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Application of Fixed Point Results on Rational F-Contraction Mappings to Solve Boundary Value Problems

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Abstract: The notion of rational F-contractions using α-admissibility of type-S in b-metric-like spaces is introduced and the new fixed and periodic point theorems are proved for such mappings. Numerical examples are illustrated to check the efficiency and applicability of our fresh findings. It is also observed that some of the works reported in the literature are the particular cases of the present study.

Keywords: b-metric-like space; admissibility type-S; rational F-contraction; periodic point; boundary value problem

1. Introduction

The notion of F-contraction mapping was introduced by Wardowski [1] in fixed point theory and proved the related results. These results are the generalization of Banach contraction mapping principle as well as various fixed point theorems appearing in the literature, for instance [2]. On the other hand, Alghamdi et al. [3] found existence and uniqueness of fixed points for the mappings in b-metric-like and partially ordered b-metric-like spaces.

The notion of α-admissible maps was introduced that provided a beautiful class of mapping by Samet et al. [4] to observe the existence as well as uniqueness of fixed point. Using the same concept or slight modifications, a lot of work has been done in that direction. Sintunavarat [5] introduced the concept of α-admissible type-S in partial b-metric space and derived based fixed point results.

In the present paper, we introduce different types of rational F-contraction with α-admissibility type-S and examine the existence and uniqueness of fixed points in b-metric-like spaces.

Throughout this paper, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ are denoted as real numbers, nonnegative real numbers and positive integers, respectively.

2. Prerequisites

Definition 1 ([1]). *Let $\mathcal{F}$ be the family of all functions $F\colon (0,\infty) \to \mathbb{R}$ such that*

(F_1) *F is strictly increasing, i.e., for all $u, v \in \mathbb{R}_+$ such that $u < v$, $F(u) < F(v)$;*

(F_2) *for each sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *of positive numbers,* $\lim_{\alpha \to 0^+} \alpha_n = 0$ *if and only if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F_3) *there exists* $k \in (0,1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2 ([1])**.** *Suppose* (U,d) *is a metric space. The mapping* $T\colon U \to U$ *is said to be F-contraction on* (U,d) *if there exist* $T \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\forall\, u,v \in U, [d(Tu,Tv) > 0 \Rightarrow \tau + F(d(Tu,Tv)) \leq F(d(u,v))]. \tag{1}$$

Definition 3. *Let* (U,d) *be a metric space. A mapping* $T\colon U \to U$ *is said to be an F-weak contraction on* (U,d) *if there exist* $T \in \mathcal{F}$ *and* $\tau > 0$ *such that* $\forall\, u,v \in U$ *with* $Tu \neq Tv$

$$\tau + F(d(u,Tu)) \leq \left(F \max\left\{ d(u,v), d(u,Tu)d(v,Tv)\frac{d(u,Tv)+d(v,Tu)}{2} \right\}\right). \tag{2}$$

Definition 4 ([6])**.** *Let* U *be a nonempty set, let* $k \geq 1$ *be a given real number. A function* $d : U \times U \to [0,\infty)$ *is called a b-metric if the following conditions hold:* $\forall\, u,v,w \in U$.

(S_1) $d(u,v) = 0$ *if and only if* $u = v$,
(S_2) $d(u,v) = d(v,u)$,
(S_3) $d(u,v) \leq k[d(u,w) + d(z,w)]$.

Then, (U,d) *is said to be a b-metric space.* $k \geq 1$ *is the coefficient of* (U,d).

Definition 5 ([7])**.** *Let* U *be a nonempty set, a mapping* $\sigma : U \times U \to \mathbb{R}_+$ *such that* $\forall\, u,v,w \in U$

(σ_1) $\sigma(u,v) = 0$ *implies* $u = v$,
(σ_2) $\sigma(u,v) = \sigma(v,u)$,
(σ_3) $\sigma(u,v) \leq \sigma(u,w) + \sigma(w,v)$.

Then (U,σ) *is said to be a metric-like space.*

Definition 6 ([3])**.** *Let* U *be a nonempty set and a real number* $k \geq 1$ *be given. A function* $\sigma_b : U \times U \to \mathbb{R}_+$ *such that the following assertions hold* $\forall\, u,v,w \in U$:

(σ_{b1}) $\sigma_b(u,v) = 0$ *implies* $u = v$,
(σ_{b2}) $\sigma_b(u,v) = \sigma_b(v,u)$,
(σ_{b3}) $\sigma_b(u,v) \leq k[\sigma_b(u,w) + \sigma_b(w,v)]$.

Then, (U,σ_b) *is said to be a b-metric-like space.*

Ref. [8] recommended that the converses of the below facts need not be held.

- Let U be a nonempty set and σ_b is b-metric-like on U such that the pair (U,σ_b) be a b-metric-like space.
- In a b-metric-like space (U,σ_b), if $u,v \in U$ and $\sigma_b(u,v) = 0$, then $u = v$, and $\sigma_b(u,u)$ may be positive for $u \in U$.
- It can be easily observed that every b-metric and partial b-metric spaces are b-metric-like spaces with the same k.

Every b-metric-like σ_b on U generates a topology τ_{σ_b} whose base is the family of all open σ_b-balls $\{B_{\sigma_b}(u,\delta) : u \in U, \delta > 0\}$, where $\{B_{\sigma_b}(u,\delta) = \{v \in U : |\sigma_b(u,v) - \sigma_b(u,u)| < \delta\}, \forall\, u \in U$ and $\delta > 0$.

Definition 7 ([3])**.** *Let* (U,σ_b) *be a b-metric-like space with coefficient k, let* $\{u_n\}$ *be any sequence in* U *and* $u \in U$. *Then,*

(i) $\{u_n\}$ *is called convergent to u w.r.t.* τ_{σ_b}, *if* $\lim_{n\to\infty} \sigma_b(u_n,u) = \sigma_b(u,u)$,

(ii) $\{u_n\}$ *is called Cauchy sequence in* (U, σ_b) *if* $\lim_{n,m\to\infty} \sigma_b(u_n, u_m)$ *exists (it is finite),*
(iii) (U, σ_b) *is called complete b-metric-like space if, for every Cauchy sequence* $\{u_n\}$ *in* U*, there exists* $u \in U$ *such that*

$$\lim_{n,m\to\infty} \sigma_b(u_n, u_m) = \lim_{n\to\infty} \sigma_b(u_n, u) = \sigma_b(u, u). \tag{3}$$

It can be noted that the limit of a sequence may not be unique in b-metric-like spaces.

Let us discuss the notion of b-convergence, b-Cauchy sequence, b-continuity and b-completeness in b-metric-like spaces.

Definition 8 ([9])**.** *Let* (U, σ_b) *be a b-metric-like space. Then, a sequence* $\{u_n\}$ *in* U *is called*

(a) *b-convergent if there exists* $u \in U$ *such that* $\sigma_b(u_n, u) \to 0$ *as* $n \to \infty$*. In this case, we write* $\lim_{n\to\infty} u_n = u$*,*
(b) *b-Cauchy if* $\sigma_b(u_n, u_m) \to 0$ *as* $n, m \to \infty$*.*

Each b-convergent sequence is b-Cauchy with a unique limit in b-metric-like spaces. The following lemma is necessary to prove main results.

Lemma 1 ([9])**.** *Let* (U, σ_b) *be a b-metric-like space with coefficient* $s \geq 1$ *and let* $\{u_n\}$ *and* $\{v_n\}$ *be b-convergent to points* $u, v \in U$*, respectively. Then,*

$$\frac{1}{s^2}\sigma_b(u, v) \leq \liminf_{n\to\infty} \sigma_b(u_n, v_n) \leq \limsup_{n\to\infty} \sigma_b(u_n, v_n) \leq s^2\sigma_b(u, v).$$

In particular, if $u = v$*, then* $\lim_{n\to\infty} \sigma_b(u_n, v_n) = 0$*. Moreover, for each* $z \in U$*, we have*

$$\frac{1}{s}\sigma_b(u, z) \leq \liminf_{n\to\infty} \sigma_b(u_n, z) \leq \limsup_{n\to\infty} \sigma_b(u_n, z) \leq s\sigma_b(u, z).$$

Remark 1 ([10])**.** *Let* $(U; \sigma_b)$ *be a b-metric-like space and let* $T : U \to U$ *be a continuous mapping. Then,*

$$\lim_{n\to+\infty} \sigma_b(u_n, u) = \sigma_b(u, u) \Rightarrow \lim_{n\to+\infty} \sigma_b(Tu_n, Tu) = \sigma_b(Tu, Tu).$$

Definition 9 ([10])**.** *Let* (U, σ_U) *and* (V, σ_V) *be two b-metric-like spaces.*

(1) *The space* (U, σ_U) *is b-complete if every b-Cauchy sequence in* U *is b-converges.*
(2) *A function* $T : U \to V$ *is b-continuous at a point* $u \in U$ *if it is b-sequentially continuous at* u*, that is, whenever* $\{u_n\}$ *is b-convergent to* u*,* $\{Tu_n\}$ *is b-convergent to* Tu*.*

Many papers related to fixed point results in b-metric-like spaces appear in literature, some of them are [3,7,10–13] and references therein.

The idea of α-admissibility was studied by [4] for the first time. After that, Ref. [5] extended this concept as α-admissibility type-S in the light of metric spaces and b-metric spaces, respectively.

Definition 10 ([4,5])**.** *For a nonempty set* U*, let* $\alpha : U \times U \to [0, \infty)$ *and* $f : U \to U$ *are mappings. Then,* $\forall u, v \in U$*,*

(i) *we say that the mapping* T *is* α*-admissible mapping if*

$$\alpha(u, v) \geq 1 \Rightarrow \alpha(Tu, Tv) \geq 1$$

and is denoted by the symbol $\mathcal{P}(U, \alpha)$*.*
(ii) *we say that the mapping* T *is* α*-admissible mapping of type* S *if*

$$\alpha(u, v) \geq s \Rightarrow \alpha(Tu, Tv) \geq s$$

and is denoted by the symbol $\mathcal{P}_s(U, \alpha)$, *where* $s \geq 1$.

Definition 11 ([5]). *Let* U *be a nonempty set. Suppose that* $\alpha : U \times U \to [0, \infty)$ *and* $T \colon U \to U$ *are mappings. Then,* $\forall u \in U$.

(i) *T is said to be a weak* α*-admissible mapping if*

$$\alpha(u, Tu) \geq 1 \Rightarrow \alpha(Tu, TTu) \geq 1,$$

and is denoted by $\mathcal{WP}(U, \alpha)$

(ii) *T is said to be a weak* α*-admissible mapping of type S if*

$$\alpha(u, Tu) \geq s \Rightarrow \alpha(Tu, TTu) \geq s,$$

and is denoted by $\mathcal{WP}_s(U, \alpha)$, *where* $s \geq 1$.

Ref. [5] presented some examples to show that the class of [] α-admissible mappings and the class of α-admissible mappings of type S are independent; that is, $\mathcal{P}(X, \alpha) \neq \mathcal{P}_s(U, \alpha)$.

Remark 2 ([5]). *It is easy to see that the following assertions hold:*

(i) α*-admissibility* $\Rightarrow$ *weak* α*-admissibility, that is,*

$$\mathcal{P}(U, \alpha) \subseteq \mathcal{WP}(U, \alpha),$$

(ii) α*-admissibility type S* $\Rightarrow$ *weak* α*-admissibility of type S, that is,*

$$\mathcal{P}_s(U, \alpha) \subseteq \mathcal{WP}_s(U, \alpha).$$

3. Results

In this section, we investigate some fixed point results for rational F-contractions mapping with α-admissibility type-S and for the classes of $\mathcal{WP}_s(U, \alpha)$ and $\mathcal{P}_s(U, \alpha)$:

$$Fix(T) := \{u \in U | Tu = u\}.$$

In addition, for each elements u and v in a b-metric-like space (U, σ_b) with coefficient $s \geq 1$. Let

$$F(\Delta_s(u,v)) = \max \left\{ \begin{array}{c} \sigma_b(u,v), \sigma_b(u,Tu), \sigma_b(v,Tv), \dfrac{\sigma_b(u,Tv) + \sigma_b(v,Tu)}{2s}, \\ \dfrac{\sigma_b(u,Tu)\sigma_b(v,Tv)}{1 + s[\sigma_b(u,v) + \sigma_b(u,Tv) + \sigma_b(v,Tu)]}, \\ \dfrac{\sigma_b(u,Tv)\sigma_b(u,v)}{1 + s\sigma_b(u,Tu) + s^3[\sigma_b(v,Tu) + \sigma_b(v,Tv)]} \end{array} \right\}, \tag{4}$$

where T is a self-mapping on U, we write $\Delta(u,v)$ instead of $\Delta_s(u,v)$ when $s = 1$, i.e.,

$$F(\Delta(u,v)) = \max \left\{ \begin{array}{c} \sigma_b(u,v), \sigma_b(u,Tu), \sigma_b(v,Tv), \dfrac{\sigma_b(u,Tv) + \sigma_b(v,Tu)}{2}, \\ \dfrac{\sigma_b(u,Tu)\sigma_b(v,Tv)}{1 + \sigma_b(u,v) + \sigma_b(u,Tv) + \sigma_b(v,Tu)}, \\ \dfrac{\sigma_b(u,Tv)\sigma_b(u,v)}{1 + \sigma_b(u,Tu) + \sigma_b(v,Tu) + \sigma_b(v,Tv)} \end{array} \right\}. \tag{5}$$

Definition 12. *Let (U, σ_b) be a b-metric-like space with coefficient $s \geq 1$, let $\alpha : U \times U \to [0, \infty)$ be given mappings. Then, $T : U \to U$ is called rational F-contraction if the following condition holds:*

$$u, v \in U \text{ with } \alpha(u, v) \geq s \text{ and } \sigma_b(Tu, Tv) > 0 \Rightarrow \tau + F(s^3 \sigma_b(Tu, Tv)) \leq F(\Delta_s(u, v)). \tag{6}$$

We denote by $\Theta_s(U, \alpha, F)$ the collection of all rational F-contractions on a b-metric-like space (U, σ_b) with coefficient $s \geq 1$.

Theorem 1. *Let (U, σ_b) be a b-complete b-metric-like space with coefficient $s \geq 1$, let $\alpha : U \times U \to [0, \infty)$, and $T : U \to U$ be given mappings. Suppose that the following conditions hold:*

(S_1) *$T \in \Theta_s(U, \alpha, F) \cap \mathcal{WS}_s(U, \alpha)$,*
(S_2) *there exists $u_0 \in U$ such that $\alpha(u_0, Tu_0) \geq s$,*
(S_3) *α has a transitive property type S, that is, for $u, v, w \in U$,*

$$\alpha(u, v) \geq s \text{ and } \alpha(v, w) \geq s \Rightarrow \alpha(u, w) \geq s.$$

(S_4) *T is b-continuous.*

Then, $Fix(T) \neq \emptyset$.

Proof. By the given condition (S_2), there exists $u_0 \in U$ such that

$$\alpha(u_0, Tu_0) \geq s.$$

Define the sequence $\{u_n\}$ by $u_{n+1} = Tu_n$. If there exists $n_0 \in \mathbb{N}$, such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} \in Fix(T)$ and hence the proof is completed. Thus, we assume that $u_n \neq u_{n+1}$, for all $n \in \mathbb{N}$.

It follows that

$$\sigma_b(u_n, u_{n+1}) > 0, \ \forall\, n \in \mathbb{N}.$$

Hence, we have

$$\frac{1}{2s} \sigma_b(u_n, Tu_n) < \sigma_b(u_n, Tu_n), \ \forall\, n \in \mathbb{N}. \tag{7}$$

Now, we need to prove that

$$\lim_{n \to \infty} \sigma_b(u_n, u_{n+1}) = 0. \tag{8}$$

It follows from $T \in \mathcal{WS}_s(U, \alpha)$ and $\alpha(u_0, Tu_0) \geq s$ that

$$\alpha(u_1, u_2) = \alpha(Tu_0, TTu_0) \geq s. \tag{9}$$

By induction, we obtain

$$\alpha(u_{n+1}, u_{n+2}) \geq s. \tag{10}$$

As we have

$$F(\sigma_b(u_{n+1}, u_{n+2})) = F(\sigma_b(Tu_n, Tu_{n+1})) \leq F(s^3 \sigma_b(Tu_n, u_{n+1})), \ \forall\, n \in \mathbb{N}, \tag{11}$$

it follows from $T \in \Theta_s(U, \alpha, F)$ that the inequalities (6) and (11) imply that

$$\tau + F(\sigma_b(Tu_n, Tu_{n+1})) \leq \tau + F(s^3 \sigma_b(Tu_n, Tu_{n+1})) \leq F(\Delta_s(u_n, u_{n+1})) \tag{12}$$

for all $n \in \mathbb{N}$. Note that, for each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&F(\Delta_s(u_n, u_{n+1})) \\
&= F\left(\max\left\{\begin{array}{c}
\sigma_b(u_n, Tu_n), \sigma_b(Tu_n, T^2u_n), \dfrac{\sigma_b(u_n, T^2u_n) + \sigma_b(Tu_n, Tu_n)}{2s}, \\
\dfrac{\sigma_b(u_n, Tu_n)\sigma_b(Tu_n, T^2u_n)}{1 + s[\sigma_b(u_n, Tu_n) + \sigma_b(u_n, T^2u_n) + \sigma_b(Tu_n, Tu_n)]}, \\
\dfrac{\sigma_b(u_n, T^2u_n)\sigma_b(u_n, Tu_n)}{1 + s\sigma_b(u_n, Tu_n) + s^3[\sigma_b(Tu_n, Tu_n) + \sigma_b(Tu_n, T^2u_n)]}
\end{array}\right\}\right) \\
&= F\left(\max\left\{\begin{array}{c}
\sigma_b(u_n, Tu_n), \sigma_b(Tu_n, T^2u_n), \\
\dfrac{\sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n) + \sigma_b(Tu_n, u_n) + \sigma_b(u_n, Tu_n)}{2s}, \\
\dfrac{\sigma_b(u_n, Tu_n)\sigma_b(Tu_n, T^2u_n)}{1 + s[\sigma_b(u_n, Tu_n) + \sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n) + \sigma_b(Tu_n, u_n) + \sigma_b(Tu_n, u_n)]}, \\
\dfrac{[\sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n)]\sigma_b(u_n, Tu_n)}{1 + s\sigma_b(u_n, Tu_n) + s^3[\sigma_b(Tu_n, u_n) + \sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n)]}
\end{array}\right\}\right) \\
&= F\left(\max\left\{\begin{array}{c}
\sigma_b(u_n, Tu_n), \sigma_b(Tu_n, T^2u_n), \\
\dfrac{3\sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n)}{2s}, \\
\dfrac{\sigma_b(u_n, Tu_n)\sigma_b(Tu_n, T^2u_n)}{1 + s[4\sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n)]}, \\
\dfrac{[\sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n)]\sigma_b(u_n, Tu_n)}{1 + s\sigma_b(u_n, Tu_n) + s^3[2\sigma_b(u_n, Tu_n) + \sigma_b(Tu_n, T^2u_n)]}
\end{array}\right\}\right) \\
&< F\left(\max\left\{\sigma_b(u_n, Tu_n), \sigma_b(Tu_n, T^2u_n)\right\}\right).
\end{aligned} \tag{13}
$$

If $\Delta_s(u_n, Tu_n) = \sigma_b(Tu_n, T^2u_n)$ for some $n \in \mathbb{N}$, then inequality (12) implies that

$$\tau + F(\sigma_b(Tu_n, T^2u_n)) \leq \tau + F(s^3\sigma_b(Tu_n, T^2u_n)) < F(\sigma_b(Tu_n, T^2u_n)),$$

which contradicts $\tau > 0$. Hence,

$$\Delta_s(u_n, u_{n+1}) = \sigma_b(u_n, Tu_n), \ \ \forall\, n \in \mathbb{N}.$$

From (12), we have

$$
\begin{aligned}
\tau + F(\sigma_b(Tu_n, T^2u_n)) \leq & \tau + F(s^3\sigma_b(Tu_n, T^2u_n)) \\
< & F(\sigma_b(u_n, Tu_n))
\end{aligned} \tag{14}
$$

for all $n \in \mathbb{N}$. Therefore, the above inequality becomes

$$F(\sigma_b(Tu_n, T^2u_n)) < F(\sigma_b(u_n, Tu_n)) - \tau, \tag{15}$$

which is equivalent to

$$F(\sigma_b(u_{n+1}, Tu_{n+1})) < F(\sigma_b(u_n, Tu_n)) - \tau, \ \ \forall\, n \in \mathbb{N}.$$

Iteratively, we find that

$$\begin{aligned} F(\sigma_b(u_n, Tu_n)) &\leq F(\sigma_b(u_{n-1}, Tu_{n-1})) - \tau \\ &\leq F(\sigma_b(u_{n-2}, Tu_{n-2})) - 2\tau \\ &\leq F(\sigma_b(u_{n-3}, Tu_{n-3})) - 3\tau \\ &\vdots \\ &\leq F(\sigma_b(u_0, Tu_0)) - n\tau. \end{aligned} \tag{16}$$

From (16), we obtain $\lim_{n\to\infty} F(\sigma_b(u_n, Tu_n)) = -\infty$, which, together with (F_2), gives

$$\lim_{n\to\infty} \sigma_b(u_n, Tu_n) = \lim_{n\to\infty} \sigma_b(u_n, u_{n+1}) = 0. \tag{17}$$

Using the method of contradiction, let us prove that $\{u_n\}$ is a b-Cauchy sequence in U. Assume that there exists $\epsilon_0 > 0$ and sequences $\{u_{p(k)}\}$ and $\{u_{q(k)}\}$ of $\{u_n\}$ such that $p(k) > q(k) \geq k$ and

$$\sigma_b(u_{p(k)}, u_{q(k)}) \geq \epsilon_0 \tag{18}$$

and $q(k)$ is the smallest number such that (18) holds:

$$\sigma_b(u_{p(k)}, u_{q(k)-1}) < \epsilon_0. \tag{19}$$

By (σ_{b3}), (18) and (19), we get

$$\begin{aligned} \epsilon_0 \leq \sigma_b(u_{p(k)}, u_{q(k)}) &\leq s\sigma_b(u_{p(k)}, u_{q(k)-1}) + s\sigma_b(u_{q(k)-1}, u_{q(k)}) \\ &< s\epsilon_0 + s\sigma_b(u_{q(k)-1}, u_{q(k)}). \end{aligned} \tag{20}$$

Owing to (17), there exists $N_1 \in \mathbb{N}$ such that

$$\sigma_b(u_{p(k)-1}, Tu_{p(k)-1}) < \epsilon_0, \quad \sigma_b(u_{q(k)}, Tu_{q(k)}) < \epsilon_0, \quad \sigma_b(u_{p(k)}, Tu_{p(k)}) < \epsilon_0, \ \forall\, k > N_1, \tag{21}$$

which together with (20) shows

$$\sigma_b(u_{p(k)}, u_{q(k)}) < 2s\epsilon_0, \ \forall\, k > N_1, \tag{22}$$

hence

$$F(\sigma_b(u_{p(k)}, u_{q(k)})) < F(2s\epsilon_0), \ \forall\, k > N_1, \tag{23}$$

From (17), (18) and (21), we get

$$\frac{1}{2s}\sigma_b(u_{p(k)}, Tu_{p(k)}) < \frac{\epsilon_0}{2s} < \sigma_b(u_{p(k)}, u_{q(k)}) \ \forall\, k > N_1. \tag{24}$$

Using the triangular inequality, we deduce that

$$\sigma_b(u_{p(k)}, u_{q(k)}) \leq s\sigma_b(u_{p(k)}, u_{p(k)+1}) + s^2[\sigma_b(u_{p(k)+1}, u_{q(k)+1}) + \sigma_b(u_{q(k)+1}, u_{q(k)})]. \tag{25}$$

Passing to the limit $k \to +\infty$ in (25), by (16) yields

$$\frac{\epsilon_0}{s^2} \leq \limsup_{k\to\infty} \sigma_b(u_{p(k)+1}, u_{q(k)+1}), \tag{26}$$

hence there exists $N_2 \in \mathbb{N}$, such that $\sigma_b(u_{p(k)+1}, u_{q(k)+1}) > 0$ for $k > N_2$,
i.e., $\sigma_b(Tu_{p(k)}, Tu_{q(k)}) > 0$. Using the transitivity property type S of α, we get

$$\alpha(u_{p(k)}, u_{q(k)}) \geq s.$$

Since $T \in \Theta_s(U, \alpha, F)$, we have

$$\begin{aligned} \tau + F(\sigma_b(u_{p(k)+1}, u_{q(k)+1})) &\leq \tau + F(s^3\sigma_b(Tu_{p(k)}, Tu_{q(k)})) \\ &\leq F(\Delta_s(u_{p(k)}, u_{q(k)})). \end{aligned} \tag{27}$$

By (6), (21), (23), and (25), we obtain

$$\begin{aligned} &F(\Delta_s(u_{p(k)}, u_{q(k)})) \\ &= F\left(\max\left\{\begin{array}{c} \sigma_b(u_{p(k)}, u_{q(k)}), \sigma_b(u_{p(k)}, Tu_{p(k)}), \sigma_b(u_{q(k)}, Tu_{q(k)}), \\ \frac{\sigma_b(u_{p(k)}, Tu_{q(k)}) + \sigma_b(u_{q(k)}, Tu_{p(k)})}{2s}, \\ \frac{\sigma_b(u_{p(k)}, Tu_{p(k)})\sigma_b(u_{q(k)}, Tu_{q(k)})}{1 + s[\sigma_b(u_{p(k)}, u_{q(k)}) + \sigma_b(u_{p(k)}, Tu_{q(k)}) + \sigma_b(u_{q(k)}, Tu_{p(k)})]}, \\ \frac{\sigma_b(u_{p(k)}, Tu_{q(k)})\sigma_b(u_{p(k)}, u_{q(k)})}{1 + s\sigma_b(u_{p(k)}, Tu_{p(k)}) + s^3[\sigma_b(u_{q(k)}, Tu_{p(k)}) + \sigma_b(u_{q(k)}, Tu_{q(k)})]} \end{array}\right\}\right) \\ &\leq F\left(\max\left\{\begin{array}{c} \sigma_b(u_{p(k)}, u_{q(k)}), \sigma_b(u_{p(k)}, Tu_{p(k)}), \sigma_b(u_{q(k)}, Tu_{q(k)}), \\ \frac{\sigma_b(u_{p(k)}, u_{q(k)}) + \sigma_b(u_{q(k)}, Tu_{q(k)}) + \sigma_b(u_{q(k)}, u_{p(k)}) + \sigma_b(u_{p(k)}, Tu_{p(k)})}{2s}, \\ \frac{\sigma_b(u_{p(k)}, Tu_{p(k)})\sigma_b(u_{q(k)}, Tu_{q(k)})}{1 + s[2\sigma_b(u_{p(k)}, u_{q(k)}) + \sigma_b(u_{q(k)}, Tu_{q(k)}) + \sigma_b(u_{q(k)}, u_{p(k)}) + \sigma_b(u_{p(k)}, Tu_{p(k)})]}, \\ \frac{\sigma_b(u_{p(k)}, u_{q(k)+1})\sigma_b(u_{p(k)}, u_{q(k)})}{1 + s\sigma_b(u_{p(k)}, Tu_{p(k)}) + s^3[\sigma_b(u_{q(k)}, u_{p(k)}) + \sigma_b(u_{p(k)}, Tu_{p(k)}) + \sigma_b(u_{q(k)}, Tu_{q(k)})]} \end{array}\right\}\right) \\ &\leq F\left(\max\left\{\begin{array}{c} 2s\epsilon_0, \sigma_b(u_{p(k)}, Tu_{p(k)}), \sigma_b(u_{q(k)}, Tu_{q(k)}), \\ \frac{2s\epsilon_0 + \epsilon_0 + 2s\epsilon_0 + \epsilon_0}{2s}, \frac{\sigma_b(u_{p(k)}, Tu_{p(k)})\sigma_b(u_{q(k)}, Tu_{q(k)})}{1 + s[2s\epsilon_0 + 2s\epsilon_0 + \epsilon_0 + 2s\epsilon_0 + \epsilon_0]}, \\ \frac{(2s\epsilon_0 + \epsilon_0)(2s\epsilon_0)}{1 + s\sigma_b(u_{p(k)}, Tu_{p(k)}) + s^3[2s\epsilon_0 + \sigma_b(u_{p(k)}, Tu_{p(k)}) + \sigma_b(u_{q(k)}, Tu_{q(k)})]} \end{array}\right\}\right) \end{aligned} \tag{28}$$

for $k > \max\{N_1, N_2\}$. Passing to the $k \to +\infty$ in (28) and using (27), we obtain

$$\tau + F(\epsilon_0) \leq F(\epsilon_0), \tag{29}$$

which contradicts $\tau > 0$. Therefore, $\{u_n\}$ is a b-Cauchy sequence in U. Now, (U, σ_b) is a b-completeness b-metric-like space; there exists $u^* \in U$ such that

$$\sigma_b(u^*, u^*) = \lim_{n\to\infty} \sigma_b(u_n, u^*) = \lim_{n,m\to\infty} \sigma_b(u_n, u_m) = 0. \tag{30}$$

By b-continuity of T, we get

$$\lim_{n\to\infty} \sigma_b(Tu_n, Tu^*) = 0.$$

From the triangle inequality, we have

$$\sigma_b(u^*, Tu^*) \leq s[\sigma_b(u^*, Tu_n) + \sigma_b(Tu_n, Tu^*)] \;\; \forall\, n \in \mathbb{N}. \tag{31}$$

Passing to the limit as $n \to \infty$ in the above inequality, we obtain

$$\sigma_b(u^*, Tu^*) = 0.$$

Then, $Tu^* = u^*$. This shows that $Fix(T) \neq \emptyset$. □

Considering different cases of condition (6) in Theorem 1, we have the following contraction results.

(I) Take $F(\alpha) = \ln \alpha$ ($\alpha > 0$) and $\tau = \ln(\frac{1}{\lambda})$ where $\lambda \in (0,1)$, then

$$\forall\, u,v \in U \text{ with } \alpha(u,v) \geq s \text{ and } \sigma_b(Tu,Tv) > 0 \Rightarrow s^3\sigma_b(Tu,Tv) \leq \lambda\Delta_s(u,v). \tag{32}$$

(II) Taking $F(\alpha) = \ln \alpha + \alpha$ ($\alpha > 0$) and $\tau = \ln(\frac{1}{\lambda})$ where $\lambda \in (0,1)$, then

$$\forall\, u,v \in U \text{ with } \alpha(u,v) \geq s \text{ and } \sigma_b(Tu,Tv) > 0 \Rightarrow s^3\sigma_b(Tu,Tv))e^{s^3\sigma_b(Tu,Tv)-\Delta_s(u,v)} \leq \lambda\Delta_s(u,v). \tag{33}$$

(III) Take $F(\alpha) = -\frac{1}{\sqrt{\alpha}}$ ($\alpha > 0$) and $\tau = \lambda$ where $\lambda > 0$, then

$$\forall\, u,v \in U \text{ with } \alpha(u,v) \geq s \text{ and } \sigma_b(Tu,Tv) > 0 \Rightarrow s^3\sigma_b(Tu,Tv)) \leq \frac{1}{(1+\lambda\sqrt{\Delta_s(u,v)})^2}\Delta_s(u,v). \tag{34}$$

(IV) Taking $F(\alpha) = \ln(\alpha^2 + \alpha)$ ($\alpha > 0$) and $\tau = \ln(\frac{1}{\lambda})$ where $\lambda > 0$, then

$$\forall\, u,v \in U \text{ with } \alpha(u,v) \geq s \text{ and } \sigma_b(Tu,Tv) > 0 \Rightarrow s^3\sigma_b(Tu,Tv))[s^3\sigma_b(Tu,Tv)+1] \leq \lambda\Delta_s(u,v)[\Delta_s(u,v)+1]. \tag{35}$$

Now, we verify Theorem 1 by the two following examples:

Example 1. *Let $U = [0,1]$ and $\sigma_b : U \times U \to [0,\infty)$ be defined by*

$$\sigma_b(u,v) = (\max\{u,v\})^2, \ \forall\ u,v \in U.$$

Then, (U,σ_b) is a b-complete b-metric-like space with constant $s = 2$. Define mappings $T : U \to U$ and $\alpha : U \times U \to [0,\infty)$ by

$$\alpha(u,v) = \begin{cases} 1+e^{u+v}, & u,v \in [0,1], \\ ln(2u+3), & otherwise, \end{cases} \quad and\ T(u) = \begin{cases} \frac{u}{8}, & u,v \in [0,1), \\ \frac{1}{32}, & if\ u = 1. \end{cases}$$

Now, we need to show that $T \in \Theta_s(U,\alpha,F)$.

Proof. Supposing that $u,v \in U$, so that $\alpha(u,v) \geq s = 2$. Defining the function $F(\alpha) = \ln(\alpha)$ for $\alpha \in \mathbb{R}_+$, then we get

$$\tau + F(s^3\sigma_b(Tu,Tv)) \leq F(\Delta_s(u,v)) \Leftrightarrow \ln\frac{\Delta_s(u,v)}{8\sigma_b(Tu,Tv)} \geq \tau. \tag{36}$$

We distinguish it into four cases:
Case I: For $u = 1$ and $v \in [0,1]$, we have

$$8\sigma_b(Tu,Tv) = 8\sigma_b\left(\frac{1}{32},Tv\right) = 8\begin{cases} \frac{1}{1024}, & v \in [0,\frac{1}{4}] \\ \frac{v^2}{64}, & v \in (\frac{1}{4},1) \\ \frac{1}{1024}, & v = 1 \end{cases} = \begin{cases} \frac{1}{128}, & v \in [0,\frac{1}{4}] \\ \frac{v^2}{8}, & v \in (\frac{1}{4},1) \\ \frac{1}{128}, & v = 1 \end{cases}$$

and

$$\begin{aligned}\Delta_s(u,v) &= \max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(v,fv),\sigma_b(u,Tu),\dfrac{\sigma_b(u,Tv)+\sigma_b(v,Tu)}{2s},\\ \dfrac{\sigma_b(u,Tu)\sigma_b(v,Tv)}{1+s[\sigma_b(u,v)+\sigma_b(u,Tv)+\sigma_b(v,Tu)]},\\ \dfrac{\sigma_b(u,Tv)\sigma_b(u,v)}{1+s\sigma_b(u,Tu)+s^3[\sigma_b(v,Tu)+\sigma_b(v,Tv)]}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}u^2,v^2,u^2,\dfrac{(\max\{u,Tv\})^2+(\max\{v,Tu\})^2}{4},\\ \dfrac{u^2v^2}{1+2[u^2+(\max\{u,Tv\})^2+(\max\{v,Tx\})^2]},\\ \dfrac{(\max\{u,Tv\})^2u^2}{1+2u^2+8[(\max\{v,Tu\})^2+v^2]}\end{array}\right\}\\ &= 1.\end{aligned} \tag{37}$$

Therefore, from (37), we get

$$\frac{\Delta_s(u,v)}{8\sigma_b(Tu,Tv)} = \begin{cases}128, & v\in[0,\frac{1}{4}]\\ \frac{8}{v^2}, & v\in(\frac{1}{4},1)\\ 128, & v=1.\end{cases} \tag{38}$$

Case II: For $u<1$ and $v=1$, we have

$$8\sigma_b(Tu,Tv) = 8\sigma_b\left(Tu,\frac{1}{32}\right) = 8\begin{cases}\frac{1}{1024}, & u\in[0,\frac{1}{4}]\\ \frac{u^2}{64}, & u\in(\frac{1}{4},1)\\ \frac{1}{1024}, & u=1\end{cases} = \begin{cases}\frac{1}{128}, & u\in[0,\frac{1}{4}]\\ \frac{u^2}{8}, & u\in(\frac{1}{4},1)\\ \frac{1}{128}, & u=1\end{cases}$$

and

$$\Delta_s(u,v)=1.$$

Hence, we get

$$\frac{\Delta_s(u,v)}{8\sigma_b(Tu,Tv)} = \begin{cases}128, & u\in[0,\frac{1}{4}]\\ \frac{8}{u^2}, & u\in(\frac{1}{4},1)\\ 128, & u=1.\end{cases} \tag{39}$$

Case III: For $u<v<1$, we have

$$8\sigma_b(Tu,Tv) = 8\sigma_b\left(\frac{u}{8},\frac{v}{8}\right) = 8\left\{\frac{v^2}{64}\right\} = \frac{v^2}{8},\text{ and } \Delta_s(u,v)=v^2.$$

Hence, we get

$$\frac{\Delta_s(u,v)}{8\sigma_b(Tu,Tv)} = 8. \tag{40}$$

Case VI: For $v\le u<1$, we have

$$8\sigma_b(Tu,Tv) = 8\sigma_b\left(\frac{u}{8},\frac{v}{8}\right) = 8\left\{\frac{u^2}{64}\right\} = \frac{u^2}{8},\text{ and } \Delta_s(u,v)=u^2.$$

Hence, we get

$$\frac{\Delta_s(u,v)}{8\sigma_b(Tu,Tv)} = 8, \tag{41}$$

if $0 < \tau \leq \ln 8$ and from the Equations (38)–(41), we obtain $\ln \dfrac{\Delta_s(u,v)}{8\sigma_b(Tu,Tv)} \geq \tau$. This implies that (6) holds and thus $T \in \Theta_s(U,\alpha,F)$. It is easy to see that $T \in \mathcal{WS}_s(U,\alpha)$. Indeed, if $u \in U$ is such that

$$\alpha(u,Tu) \geq s = 2,$$

then $u, Tu \in [0,1)$. This implies that $TTu \in [0,1)$ and hence

$$\alpha(Tu,TTu) \geq s.$$

In addition, we can see that T is b-continuous and there is $u_0 = 1$ such that

$$\begin{aligned}\alpha(u_0,Tu_0) &= \alpha(1,T(1)) = \alpha(1,\frac{1}{32})\\ &= 1 + e^{\left(1+\frac{1}{32}\right)} \geq 2 = s.\end{aligned}$$

Hence, all the conditions of Theorem 1 are fulfilled and $Fix(T) \neq \emptyset$. This example is verified that $0 \in Fix(T)$. □

Example 2. *Consider $U = \{0,1,2\}$. Let $\sigma_b : U \times U \to [0,\infty)$ be defined by*

$$\sigma_b(0,0) = 0, \qquad \sigma_b(1,1) = \frac{1}{4}, \qquad \sigma_b(2,2) = \frac{5}{2}, \quad \sigma_b(0,1) = \sigma_b(1,0) = \frac{1}{2},$$

$$\sigma_b(0,2) = \sigma_b(2,0) = \frac{9}{2}, \quad \sigma_b(1,2) = \sigma_b(2,1) = 5.$$

It is clear that (U,σ_b) is a b-complete b-metric like space with constant $s = \frac{17}{8}$. The mappings $T : U \to U$ and $\alpha : U \times U \to [0,\infty)$ defined by

$$T0 = 0, \qquad T1 = 0, \qquad T2 = 1,$$

and

$$\alpha(u,v) = \begin{cases} \sinh(u+v) + e^{uv} + \frac{5}{4}, & u, \in [0,1] \\ \ln(5u+3), & \text{otherwise.} \end{cases}$$

Now, we need to prove that $T \in \Theta_s(U,\alpha,F)$.

Proof. Supposing that $u,v \in U$, so that $\alpha(u,v) \geq s = \frac{17}{4}$. Define the function $F(\alpha) = \frac{-1}{\alpha} + \alpha$ for $\alpha \in \mathbb{R}_+$, and $\tau = \frac{1}{10}$.

Now, $\sigma_b(T0,T0) = \sigma_b(T0,T1) = 0$, so it can be distinguished in three cases:
Case I: For $u = 0$ and $v = 2$,

$$\begin{aligned} F(\Delta_s(0,2)) &= F\left(\max\left\{\begin{array}{c} \sigma_b(0,2), \sigma_b(2,T2), \sigma_b(0,T0), \dfrac{\sigma_b(0,T2)+\sigma_b(2,T0)}{2s}, \\ \dfrac{\sigma_b(0,T0)\sigma_b(2,T2)}{1+\frac{17}{8}[\sigma_b(0,2)+\sigma_b(0,T2)+\sigma_b(2,T0)]}, \\ \dfrac{\sigma_b(0,T2)\sigma_b(0,2)}{1+\frac{17}{8}\sigma_b(0,T0)+\left(\frac{17}{8}\right)^3[\sigma_b(2,T0)+\sigma_b(2,T2)]} \end{array}\right\}\right) \\ &= F\left(\max\left\{\frac{9}{2}, 5, 0, \frac{5}{4}, 0, 0.02\right\}\right) \\ &= 4.8 \end{aligned} \tag{42}$$

and

$$\begin{aligned} \tau + F(s^3\sigma_b(Tu, Tv)) &= \frac{1}{10} + F\left(\left(\frac{17}{8}\right)^3 \sigma_b(T0, T2)\right) \\ &= 4.58. \end{aligned} \tag{43}$$

Therefore, from (42) and (43), inequality (6) was satisfied.

In addition, it has been observed that $u = 1$ and $v = 2$ was similar to case I, since $\sigma_b(T0, T2) = \sigma_b(T1, T2) = \sigma_b(0, 1)$.

Case II: For $u = 2$ and $v = 2$,

$$\begin{aligned} F(\Delta_s(2,2)) &= F\left(\max\left\{\begin{array}{c} \sigma_b(2,2), \sigma_b(2,T2), \sigma_b(2,T2), \dfrac{\sigma_b(2,T2)+\sigma_b(2,T2)}{2s}, \\ \dfrac{\sigma_b(2,T2)\sigma_b(2,T2)}{1+\frac{17}{8}[\sigma_b(2,2)+\sigma_b(2,T2)+\sigma_b(2,T2)]}, \\ \dfrac{\sigma_b(2,T2)\sigma_b(2,2)}{1+\frac{17}{8}\sigma_b(2,T2)+\left(\frac{17}{8}\right)^3[\sigma_b(2,T2)+\sigma_b(2,T2)]} \end{array}\right\}\right) \\ &= F\left(\max\left\{\frac{5}{2}, 5, 5, \frac{5}{2}, \frac{9}{29}, \frac{29}{250}\right\}\right) \\ &= 4.8 \end{aligned} \tag{44}$$

and

$$\begin{aligned} \tau + F(s^3\sigma_b(Tu, Tv)) &= \frac{1}{10} + F\left(\left(\frac{17}{8}\right)^3 \sigma_b(T2, T2)\right) \\ &= 2.07. \end{aligned} \tag{45}$$

Therefore, from (44) and (45), inequality (6) was satisfied.

Case III: For $u = 2$ and $v \in (0, 1)$, it is obvious.

(a) For $u = 2$ and $v = 0$, it follows as Case I.
(b) For $u = 2$ and $v = 1$, it follows as Case II.

This implies that (6) holds for all the cases—thus $f \in \Theta_s(U, \alpha, F)$. It is easy to see that $f \in \mathcal{WS}_s(U, \alpha)$. Indeed, if $u \in U$ is such that

$$\alpha(u, Tu) \geq s = \frac{17}{8},$$

then $u, Tu \in [0, 1)$. This implies that $TTu \in [0, 1)$ and hence

$$\alpha(Tu, TTu) \geq s.$$

In addition, we can see that T is b-continuous and there is $u_0 = 1$ such that

$$\begin{aligned} \alpha(u_0, Tu_0) &= \alpha(1, T1) = \alpha(1, 0) \\ &= \sinh(1+0) + e^0 + \frac{5}{4} \\ &\geq \frac{17}{8} = s. \end{aligned}$$

All the requirements of Theorem 1 are satisfied. Hence, it can be concluded that $Fix(T) \neq \emptyset$. In this example, it shows that that $0 \in Fix(T)$. □

In the following theorem, we derive fixed point results by replacing assumption (S_4) of Theorem 1 by α_s-regularity of U.

Theorem 2. *Let (U, σ_b) be a b-complete b-metric-like space with coefficient $s \geq 1$, let $\alpha : U \times U \to [0, \infty)$, and $T : U \to U$ be a rational F contraction mapping with α-admissibility type-S. Again, assume the following conditions:*

(S_1) *$T \in \Theta_s(U, \alpha, F) \cap \mathcal{WS}_s(U, \alpha)$,*
(S_2) *there exists $u_0 \in U$ such that $\alpha(u_0, Tu_0) \geq s$,*
(S_3) *α has a transitive property type S,*
(S_4) *U is α_s-regular, that is if $\{u_n\}$ is a sequence in U such that*

$$\alpha(u_n, u_{n+1}) \geq s,$$

for all $n \in \mathbb{N}$ and $u_n \to u \in U$ as $n \to \infty$, then $\alpha(u_n, u) \geq s$, for all $n \in \mathbb{N}$.

Then, $Fix(T) \neq \emptyset$.

Proof. Following the proof in Theorem 1, we obtain that $\{u_n\}$ is a b-Cauchy sequence in the b-complete b-metric-like space (U, σ_b). By b-completeness of U, there exists $u \in U$ such that

$$\lim_{n\to\infty} \sigma_b(u_n, u) = 0, \tag{46}$$

that is, $u_n \to u$ as $n \to \infty$. By α_s-regularity of U, we have

$$\alpha(u_n, u) \geq s,$$

for all $n \in \mathbb{N}$. It follows from $T \in \Theta_s(U, \alpha, F)$ that

$$\tau + F(\sigma_b(Tu_n, Tu)) \leq \tau + F(s^3\sigma_b(Tu_n, Tu)) \leq F(\Delta_s(u_n, u)), \tag{47}$$

where

$$\Delta_s(u_n, u) = \max \left\{ \begin{array}{c} \sigma_b(u_n, u), \sigma_b(u_n, Tu_n), \sigma_b(u, Tu), \dfrac{\sigma_b(u_n, Tu) + \sigma_b(Tu_n, u)}{4k}, \\ \dfrac{\sigma_b(u_n, Tu_n)\sigma_b(u, Tu)}{1 + \sigma_b(u_n, u)}, \dfrac{\sigma_b(u_n, Tu_n)\sigma_b(u, Tu)}{1 + \sigma_b(Tu_n, Tu)} \\ \dfrac{1 + \sigma_b(u, Tu_n)}{1 + \sigma_b(u_n, Tu)}\sigma_b(u, Tu_n) \end{array} \right\}. \tag{48}$$

Taking the limit supremum as $n \to \infty$ in (48) and using Lemma (1), we get

$$\begin{aligned} \tau + F(\sigma_b(u, Tu)) &\leq \tau + F(s^2\sigma_b(u, Tu)) \\ &= \tau + F\left(s^3 \frac{1}{s} \sigma_b(u, Tu)\right) \\ &\leq \tau + F\left(s^3 \limsup_{n\to\infty} \sigma_b(u_{n+1}, Tu)\right) \\ &\leq F\left(\limsup_{n\to\infty} \Delta_s(u_n, u)\right) \\ &\leq F(\sigma_b(u, Tu)), \end{aligned}$$

which is a contradiction since $\tau > 0$, which is possible only if $F(\sigma_b(u, Tu)) = 0$. It follows that $\sigma_b(u, Tu) = 0$, equivalently, $u = Tu$ and thus $Fix(T) \neq \emptyset$. This completes the proof. $\square$

Next, we use Remark 2 to establish the following results for the class $\mathcal{S}_s(U, \alpha)$.

Corollary 1. *Supposing all the conditions of Theorem 1 are fulfilled, except the condition (S_1), i.e.,*

$(\widehat{S_1})$ $T \in \Theta_s(U, \alpha, F) \cap \mathcal{S}_s(U, \alpha)$,
(S_2) *there exists* $u_0 \in U$ *such that* $\alpha(u_0, Tu_0) \geq s$,
(S_3) α *has a transitive property type* S,
(S_4) T *is b-continuous.*

Then, $Fix(T) \neq \emptyset$.

Corollary 2. *Suppose all the conditions of Corollary 1 are satisfied, apart from the condition* (S_4), *i.e.,*

$(\widehat{S_1})$ $T \in \Theta_s(U, \alpha, F) \cap \mathcal{S}_s(U, \alpha)$,
(S_2) *there exists* $u_0 \in U$ *such that* $\alpha(u_0, Tu_0) \geq s$,
(S_3) α *has a transitive property type* S,
$(\widehat{S_4})$ U *is* α_s*-regular.*

Then, $Fix(T) \neq \emptyset$.

4. Periodic Point Results

Now, we discuss periodic point theorems for self-mappings on a b-metric-like space for which the following definition is required.

Definition 13 ([14]). *A mapping* $T\colon U \to U$ *is said to have the property-*(P) *if* $Fix(T^n)$= Fix(T), *for every* $n \in \mathbb{N}$.

Theorem 3. *Let* (U, σ_b) *be a b-complete b-metric-like space with coefficient* $s \geq 1$, *let* $\alpha : U \times U \to [0, \infty)$, *and* $T : U \to U$ *be given mappings. Suppose that the following conditions hold:*

(S_1) $T \in \Theta_s(U, \alpha, F) \cap \mathcal{WS}_s(U, \alpha)$;
(S_2) *there exists* $u_0 \in U$ *such that* $\alpha(u_0, Tu_0) \geq s$;
(S_3) α *has a transitive property type* S,
(S_4) T *is b-continuous;*
(S_5) *If* $z \in Fix(T^n)$ *and* $z \notin Fix(T)$, *then* $\alpha(T^{n-1}z, T^n z) \geq s$.

Then, $Fix(T)$ *has propertv-*(P).

Proof. Following Theorem 1, we have $Tu^* = u^*$. This shows that $Fix(T^n)$=*Fix(T)* for $n = 1$. Let $n > 1$ and assume, by contradiction, that $z \in Fix(T^n)$ and $z \notin Fix(T)$, such that $\sigma_b(z, Tz) \geq 0$. Now, applying (S_5) and (11), we have

$$\begin{aligned} F(\sigma_b(z, Tz)) &\leq F(\sigma_b(T(T^{n-1}z), T^2(T^{n-1}z))) \\ &\leq F(s^3\sigma_b(T(T^{n-1}z), T^2(T^{n-1}z))) \\ &\leq F(\Delta_s(T^{n-1}z, T^n z)) \end{aligned} \tag{49}$$

and, by using the inequality (12) and (49), we get

$$\begin{aligned} \tau + F(\sigma_b(z, Tz)) &\leq \tau + F(\sigma_b(T(T^{n-1}z), T^2(T^{n-1}z))) \\ &\leq \tau + F(s^3\sigma_b(T(T^{n-1}z), T^2(T^{n-1}z))) \\ &\leq F(\Delta_s(T^{n-1}z, T^n z)) \\ &\leq F(\sigma_b(T^{n-1}z, T^n z)). \end{aligned} \tag{50}$$

Hence, the above inequality turns into

$$F(\sigma_b(z, Tz)) < F(\sigma_b(T^{n-1}z, T^n z)) - \tau. \tag{51}$$

Iteratively, we find that

$$\begin{aligned} F(\sigma_b(z,Tz)) &\leq F(\sigma_b(T^{n-1}z,T^n z)) - \tau \\ &\leq F(\sigma_b(T^{n-2}z,T^{n-1}z)) - 2\tau \\ &\leq F(\sigma_b(T^{n-3}z,T^{n-2}z)) - 3\tau \\ &\vdots \\ &\leq F(\sigma_b(z,Tz)) - n\tau. \end{aligned} \tag{52}$$

From (52), we obtain $\lim_{n\to\infty} F(\sigma_b(z,Tz)) = -\infty$, which together with (F_2) gives

$$\lim_{n\to\infty} \sigma_b(z,Tz) = 0, \tag{53}$$

which implies that $\sigma_b(z,Tz) = 0$. Hence, $Fix(T^n) = Fix(T)$, $\square$

5. Application to First-Order Periodic Boundary Value Problem

Consider the first-order periodic boundary value problem

$$\begin{cases} u'(t) = T(t,u(t)), & t \in I = [0,T], \\ u(0) = u(T), \end{cases} \tag{54}$$

where $T > 0$ and $T : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. We prove an existence theorem for the solution of (54) as an application of Theorem 1. Consider the space

$$\mathcal{F} = C(I,\mathbb{R}) := \{u : I \to \mathbb{R} \mid u \text{ is continuous on } I\}.$$

Define $\sigma_b : U \times U \to \mathbb{R}^+$ by

$$\sigma_b(u,v) = \sup_{t\in[0,\tau]} (|u(t)| + |v(t)|)^2 \ \forall\, u,v \in \mathcal{F}. \tag{55}$$

Obviously, $(\mathcal{F},\sigma_b,2)$ is a b-complete b-metric like space. Then, $(\mathcal{F},\sigma_b,2)$ is a b-complete b-metric like space. This problem (54) is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)[T(s,u(s)) + \lambda u(s)]\, ds, \text{ for all } t \in [0,T], \tag{56}$$

where $G(t,s)$ is the Green function given by

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s < t \leq T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t < s \leq T. \end{cases}$$

Define the mapping $T\colon C(I,\mathbb{R}) \to C(I,\mathbb{R})$ by

$$Tu(t) = \int_0^T G(t,s)[T(s,u(s)) + \lambda u(s)]\, ds, \text{ for all } t \in I. \tag{57}$$

Note that, if $u^* \in C^1(I,\mathbb{R})$ is a fixed point of T, then $u^* \in C^1(I,\mathbb{R})$ is a solution of (54). Next, we give the following notions which are required to complete this section.

Definition 14.

1. *A solution to* (54) *is a function* $u \in C^1(I,\mathbb{R})$ *satisfying conditions in* (54).

2. *A lower solution for* (54) *is a function* $\gamma \in C^1(I, \mathbb{R})$ *such that*

$$\begin{cases} \gamma'(t) \leq T(t, \gamma(t)), & \textit{for } t \in I, \\ \gamma(0) \leq \gamma(T). \end{cases}$$

3. *An upper solution for* (54) *is a function* $\beta \in C^1(I, \mathbb{R})$ *such that*

$$\begin{cases} \beta'(t) \geq T(t, \beta(t)), & \textit{for } t \in I, \\ \beta(0) \geq \beta(T). \end{cases}$$

Theorem 4. *Assuming that the following assertions hold*

(H1) $T : I \times \mathbb{R} \to \mathbb{R}$ *is continuous,*
(H2) A nondecreasing function $\xi(k, \cdot)$, $\forall\, k \in [0, \tau]$, *i.e.,*

$$u, v \in \mathbb{R},\ u \leq v \ \Rightarrow\ \xi(r, u) \leq \xi(r, v);$$

(H3) there exists $u_0 \in U$ *such that* $\xi(u_0(t), Tu_0(t)) \geq s$ *for all* $t \in I$,
(H4) for each $t \in I$ *and* $(u, v) \in U$, $\xi(u(t), v(t)) \geq s$ *implies that* $\xi(Tu(t), Tv(t)) \geq s$,
(H5) for each $t \in I$, *if* $\{u_n\}$ *is a sequence in* U *such that* $u_n \to u$ *in* U *and* $\xi(u_n(t), u_{n+1}(t)) \geq 0$, *for all* $n \in \mathbb{N}$, *then*

$$\xi(u_n(t), u(t)) \geq 0, \ \ \forall n \in \mathbb{N},$$

(H6) there exist $\lambda > 0$, *such that for* $(u, v) \in U$ *and* $t \in I$ *with* $\xi(u, v) \geq 0$,

$$0 \leq |T(t, u) + \lambda u(t)| + |T(t, v) + \lambda v(t)| \leq \frac{3\lambda}{4} \Delta_s(u, v),$$

where

$$\Delta_s(u, v) = \max \left\{ \begin{array}{c} (|u| + |v|)^2, (|u| + |Tu|)^2, (|v| + |Tv|)^2, \\ \dfrac{(|u| + |Tv|)^2 + (|v| + |Tu|)^2}{2s}, \\ \dfrac{(|u| + |Tu|)^2 (|v| + |Tv|)^2}{1 + s[(|u| + |v|)^2 + (|u| + |Tv|)^2 + (|v| + |Tu|)^2]}, \\ \dfrac{(|u| + |Tv|)^2 (|u| + |v|)^2}{1 + s(|u| + |Tu|)^2 + s^3[(|v| + |Tu|)^2 + (|v| + |Tv|)^2]} \end{array} \right\}^{\frac{1}{2}},$$

(H7) there exist $\beta \in U$, *a lower solution of* (54) *such that for all* $t \in I$,

$$\xi(\beta(t), T\beta(t)) \geq 0.$$

Then, the existence of a lower solution for (54) *implies the existence of a unique solution of* (54). *Then,* $u^* \in C(I, \mathbb{R})$ *is a solution of the integral Equation* (56).

Proof. From (H1)-(H2), it follows that f is continuous and non-decreasing mapping. In addition, for (57), there exists $u_0 \in f$ such that $u_0 \leq f u_0$. For all $t \in [0, \tau]$, and conditions (H6) and (56), we get

$$
\begin{aligned}
&\sigma_b(T(u),T(v)) \\
&= \sup_{p\in[0,\tau]} (|T(u)|+|T(v)|)^2 \\
&= \sup_{t\in[0,\tau]} \left(\left| \int_0^\tau G(t,s)(T(t,u)+\lambda u(t))dr \right| + \left| \int_0^\tau G(t,s)(T(t,v)+\lambda v(t))dr \right| \right)^2 \\
&\leq \sup_{t\in[0,\tau]} \left(\int_0^\tau G(t,s)|(T(t,u)+\lambda u(t))|dr + \int_0^\tau G(t,s)\,|(T(t,v)+\lambda v(t))|\,dr \right)^2 \\
&= \sup_{t\in[0,\tau]} \left(\int_0^\tau G(t,s)(|(T(t,u)+\lambda u(t))|+|(T(t,v)+\lambda v(t))|)dr \right)^2 \\
&\leq \sup_{t\in[0,\tau]} \left(\int_0^\tau G(t,s)(\Delta_s(u,v))dr \right)^2 \\
&\leq \sup_{t\in[0,\tau]} \left(\int_0^\tau G(t,s)dr \right)^2 \\
&\times \left(\frac{3\lambda}{4} \max \left\{ \begin{array}{c} (|u|+|v|)^2, (|u|+|Tu|)^2, (|v|+|Tv|)^2, \\ \frac{(|u|+|Tv|)^2+(|v|+|Tu|)^2}{2s}, \\ \frac{(|u|+|Tu|)^2(|v|+|Tv|)^2}{1+s[(|u|+|v|)^2+(|u|+|Tv|)^2+(|v|+|Tu|)^2]}, \\ \frac{(|u|+|Tv|)^2(|u|+|v|)^2}{1+s(|u|+|Tu|)^2+s^3[(|v|+|Tu|)^2+(|v|+|Tv|)^2]} \end{array} \right\}^{\frac{1}{2}} \right)^2 \\
&\leq \left(\sup_{t\in[0,\tau]} \frac{1}{e^{\lambda T-1}} \left(\left[\frac{1}{\lambda} e^{\lambda(T+s-t)} \right]_0^t + \left[\frac{1}{\lambda} e^{\lambda(s-t)} \right]_t^T \right) \right)^2 \\
&\times \frac{9\lambda}{16} \left(\max \left\{ \begin{array}{c} (|u|+|v|)^2, (|u|+|Tu|)^2, (|v|+|Tv|)^2, \\ \frac{(|u|+|Tv|)^2+(|v|+|Tu|)^2}{2s}, \\ \frac{(|u|+|Tu|)^2(|v|+|Tv|)^2}{1+s[(|u|+|v|)^2+(|u|+|Tv|)^2+(|v|+|Tu|)^2]}, \\ \frac{(|u|+|Tv|)^2(|u|+|v|)^2}{1+s(|u|+|Tu|)^2+s^3[(|v|+|Tu|)^2+(|v|+|Tv|)^2]} \end{array} \right\}^{\frac{1}{2}} \right)^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\sigma_b(T(u),T(v)) \\
&\leq \frac{1}{16} \times \frac{9\lambda}{16} \left(\max \left\{ \begin{array}{c} (|u|+|v|)^2, (|u|+|Tu|)^2, (|v|+|Tv|)^2, \\ \frac{(|u|+|Tv|)^2+(|v|+|Tu|)^2}{2s}, \\ \frac{(|u|+|Tu|)^2(|v|+|Tv|)^2}{1+s[(|u|+|v|)^2+(|u|+|Tv|)^2+(|v|+|Tu|)^2]}, \\ \frac{(|u|+|Tv|)^2(|u|+|v|)^2}{1+s(|u|+|Tu|)^2+s^3[(|v|+|Tu|)^2+(|v|+|Tv|)^2]} \end{array} \right\}^{\frac{1}{2}} \right)^2.
\end{aligned}
$$

Now, by considering the F-contraction function $T : [0,+\infty)$ into itself defined by:

$$F(\alpha) = \frac{-1}{\alpha} + \alpha, \text{ for } t \geq 0,$$

we get

$$\begin{aligned}&\tau + F(s^3\sigma_b(Tu,Tv))\\ &\leq F\left(\max\left\{\begin{array}{c}\sigma_b(u,v),\sigma_b(u,Tu),\sigma_b(v,Tv),\dfrac{\sigma_b(u,Tv)+\sigma_b(v,Tu)}{2s},\\ \dfrac{\sigma_b(u,Tu)\sigma_b(v,Tv)}{1+s[\sigma_b(u,v)+\sigma_b(u,Tv)+\sigma_b(v,Tu)]},\\ \dfrac{\sigma_b(u,Tv)\sigma_b(u,v)}{1+s\sigma_b(u,Tu)+s^3[\sigma_b(v,Tu)+\sigma_b(v,Tv)]}\end{array}\right\}\right).\end{aligned}$$

Now, we define the function $\alpha : U \times U \to [0,\infty)$ by

$$\alpha(u,v) = \begin{cases} 1+e^{u+v}, & \text{if } \ \xi(u(t),v(t)) \geq 0, \text{ for } (u,v) \in U, \text{ for all } t \in [0,1], \\ \ln(2u+3), & \text{otherwise}. \end{cases} \tag{58}$$

From (6), we have

$$\alpha(u,v) \geq s \text{ and } \sigma_b(Tu,Tv) > 0 \Rightarrow \tau + F(s^3\sigma_b(Tu,Tv)) \leq F(\Delta_s(u,v)), \tag{59}$$

with coefficient $s \geq 1$ and for each $(u,v) \in U$. From condition (H3) and (57), there exists $u_0 \in U$ such that $(u_0, Tu_0) \in U$ with $\alpha(u_0, Tu_0) \geq s$.

Again, by using (58) and condition (H2), the following assertions hold $\forall \ (u,v) \in U$:

$$\begin{aligned}\alpha(u,v) \geq s &\Longrightarrow \xi(u(t),v(t)) \geq s, \forall \ t \in [0,1]\\ &\Longrightarrow \xi(Tu(t),Tv(t)) \geq s, \forall \ t \in [0,1]\\ &\Longrightarrow \alpha(Tu,Tv) \geq s, \text{ for } (u,v) \in U.\end{aligned}$$

Therefore, T is a α-admissibility of type -S. Next, from (57) and condition (H5), we get easily that

$$\begin{cases} \text{for any sequence } \{u_n\} \text{ in } U \text{ if } \alpha(u_n,u_{n+1}) \geq s, \\ \forall \ n \in \mathbb{N} \text{ and } u_n \to u \in U \text{ as } n \to \infty, \text{ then } \alpha(u_n,u) \geq s, \forall \ n \in \mathbb{N}. \end{cases}$$

Finally, let $\beta(t)$ be a lower solution for (54). We claim that $\beta \leq T\beta$.

In fact,

$$\beta'(t) + \lambda\beta(t) \leq T(t,\beta(t)) + \lambda\beta(t), \ t \in I.$$

Multiplying by $e^{\lambda t}$,

$$(\beta(t)e^{\lambda t})' \leq [T(t,\beta(t)) + \lambda\beta(t)]e^{\lambda t}, \ t \in I,$$

and this gives us

$$\beta(t)e^{\lambda t} \leq \beta(0) + \int_0^t [T(s,\beta(s)) + \lambda\beta(s)]e^{\lambda s}\,ds, \quad t \in I. \tag{60}$$

As $\beta(0) \leq \beta(T)$, the last inequality gives us

$$\beta(0)e^{\lambda T} \leq \beta(T)e^{\lambda T} \leq \beta(0) + \int_0^T [T(s,\beta(s)) + \lambda\beta(s)]e^{\lambda s}ds,$$

and so

$$\beta(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[T(s,\beta(s)) + \lambda\beta(s)]\,ds. \tag{61}$$

The above equation and (60) give us

$$\beta(t)e^{\lambda t} \leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T}-1}[T(s,\beta(s)) + \lambda\beta(s)]\,ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T}-1}[T(s,\beta(s)) + \lambda\beta(s)]\,ds,$$

and, consequently,

$$\begin{aligned}\beta(t) &\leq \left(\int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}\,ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1} \right) [T(s,\beta(s)) + \lambda\beta(s)]\,ds \\ &= \int_0^T G(t,s)[T(s,\beta(s)) + \lambda\beta(s)]\,ds \\ &= T\beta(t), \text{ for } t \in I.\end{aligned}$$

Therefore, from (57) and condition (H7), we get $\alpha(\beta, T\beta) > s$. Finally, Theorem 1 gives that T has a unique fixed point. Hence, the problem (54) has a unique solution. □

Remark 3. *Similarly, we can get the upper solution of (54) if we prove the upper condition in place of a lower condition.*

6. Conclusions

The notion of rational F-contractions using α-admissibility of type-S is considered in b-metric-like spaces and the new fixed point and periodic point results are studied for such mappings. Some new theorems have been established on existence of solutions for rational F-contractions mapping with α-admissibility type-S, for the classes $\mathcal{WP}_s(X,\alpha)$ and $\mathcal{P}_s(X,\alpha)$. Numerical examples are illustrated in order to check the effectiveness and applicability of results. Furthermore, as an application to our results, the solution of first-order periodic boundary value problem is discussed.

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References

1. Wardowski, D. Fixed point theory of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *94*. [CrossRef]
2. Ali, M.U.; Kamran, T.; Postolache, M. Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem. *Nonlinear Anal. Model. Control* **2017**, *22*, 17–30. [CrossRef]
3. Alghamdi, M.A.; Hussain, N.; Salimi, P. Fixed point and coupled fixed point theorems on b-metric-like spaces. *J. Inequal. Appl.* **2013**. [CrossRef]
4. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for α-ψ-contactive type mappings. *Nonlinear Anal.* **2012**, *75*, 2154–2165. [CrossRef]
5. Sintunavarat, W. Nonlinear integral equations with new admissibility types in b-metric spaces. *J. Fixed Point Theory Appl.* **2015**, *17*, 1–32. [CrossRef]
6. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Inf. Univ. Ostrav.* **1993**, *5*, 5–11.
7. Amini-Harandi, A. Metric-like spaces, partial metric spaces and fixed points. *Fixed Point Theory Appl.* **2012**. [CrossRef]
8. Shukla, S. Partial b-metric spaces and fixed point theorems. *Mediterr. J. Math.* **2014**, *11*, 703–711. [CrossRef]
9. Hussain, N.; Roshan, J.R.; Parvaneh, V.; Kadelburg, Z. Fixed points of contractive mappings in b-metric-like spaces. *Sci. World J.* **2014**, *2014*. [CrossRef] [PubMed]
10. Chen, C.F.; Dong, J.; Zhu, C.X. Some fixed point theorems in b-metric-like spaces. *Fixed Point Theory Appl.* **2015**, *122*. [CrossRef]
11. Karapınar, E.; Kutbi, MA.; Piri, H.; O'Regan, D. Fixed points of conditionally F-contractions in complete metric-like spaces. *Fixed Point Theory Appl.* **2015**, *126*. [CrossRef]

12. Piri, H.; Kumam, P. Some fixed point theorems concerning F-contraction in complete metric spaces. *Fixed Point Theory Appl.* **2014**, *210*. [CrossRef]
13. Minak, G.; Helvaci, A.; Altun, I. Ćirić type generalized F-contractions on complete metric spaces and fixed point results. *Filomat* **2014**, *28*, 1143–1151. [CrossRef]
14. Abbas, M.; Ali, B.; Romaguera, S. Fixed and period points of generalized contractions in metric space. *Fixed Point Theory Appl.* **2013**, *243*. [CrossRef]