1$ *and satisfies*:

$$\lambda_0 (q(z))^a + \lambda q(z) + \frac{zq'(z)}{q(z)} \prec h(z) \quad (z \in \mathbb{U}), \tag{6}$$

where:

$$h(z) = \lambda_0 \left(\frac{1+z}{1-z}\right)^{a\beta} + \lambda \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\beta z}{1-z^2} \tag{7}$$

is (close-to-convex) univalent in $\mathbb{U}$*, then:*

$$|\arg\{q(z)\}| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}). \tag{8}$$

The bound β *in (8) is sharp for the function* q *defined by:*

$$q(z) = \left(\frac{1+z}{1-z}\right)^{\beta}. \tag{9}$$

Proof. We choose:

$$g(z) = \left(\frac{1+z}{1-z}\right)^{\beta}, \quad \theta(w) = \lambda_0 w^a + \lambda w \quad \text{and} \quad \varphi(w) = \frac{1}{w}$$

in Lemma 1. Then, the function g is analytic and convex univalent in $\mathbb{U}$ and:

$$|\arg\{g(z)\}| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}). \tag{10}$$

It is clear that φ and θ are analytic in a domain $\mathbb{D}$, which contains $g(\mathbb{U})$ and $q(\mathbb{U})$ with $\varphi(w) \neq 0$ for $w \in g(\mathbb{U})$. The function Q given by:

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{2\beta z}{1-z^2}$$

is univalent starlike. Further, we have:

$$\begin{aligned} \theta(g(z)) + Q(z) &= \lambda_0 \left(\frac{1+z}{1-z}\right)^{a\beta} + \lambda \left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\beta z}{1-z^2} \\ &= h(z), \end{aligned}$$

and so:

$$\begin{aligned}\frac{zh'(z)}{Q(z)} &= \frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)} \\ &= \lambda_0 a(g(z))^a + \lambda g(z) + \frac{zQ'(z)}{Q(z)}.\end{aligned} \tag{11}$$

Furthermore, for $|a| \leq \frac{1}{\beta}$, we find that:

$$|\arg\{(g(z))^a\}| = \frac{|a|\beta\pi}{2} \leq \frac{\pi}{2} \quad (z \in \mathbb{U}). \tag{12}$$

Therefore, it follows from (10)–(12) that:

$$\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

The other conditions of Lemma 1 are also satisfied. Hence, we conclude that:

$$q(z) \prec g(z) = \left(\frac{1+z}{1-z}\right)^\beta \quad (z \in \mathbb{U})$$

and the function g is the best dominant of (6).

Furthermore, for the function q defined by (9), we have:

$$\lambda_0(q(z))^a + \lambda q(z) + \frac{zq'(z)}{q(z)} = h(z)$$

and it follows that the bound β in (8) is sharp. The proof of Theorem 1 is completed. □

Theorem 2. *Let $\alpha > 0$, $0 < \beta < 1$ and $\delta > 0$. If $f \in \mathcal{A}(p)$ satisfies $f(z)f'(z) \neq 0$ $(0 < |z| < 1)$ and:*

$$\left|\arg\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) + \left(\frac{zf'(z)}{pf(z)}\right)^\delta\right\}\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}), \tag{13}$$

then:

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\beta\pi}{2\delta} \quad (z \in \mathbb{U}). \tag{14}$$

In particular, if $\delta \geq 1$, then f is p-valently strongly starlike of order $\frac{\beta}{\delta}$. The bound $\frac{\beta\pi}{2}$ in (13) is the largest number such that (14) holds true.

Proof. One can see that the condition (13) is a generalization of the condition (4). For $f \in \mathcal{A}(p)$ satisfying $f(z)f'(z) \neq 0$ $(0 < |z| < 1)$, we define the function $p(z)$ by:

$$p(z) = \left(\frac{zf'(z)}{pf(z)}\right)^\delta \quad (z \in \mathbb{U}). \tag{15}$$

Then, $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$. The condition (13) becomes:

$$\left|\arg\left\{p(z) + \frac{\alpha}{\delta}\frac{zp'(z)}{p(z)}\right\}\right| < \frac{\beta\pi}{2} \quad (z \in \mathbb{U}). \tag{16}$$

Putting:

$$\lambda_0 = 0 \quad \text{and} \quad \lambda = \frac{\delta}{\alpha}$$

in Theorem 1 and using (16), we find that if:

$$\begin{aligned}
&\frac{\alpha}{\delta}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\left(\frac{zf'(z)}{pf(z)}\right)^{\delta}\right\} \\
&=\frac{\delta}{\alpha}p(z)+\frac{zp'(z)}{p(z)}\prec h(z), \qquad (17)
\end{aligned}$$

where:

$$h(z)=\frac{\delta}{\alpha}\left(\frac{1+z}{1-z}\right)^{\beta}+\frac{2\beta z}{1-z^2} \qquad (18)$$

is (close-to-convex) univalent in $\mathbb{U}$, then (14) is true.

Letting $0<\theta<\pi$ and $x=\cot\frac{\theta}{2}$, we deduce that:

$$\begin{aligned}
\arg\{h(e^{i\theta})\} &= \arg\left\{\frac{\delta}{\alpha}\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^{\beta}+\frac{2\beta e^{i\theta}}{1-e^{2i\theta}}\right\} \\
&= \arg\left\{\frac{\delta}{\alpha}x^{\beta}e^{\frac{\beta\pi i}{2}}+\frac{\beta i}{2}\left(x+\frac{1}{x}\right)\right\} \\
&= \arctan\left\{\frac{\delta x^{\beta}\sin\left(\frac{\beta\pi}{2}\right)+\frac{\alpha\beta}{2}\left(x+\frac{1}{x}\right)}{\delta x^{\beta}\cos\left(\frac{\beta\pi}{2}\right)}\right\}\geq\frac{\beta\pi}{2}. \qquad (19)
\end{aligned}$$

Hence, in view of $h(e^{-i\theta})=\overline{h(e^{i\theta})}$, we deduce from (19) that $h(\mathbb{U})$ contains the sector $|\arg w|<\frac{\beta\pi}{2}$. Consequently, if $f\in\mathcal{A}(p)$ satisfies (13), then the subordination (17) holds true.

For the function f defined by:

$$f(z)=\exp\left(p\int_0^z\frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\frac{\beta}{\delta}}dt\right)\in\mathcal{A}(p),$$

we find after some computations that f satisfies (14) and:

$$\frac{\alpha}{\delta}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\left(\frac{zf'(z)}{pf(z)}\right)^{\delta}\right\}=h(z),$$

which shows that the bound $\frac{\beta\pi}{2}$ in (13) is the largest number such that (14) holds true. The proof of Theorem 2 is completed. □

Theorem 3. *Let $\delta>0$ and $\alpha\in\mathbb{C}$. Assume that $-\frac{\pi}{2}<\varphi=\arg\{\alpha\}\leq 0$. If $f\in\mathcal{A}(p)$ satisfies $f(z)f'(z)\neq 0$ $(0<|z|<1)$ and:*

$$-\frac{\pi}{2}+\arctan\left\{\frac{|\alpha|\sin\varphi}{2\delta+|\alpha|\cos\varphi}\right\}<\arg\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\left(\frac{zf'(z)}{pf(z)}\right)^{\delta}\right\}<\frac{\pi}{2}+\varphi \qquad (20)$$

for $z\in\mathbb{U}$, then:

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{2\delta}\quad(z\in\mathbb{U}).$$

In particular, if $\delta\geq 1$, then f is p-valently strongly starlike of order $\frac{1}{\delta}$.

Proof. Define the function $p(z)$ by (15). Then, the condition (20) becomes:

$$-\frac{\pi}{2}+\arctan\left\{\frac{|\alpha|\sin\varphi}{2\delta+|\alpha|\cos\varphi}\right\}<\arg\left\{p(z)+\frac{\alpha}{\delta}\frac{zp'(z)}{p(z)}\right\}<\frac{\pi}{2}+\varphi \quad (z\in\mathbb{U}). \tag{21}$$

We want to prove that:

$$|\arg\{p(z)\}|<\frac{\pi}{2} \quad (z\in\mathbb{U}). \tag{22}$$

If there exists a point z_0 $(|z_0|<1)$ such that:

$$|\arg\{p(z)\}|<\frac{\pi}{2} \quad (|z|<|z_0|)$$

and:

$$|\arg\{p(z_0)\}|=\frac{\pi}{2},$$

then from Lemma 2, we have:

$$\frac{z_0p'(z_0)}{p(z_0)}=il,$$

where $p(z_0)=\pm ai$, $a>0$ and:

$$l\geq\frac{m}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad \arg\{p(z_0)\}=\frac{\pi}{2}$$

and:

$$l\leq-\frac{m}{2}\left(a+\frac{1}{a}\right) \quad \text{when} \quad \arg\{p(z_0)\}=-\frac{\pi}{2}.$$

For the case $\arg\{p(z_0)\}=-\frac{\pi}{2}$, we have $l<0$ and:

$$\begin{aligned}
&\arg\left\{p(z_0)+\frac{\alpha}{\delta}\frac{z_0p'(z_0)}{p(z_0)}\right\}=\arg\left\{-ai+\frac{l\alpha}{\delta}i\right\}=-\frac{\pi}{2}+\arg\left\{a-\frac{l\alpha}{\delta}\right\}\\
&=-\frac{\pi}{2}+\arctan\left\{\frac{\operatorname{Im}\left(a-\frac{l\alpha}{\delta}\right)}{\operatorname{Re}\left(a-\frac{l\alpha}{\delta}\right)}\right\}=-\frac{\pi}{2}+\arctan\left\{\frac{-l|\alpha|\sin\varphi}{a\delta-l|\alpha|\cos\varphi}\right\}\\
&\leq-\frac{\pi}{2}+\arctan\left\{\frac{\frac{1}{2}\left(a+\frac{1}{a}\right)|\alpha|\sin\varphi}{a\delta+\frac{1}{2}\left(a+\frac{1}{a}\right)|\alpha|\cos\varphi}\right\}\\
&\leq-\frac{\pi}{2}+Q(\alpha,\varphi),
\end{aligned} \tag{23}$$

where:

$$Q(\alpha,\varphi)=\max_{a>0}\left\{\arctan\left(\frac{|\alpha|(a^2+1)\sin\varphi}{(2\delta+|\alpha|\cos\varphi)a^2+|\alpha|\cos\varphi}\right)\right\}.$$

The function:

$$g(a)=\frac{|\alpha|(a^2+1)\sin\varphi}{(2\delta+|\alpha|\cos\varphi)a^2+|\alpha|\cos\varphi}, \quad a>0$$

has a positive derivative:

$$g'(a)=\frac{-4a\delta|\alpha|\sin\varphi}{((2\delta+|\alpha|\cos\varphi)a^2+|\alpha|\cos\varphi)^2}\geq 0 \quad \text{for} \quad -\frac{\pi}{2}<\varphi\leq 0,$$

hence:

$$Q(\alpha,\varphi)=\lim_{a\to\infty}\arctan\{g(a)\}=\arctan\frac{|\alpha|\sin\varphi}{2\delta+|\alpha|\cos\varphi}.$$

Therefore, (23) becomes:

$$\arg\left\{p(z_0)+\frac{\alpha}{\delta}\frac{zp'(z_0)}{p(z_0)}\right\} \le -\frac{\pi}{2}+\arctan\left\{\frac{|\alpha|\sin\varphi}{2\delta+|\alpha|\cos\varphi}\right\},$$

which contradicts (21). Thus,

$$|\arg\{p(z)\}| < \frac{\pi}{2} \quad (z\in\mathbb{U}).$$

For the case $\arg\{p(z_0)\}=\frac{\pi}{2}$, applying the same method as the above, we have $l>0$ and:

$$\begin{aligned}\arg\left\{p(z_0)+\frac{\alpha}{\delta}\frac{z_0p'(z_0)}{p(z_0)}\right\} &= \arg\left\{ai+\frac{l\alpha}{\delta}i\right\}\\ = \frac{\pi}{2}+\arg\left\{a+\frac{l\alpha}{\delta}\right\} &\ge \frac{\pi}{2}+\varphi.\end{aligned}$$

This contradicts (21). Now, the proof of Theorem 3 is completed. □

Remark 1. *For $\varphi=0$ and $\delta=1$, Theorem 3 becomes the known result in [17] that every α-convex function is starlike.*

Applying the same method as the above, we can prove the following theorem.

Theorem 4. *Let $\delta>0$ and $\alpha\in\mathbb{C}$. Assume that $0\le\varphi=\arg\{\alpha\}<\frac{\pi}{2}$. If $f\in\mathcal{A}(p)$ satisfies $f(z)f'(z)\neq 0$ $(0<|z|<1)$ and:*

$$-\frac{\pi}{2}+\varphi<\arg\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\left(\frac{zf'(z)}{pf(z)}\right)^{\delta}\right\}<\frac{\pi}{2}+\arctan\left\{\frac{|\alpha|\sin\varphi}{2\delta+|\alpha|\cos\varphi}\right\} \quad (24)$$

for $z\in\mathbb{U}$, then:

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi}{2\delta} \quad (z\in\mathbb{U}).$$

In particular, if $\delta\ge 1$, then f is p-valently strongly starlike of order $\frac{1}{\delta}$.

Theorem 5. *Theorem 5. Let $0<\alpha<1$, $0<\beta<1$ and $\beta<\delta\le 1$. If $f\in\mathcal{A}(p)$ satisfies $f(z)f'(z)\neq 0$ $(0<|z|<1)$ and:*

$$\left|\arg\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{pf(z)}\right)^{\delta}\right\}\right|<\frac{\beta\pi}{2} \quad (z\in\mathbb{U}), \quad (25)$$

then:

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\beta\pi}{2\delta} \quad (z\in\mathbb{U}), \quad (26)$$

or f is p-valently strongly starlike of order $\frac{\beta}{\delta}$. The bound $\frac{\beta\pi}{2}$ in (25) is the largest number such that (26) holds true.

Proof. It is obvious that the condition (25) is a generalization of the condition (4). Defining the function $p(z)$ by (15), the condition (25) becomes:

$$\left|\arg\left\{p\alpha(p(z))^{\frac{1}{\delta}}+(1-\alpha)p(z)+\frac{\alpha}{\delta}\frac{zp'(z)}{p(z)}\right\}\right|<\frac{\beta\pi}{2} \quad (z\in\mathbb{U}). \quad (27)$$

Setting:

$$a = \frac{1}{\delta}, \quad \lambda_0 = p\delta \quad \text{and} \quad \lambda = \frac{\delta(1-\alpha)}{\alpha}$$

in Theorem 1 and using (27), we see that if:

$$\begin{aligned}
&\frac{\alpha}{\delta}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\alpha)\left(\frac{zf'(z)}{pf(z)}\right)^{\delta}\right\} \\
&= p\delta(p(z))^{\frac{1}{\delta}} + \frac{\delta(1-\alpha)}{\alpha}p(z) + \frac{zp'(z)}{p(z)} \prec h(z),
\end{aligned} \tag{28}$$

where:

$$h(z) = p\delta\left(\frac{1+z}{1-z}\right)^{\frac{\beta}{\delta}} + \frac{\delta(1-\alpha)}{\alpha}\left(\frac{1+z}{1-z}\right)^{\beta} + \frac{2\beta z}{1-z^2} \tag{29}$$

is (close-to-convex) univalent in $\mathbb{U}$, then (26) holds true.

Letting $0 < \theta < \pi$ and $x = \cot\frac{\theta}{2}$, we have:

$$h(e^{i\theta}) = p\delta x^{\frac{\beta}{\delta}} e^{\frac{\beta\pi i}{2\delta}} + \frac{\delta(1-\alpha)}{\alpha}x^{\beta}e^{\frac{\beta\pi i}{2}} + \frac{\beta i}{2}\left(x+\frac{1}{x}\right)$$

and:

$$\arg\left\{h(e^{i\theta})\right\} = \arctan\left\{\frac{p\delta x^{\frac{\beta}{\delta}}\sin\left(\frac{\beta\pi}{2\delta}\right) + \frac{\delta(1-\alpha)}{\alpha}x^{\beta}\sin\left(\frac{\beta\pi}{2}\right) + \frac{\beta}{2}\left(x+\frac{1}{x}\right)}{p\delta x^{\frac{\beta}{\delta}}\cos\left(\frac{\beta\pi}{2\delta}\right) + \frac{\delta(1-\alpha)}{\alpha}x^{\beta}\cos\left(\frac{\beta\pi}{2}\right)}\right\}.$$

For $x > 0$, $0 < \alpha < 1$ and $0 < \frac{\beta}{\delta} < 1$, we deduce that:

$$\begin{aligned}
\arg\left\{h(e^{i\theta})\right\} &\geq \arctan\left\{\frac{p\delta x^{\frac{\beta}{\delta}}\sin\left(\frac{\beta\pi}{2\delta}\right) + \frac{\delta(1-\alpha)}{\alpha}x^{\beta}\sin\left(\frac{\beta\pi}{2}\right)}{p\delta x^{\frac{\beta}{\delta}}\cos\left(\frac{\beta\pi}{2\delta}\right) + \frac{\delta(1-\alpha)}{\alpha}x^{\beta}\cos\left(\frac{\beta\pi}{2}\right)}\right\} \\
&= \arctan\left\{\tan\left(\frac{\beta\pi}{2}\right)\frac{p\delta x^{\frac{\beta}{\delta}}\cos\left(\frac{\beta\pi}{2\delta}\right)\tan\left(\frac{\beta\pi}{2\delta}\right)\cot\left(\frac{\beta\pi}{2}\right) + \frac{\delta(1-\alpha)}{\alpha}x^{\beta}\cos\left(\frac{\beta\pi}{2}\right)}{p\delta x^{\frac{\beta}{\delta}}\cos\left(\frac{\beta\pi}{2\delta}\right) + \frac{\delta(1-\alpha)}{\alpha}x^{\beta}\cos\left(\frac{\beta\pi}{2}\right)}\right\} \\
&\geq \frac{\beta\pi}{2},
\end{aligned} \tag{30}$$

since:

$$\tan\left(\frac{\beta\pi}{2\delta}\right)\cot\left(\frac{\beta\pi}{2}\right) \geq \tan\left(\frac{\beta\pi}{2}\right)\cot\left(\frac{\beta\pi}{2}\right) = 1 \quad \text{for} \quad \frac{1}{\delta} \geq 1.$$

In view of the proof of Theorem 2, we find from (30) that $h(\mathbb{U})$ contains the sector $|\arg w| < \frac{\beta\pi}{2}$. Hence, if $f \in \mathcal{A}(p)$ satisfies (25), then the subordination (28) holds true.

The sharpness part of the proof is similar to that in the proof of Theorem 2, and so, we omit it. The proof of Theorem 5 is completed. □

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