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Geometric Properties of Normalized Mittag–Leffler Functions

Saddaf Noreen [1], Mohsan Raza [1], Jin-Lin Liu [2,*] and Muhammad Arif [3]

1 Department of Mathematics, Government College University, Faisalabad 38000, Pakistan; saddafnoreen@yahoo.com (S.N.); mohsan976@yahoo.com (M.R.)
2 Department of Mathematics, Yangzhou University, Yangzhou 225002, China
3 Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan; marifmaths@yahoo.com
* Correspondence: jlliu@yzu.edu.cn

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Abstract: The aim of this paper is to investigate certain properties such as convexity of order μ, close-to-convexity of order $(1+\mu)/2$ and starlikeness of normalized Mittag–Leffler function. We use some inequalities to prove our results. We also discuss the close-to-convexity of Mittag–Leffler functions with respect to certain starlike functions. Furthermore, we find the conditions for the above-mentioned function to belong to the Hardy space $\mathcal{H}^p$. Some of our results improve the results in the literature.

Keywords: analytic functions; Mittag–Leffler functions; starlike functions; convex functions; Hardy space

MSC: 30C45; 33E12.

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1. introduction

The one parameter Mittag–Leffler function $\mathbb{E}_{\alpha}(z)$ defined by

$$\mathbb{E}_{\alpha}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+1)} \tag{1}$$

was introduced by Mittag–Leffler [1]. This function of complex variable is entire. The series defined by Equation (1) converges in $\mathbb{C}$ when $\mathrm{Re}(\alpha)>0$. Consider that the function $\mathbb{E}_{\alpha,\kappa}(z)$ which generalizes the function $\mathbb{E}_{\alpha}(z)$ is defined by

$$\mathbb{E}_{\alpha,\kappa}(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\alpha m+\kappa)},\alpha,\kappa\in\mathbb{C},\ z\in\mathbb{C}. \tag{2}$$

It was introduced by Wiman [2] and was named as Mittag–Leffler type function. The series in Equation (2) converges in $\mathbb{C}$ when $Re(\alpha)>0$ and $Re(\kappa)>0$. Furthermore, the functions defined in (1) and (2) are entire of order $1/Re(\alpha)$ and of type 1, for more details, see [3]. The function $\mathbb{E}_{\alpha,\kappa}(z)$ and its analysis with its generalizations is increasingly becoming a rich research area in mathematics and its related fields. A number of researchers studied and analyzed the function given in (2) (see Wiman [2,4,5]). One can find this function in the study of kinetic equation of fractional order, Lévy flights, random walks, super-diffusive transport as well as in investigations of complex systems.

In a similar manner, the advanced studies of these functions reflect and highlight many vital properties of these functions. The function $\mathbb{E}_{\alpha,\kappa}(z)$ generalizes many functions such as

$$\begin{aligned}\mathbb{E}_{1,1}(z) &= e^{z},\ \mathbb{E}_{1,2}(z)=\frac{e^{z}-1}{z},\\ \mathbb{E}_{2,1}(z) &= \cosh\left(\sqrt{z}\right),\ \mathbb{E}_{2,2}(z)=\frac{\sinh\left(\sqrt{z}\right)}{\sqrt{z}}.\end{aligned}$$

The interested readers are suggested to go through [6–9].

Let $\mathcal{A}$ be the family of all functions g having the form

$$g(z)=z+\sum_{m=2}^{\infty}a_{m}z^{m}, \tag{3}$$

and are analytic in $\mathcal{D}=\{z:|z|<1\}$ and $\mathcal{S}$ denote the family of univalent functions from $\mathcal{A}$. The families of functions which are convex, starlike and close-to-convex of order μ , respectively, are defined as:

$$\begin{aligned}\mathcal{C}(\mu) &= \left\{g:g\in\mathcal{A}\text{ and }Re\left(1+\frac{zg''(z)}{g'(z)}\right)>\mu,\quad z\in\mathcal{D};\ 0\le\mu<1\right\},\\ \mathcal{S}^{*}(\mu) &= \left\{g:g\in\mathcal{A}\text{ and }Re\left(\frac{zg'(z)}{g(z)}\right)>\mu,\quad z\in\mathcal{D};\ 0\le\mu<1\right\},\end{aligned}$$

and

$$\mathcal{K}(\mu)=\left\{g:g\in\mathcal{A}\text{ and }Re\left(\frac{g'(z)}{h'(z)}\right)>\mu,\quad z\in\mathcal{D};\ 0\le\mu<1;\ h\in\mathcal{C}\right\}.$$

It is clear that $\mathcal{C}^{*}(0)=\mathcal{C}$, $\mathcal{S}^{*}(0)=\mathcal{S}^{*}$ and $\mathcal{K}(0)=\mathcal{K}$. Consider the class $\mathcal{H}$ of all analytic functions in $\mathcal{D}$ and $\mu<1$, Baricz [10] introduced the classes

$$\mathcal{P}_{\eta}(\mu)=\left\{p:p\in\mathcal{H},\,p(0)=1,\,Re\left\{e^{i\eta}\left(p(z)-\mu\right)\right\}>0,\ z\in\mathcal{D},\ \eta\in\mathbb{R}\right\}$$

and

$$\mathcal{R}_{\eta}(\mu)=\left\{f:f\in\mathcal{A}\text{ and }Re\left\{e^{i\eta}\left(f'(z)-\mu\right)\right\}>0,\ z\in\mathcal{D},\ \eta\in\mathbb{R}\right\}.$$

For $\eta=0$, we have the classes of analytic functions $\mathcal{P}_{0}(\alpha)$ and $\mathcal{R}_{0}(\alpha)$ respectively. Also for $\eta=0$ and $\alpha=0$, we have the classes $\mathcal{P}$ and $\mathcal{R}$.

For the functions $g\in\mathcal{A}$ given by (1) and $h\in\mathcal{A}$ given by

$$h(z)=z+\sum_{m=2}^{\infty}b_{m}z^{m},$$

then the convolution (Hadamard product) of g and h is defined as:

$$(g*h)(z)=z+\sum_{m=2}^{\infty}a_{m}b_{m}z^{m},\ z\in\mathcal{D}.$$

It is clear that the function $\mathbb{E}_{\alpha,\kappa}(z)$ is not in class $\mathcal{A}$. Recently, Bansal and Prajapat [11] considered the normalization of the function $\mathbb{E}_{\alpha,\kappa}(z)$ given as

$$E_{\alpha,\kappa}(z)=\Gamma(\kappa)\,z\mathbb{E}_{\alpha,\kappa}(z)=z+\sum_{m=1}^{\infty}\frac{\Gamma(\kappa)\,z^{m+1}}{\Gamma(\alpha m+\kappa)},\quad \alpha,\ \kappa\in\mathbb{C},\ Re(\alpha)>0,\kappa\neq 0,-1,\cdots.$$

In this article, we investigate some geometric properties of function $E_{\alpha,\kappa}(z)$ with real parameters α and κ.

We need the following results in our investigations.

Lemma 1 ([12]). *If $g \in \mathcal{A}$ and*

$$\left|zg''(z)\right| < \frac{1-\mu}{4}, \quad z \in \mathcal{D};\ 0 \leq \mu < 1,$$

then

$$Re\left\{g'(z)\right\} > \frac{1+\mu}{2}, \quad z \in \mathcal{D};\ 0 \leq \mu < 1.$$

Lemma 2 ([13]). *Let $\kappa \in \mathbb{C}$ such that $Re(\kappa) > 0$, $c \in \mathbb{C}$ and $|c| \leq 1$, $c \neq -1$. If $h \in \mathcal{A}$ satisfies*

$$\left|c\,|z|^{2\beta} + \left(1 - |z|^{2\beta}\,\frac{zh''(z)}{\beta h'(z)}\right)\right| \leq 1, \quad z \in \mathcal{D},$$

then

$$C_\beta(z) = \left\{\beta \int_0^z t^{\beta-1}h'(t)dt\right\}^{1/\beta}, \quad z \in \mathcal{D}$$

is analytic and univalent in $\mathcal{D}$.

Lemma 3 ([14]). *Let $g(z) = z + a_2z^2 + ... + a_mz^m + ...$, be analytic in $\mathcal{D}$ and in addition $1 \geq 2a_2 \geq ... \geq ma_m \geq ... \geq 0$ or $1 \leq 2a_2 \leq ... \leq ma_m \leq ... \leq 2$, then $g(z)$ is in class $\mathcal{K}$ with respect to the function $z \rightarrow -\log(1-z)$. Also if the function $g(z) = z + 3a_3 + ... + a_{2m-1}z^{2m-1} + ...$, which is odd and analytic in $\mathcal{D}$ and satisfies in addition $1 \geq 3a_3 \geq ... \geq (2m+1)\,a_{2m+1} \geq ... \geq 0$ or $1 \leq 3a_3 \leq ... \leq (2m+1)\,a_{2m+1} \leq ... \leq 2$, then $g(z) \in \mathcal{S}$ in $\mathcal{D}$.*

Lemma 4 ([[15]). *If $g(z) = \sum\limits_{m=1}^{\infty} a_mz^{m-1}$, such that $a_1 = 1$ and $a_m \geq 0$, $\forall m \geq 2$, is analytic in $\mathcal{D}$ and if $\{a_m\}_{m=1}^{\infty}$ is a sequence which is decreasing, i.e., $a_{m+2} + a_m - 2a_{m+1} \geq 0$ and $a_m - a_{m+1} \geq 0$, $\forall m \geq 1$, then*

$$Re\left\{\sum_{m=1}^{\infty} a_mz^{m-1}\right\} > \frac{1}{2},\ \forall z \in \mathcal{D}.$$

Lemma 5 ([15]). *If $a_m \geq 0$, $\{ma_m\}$ and $\{ma_m - (m+1)a_{m+1}\}$ both are non-increasing, then the function g defined by (3) is in $\mathcal{S}^*$.*

2. Starlikeness, Convexity, Close-to-Convexity

Theorem 1. *Let $\alpha \geq \frac{3}{2}$ and $\kappa \geq \frac{3}{2}$. Then,*

$$Re\left\{\frac{E_{\alpha,\kappa}(z)}{z}\right\} > \frac{1}{2},\ for\ z \in \mathcal{D}.$$

Proof. For the proof of this result, we have to show that

$$\{a_m\}_{m=1}^{\infty} = \left\{\frac{\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)}\right\}_{m=1}^{\infty}$$

is a decreasing sequence. Consider

$$
\begin{aligned}
a_m - a_{m+1} &= \frac{\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)} - \frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} \\
&= \Gamma(\kappa)\left\{\frac{\Gamma(\alpha m+\kappa) - \Gamma(\alpha(m-1)+\kappa)}{\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha m+\kappa)}\right\} > 0,
\end{aligned}
$$

where $\forall\ m \geq 1$, $\alpha \geq \frac{3}{2}$ and $\kappa \geq \frac{3}{2}$. Now, to show that $\{a_m\}_{m=1}^{\infty}$ is decreasing, we prove that $a_m + a_{m+2} \geq 2a_{m+1}$.

Take

$$
a_m - 2a_{m+1} + a_{m+2} = \frac{\Gamma(\kappa)}{\Gamma(\alpha(m+1)+\kappa)} + \frac{\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)} - \frac{2\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)}
$$

$$
= \Gamma(\kappa)\left\{\frac{\begin{array}{c}\Gamma(\alpha m+\kappa)\,\Gamma(\alpha(m+1)+\kappa) - 2\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha(m+1)+\kappa)\\ +\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha m+\kappa)\end{array}}{\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha m+\kappa)\,\Gamma(\alpha(m+1)+\kappa)}\right\}
$$

$$
= \Gamma(\kappa)\left[\frac{\begin{array}{c}\Gamma(\alpha(m+1)+\kappa)\,\{\Gamma(\alpha m+\kappa) - 2\Gamma(\alpha(m-1)+\kappa)\}\\ +\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha m+\kappa)\end{array}}{\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha m+\kappa)\,\Gamma(\alpha(m+1)+\kappa)}\right].
$$

The above expression is non negative $\forall\ m \geq 1$, $\alpha \geq \frac{3}{2}$ and $\kappa \geq \frac{3}{2}$, which shows that $\{a_m\}_{m=1}^{\infty}$ is decreasing and convex sequence. Now, from the Lemma 4, we have

$$
Re\left(\sum_{m=1}^{\infty} b_m z^{m-1}\right) > \frac{1}{2},\ z \in \mathcal{D},
$$

which is equivalent to

$$
Re\left(\frac{E_{\alpha,\kappa}(z)}{z}\right) > \frac{1}{2},\ z \in \mathcal{D}.
$$

□

Theorem 2. *Let $\alpha \geq 2.67$ and $\kappa \geq 1$. Then, $E_{\alpha,\kappa}(z)$ is starlike in the open unit disc $\mathcal{D}$.*

Proof. To show that $E_{\alpha,\kappa}(z)$ is starlike in $\mathcal{D}$, we prove that $\{ma_m\}$ and $\{ma_m - (m+1)a_{m+1}\}$ both are non-increasing in view of Lemma 5. Since $a_m \geq 0$ for the normalized Mittag–Leffler function under the given conditions, consider

$$
\begin{aligned}
ma_m - (m+1)a_{m+1} &= \frac{m\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)} - \frac{(m+1)\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} \\
&= \Gamma(\kappa)\left\{\frac{m\Gamma(\alpha m+\kappa) - (m+1)\Gamma(\alpha(m-1)+\kappa)}{\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha m+\kappa)}\right\} > 0
\end{aligned}
$$

for $m \geq 1$, $\alpha \geq 2.67$ and $\kappa \geq 1$. Now,

$$
\begin{aligned}
ma_m - 2(m+1)a_{m+1} + (m+2) &= \frac{m\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)} - \frac{2(m+1)\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} + \frac{(m+2)\Gamma(\kappa)}{\Gamma(\alpha(m+1)+\kappa)} \\
&= \Gamma(\kappa)\left\{ \frac{\begin{array}{c} -2(m+1)\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha(m+1)+\kappa) + \\ m\Gamma(\alpha m+\kappa)\Gamma(\alpha(m+1)+\kappa) + (m+2)\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha m+\kappa) \end{array}}{\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha m+\kappa)\Gamma(\alpha(m+1)+\kappa)} \right\} \\
&= \Gamma(\kappa)\left[\frac{\begin{array}{c} \Gamma(\alpha(m+1)+\kappa)\{m\Gamma(\alpha m+\kappa) - 2(m+1)\Gamma(\alpha(m-1)+\kappa)\} \\ +(m+2)\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha m+\kappa) \end{array}}{\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha m+\kappa)\Gamma(\alpha(m+1)+\kappa)} \right].
\end{aligned}
$$

The above relation is non-negative $\forall\, m \geq 1$, $\alpha \geq 2.67$ and $\kappa \geq 1$. Thus, from Lemma 5, $E_{\alpha,\kappa}(z)$ is starlike in $\mathcal{D}$. □

Theorem 3. *Let $\alpha \geq 3.323$ and $\kappa \geq 1$. Then,*

$$
Re\left\{E'_{\alpha,\kappa}(z)\right\} > \frac{1}{2}, \quad (z \in \mathcal{D}).
$$

Proof. Consider

$$
\begin{aligned}
E_{\alpha,\kappa}(z) &= z + \sum_{m=2}^{\infty} \frac{\Gamma(\kappa)\, z^m}{\Gamma(\alpha(m-1)+\kappa)}, \\
E'_{\alpha,\kappa}(z) &= 1 + \sum_{m=2}^{\infty} \frac{m\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)} z^{m-1}, \\
E'_{\alpha,\kappa}(z) &= 1 + \sum_{m=2}^{\infty} A_m z^{m-1}.
\end{aligned}
$$

Here, $A_m = \frac{m\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)}$. By taking the same computations as in Theorem 2, we get the proof. □

Theorem 4. *If $\alpha \geq 1$ and $\kappa \geq 1$, then $z \to E_{\alpha,\kappa}(z)$ is in $\mathcal{K}$ with respect to the function $-\log(1-z)$.*

Proof. Set

$$
E_{\alpha,\kappa}(z) = z + \sum_{m=2}^{\infty} a_{m-1} z^m,
$$

and we have $a_{m-1} > 0$ for all $m \geq 2$ and $a_1 = \frac{\Gamma(\kappa)}{\Gamma(\alpha+\kappa)} \leq 1$. For the proof of this result, we use Lemma 3. Therefore, we have to show that $\{ma_{m-1}\}_{m\geq 2}$ is decreasing. Now,

$$
\begin{aligned}
ma_{m-1} - (m+1)\, a_m = \Gamma(\kappa)&\left[\frac{m}{\Gamma(\alpha(m-1)+\kappa)} - \frac{m+1}{\Gamma(\alpha m+\kappa)}\right], \\
= \quad \Gamma(\kappa)&\left[\frac{m\Gamma(\alpha m+\kappa) - (m+1)\Gamma(\alpha(m-1)+\kappa)}{\Gamma(\alpha(m-1)+\kappa)\Gamma(\alpha m+\kappa)}\right] > 0.
\end{aligned}
$$

By restricting parameters, we note that $ma_{m-1} - (m+1)\, a_m > 0$ for all $m \geq 2$. Thus, $\{ma_{m-1}\}_{m\geq 2}$ is a decreasing sequence—hence the result. □

Theorem 5. *If $\alpha \geq 1$ and $\kappa \geq 1$, then $z \to zE_{\alpha,\kappa}(z^2)$ is in $\mathcal{K}$ respect to the function $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.*

Proof. Set

$$zE_{\alpha,\kappa}\left(z^2\right) = z + \sum_{m=2}^{\infty} A_{2m-1} z^{2m-1}.$$

Here, $A_{2m-1} = a_{m-1} = \frac{\Gamma(\kappa)}{\Gamma(\alpha(m-1)+\kappa)}$ for all $m \geq 2$. In addition, it is clear that $a_1 \leq 1$. Mainly, we have to show that $\{(2m-1)\, a_{m-1}\}_{m\geq 2}$ is decreasing. Now,

$$\begin{aligned}(2m-1)\, a_{m-1} - (2m+1)\, a_m &= \Gamma(\kappa)\left[\frac{(2m-1)}{\Gamma(\alpha(m-1)+\kappa)} - \frac{(2m+1)}{\Gamma(\alpha m+\kappa)}\right], \\ &= \Gamma(\kappa)\left[\frac{(2m-1)\,\Gamma(\alpha m+\kappa) - (2m+1)\,\Gamma(\alpha(m-1)+\kappa)}{\Gamma(\alpha(m-1)+\kappa)\,\Gamma(\alpha m+\kappa)}\right] > 0.\end{aligned}$$

By using conditions on parameters, we observe that $(2m-1)\, a_{m-1} - (2m+1)\, a_m > 0$ for all $m \geq 2$. Thus, $\{(2m-1)\, a_{m-1}\}_{m\geq 2}$ is a decreasing sequence. By applying Lemma 3, we have the required result. □

Theorem 6. *If $\alpha \geq 1$ and $\kappa \geq 3.214319744$, then $E_{\alpha,\kappa}(z) \in \mathcal{S}^*$ in $\mathcal{D}$.*

Proof. Let $p(z) = \frac{zE'_{\alpha,\kappa}(z)}{E_{\alpha,\kappa}(z)}$, $z \in \mathcal{D}$. Then, the function p is analytic in $\mathcal{D}$ with $p(0) = 1$. To prove $E_{\alpha,\kappa}(z)$ is starlike in $\mathcal{D}$, we just prove that $Rep(z) > 0$ in $z \in \mathcal{D}$. For this, it is enough to show $|p(z) - 1| < 1$ for $z \in \mathcal{D}$. By using the inequalities

$$\begin{aligned}\frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} &\leq \frac{1}{(\kappa)_m},\ \alpha \geq 1, \kappa \geq 1,\ m \in \mathbb{N}, \\ m\,(\kappa)_m &\leq 2^{m-1}\kappa\,(\kappa+1)_{m-1}, \kappa \geq 1,\ m \in \mathbb{N},\end{aligned}$$

we have

$$\begin{aligned}\left|E'_{\alpha,\kappa}(z) - \frac{E_{\alpha,\kappa}(z)}{z}\right| &= \left|\sum_{m=1}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} m z^m\right| \\ &\leq \sum_{m=1}^{\infty} \frac{2^{m-1}}{\kappa\,(\kappa+1)^{m-1}} \\ &\leq \frac{1}{\kappa}\sum_{m=1}^{\infty}\left(\frac{2}{\kappa+1}\right)^{m-1} \\ &= \frac{\kappa+1}{\kappa(\kappa-1)}, \quad (\kappa > 1).\end{aligned} \tag{4}$$

Furthermore, using reverse triangle inequality and the inequality $(\kappa)^m \leq (\kappa)_m$, we obtain

$$\begin{aligned}\left|\frac{E_{\alpha,\kappa}(z)}{z}\right| &= \left|1 + \sum_{m=1}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} z^m\right| \\ &\geq 1 - \sum_{m=1}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} \\ &\geq 1 - \sum_{m=1}^{\infty} \frac{1}{(\kappa)_m} \\ &\geq 1 - \frac{1}{\kappa}\sum_{m=1}^{\infty}\left(\frac{1}{\kappa+1}\right)^{m-1} \\ &= \frac{\kappa^2-\kappa-1}{\kappa^2} \quad (\kappa > 0).\end{aligned} \tag{5}$$

By combining (4) and (5), we get

$$\left|\frac{zE'_{\alpha,\kappa}(z)}{E_{\alpha,\kappa}(z)}-1\right| \leq \frac{\kappa(\kappa+1)}{(\kappa-1)(\kappa^2-\kappa-1)}. \tag{6}$$

Therefore, $E_{\alpha,\kappa}(z) \in \mathcal{S}^*$ in $\mathcal{D}$ if $\frac{\kappa(\kappa+1)}{(\kappa-1)(\kappa^2-\kappa-1)} \leq 1$. In other words, we have to show that $\kappa^3-3\kappa^2-\kappa+1 \geq 0$. The inequality is satisfied for $\kappa \geq 3.214319744$. Hence, $E_{\alpha,\kappa}(z)$ is starlike in $\mathcal{D}$. □

Remark 1. *Recently, Bansal and Prajpat [11] proved that* $E_{\alpha,\kappa}(z)$ *is starlike, if* $\alpha \geq 1$ *and* $\kappa \geq (3+\sqrt{17})/2 \approx 3.56155281$. *The above result improves the result in [11].*

Theorem 7. *If* $\alpha \geq 1$ *and* $\kappa \geq 3.56155281$, *then* $E_{\alpha,\kappa}(z) \in \mathcal{C}$ *in* $\mathcal{D}$.

Proof. Let $p(z) = 1+\frac{zE''_{\alpha,\kappa}(z)}{E'_{\alpha,\kappa}(z)}$, $z \in \mathcal{D}$. Then, $p(z)$ is analytic in $\mathcal{D}$ with $p(0)=1$. To show that $E_{\alpha,\kappa}(z)$ is convex in $\mathcal{D}$, it is enough to prove that $|p(z)-1|<1$, $z \in \mathcal{D}$. By using the inequalities

$$\begin{aligned}
\frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} &\leq \frac{1}{(\kappa)_m},\ \alpha \geq 1, \kappa \geq 1,\ m \in \mathbb{N},\\
2m(m+1)(\kappa)_m &\leq 4^{m-1}\kappa(\kappa+1)_{m-1}, \kappa \geq 1,\ m \in \mathbb{N},
\end{aligned}$$

we have

$$\begin{aligned}
\left|zE''_{\alpha,\kappa}(z)\right| &= \left|\sum_{m=1}^{\infty}\frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)}m(m+1)z^m\right|\\
&\leq \sum_{m=1}^{\infty}\frac{4^{m-1}}{2\kappa(\kappa+1)^{m-1}}\\
&\leq \frac{2}{\kappa}\sum_{m=1}^{\infty}\left(\frac{4}{\kappa+1}\right)^{m-1}\\
&= \frac{2(\kappa+1)}{\kappa(\kappa-3)}, \quad (\kappa>3).
\end{aligned} \tag{7}$$

Furthermore, using the inequality $m(\kappa)^m \leq 2^{m-1}(\kappa)_m$, then we have

$$\begin{aligned}
\left|E'_{\alpha,\kappa}(z)\right| &= \left|1+\sum_{m=1}^{\infty}(m+1)\frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)}z^m\right|\\
&\geq 1-\sum_{m=1}^{\infty}(m+1)\frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)}\\
&\geq 1-\sum_{m=1}^{\infty}\frac{1}{(\kappa)_m}\\
&\geq 1-\frac{2}{\kappa}\sum_{m=1}^{\infty}\left(\frac{2}{\kappa+1}\right)^{m-1}\\
&= \frac{\kappa^2-3\kappa-2}{\kappa(\kappa-1)} \quad (\kappa>0).
\end{aligned} \tag{8}$$

From (7) and (8), we get

$$\left|\frac{zE''_{\alpha,\kappa}(z)}{E'_{\alpha,\kappa}(z)}\right| \leq \frac{2(\kappa^2-1)}{(\kappa-1)(\kappa^2-3\kappa-2)}. \tag{9}$$

This implies that $E_{\alpha,\kappa}(z) \in \mathcal{C}$ in $\mathcal{D}$ if $\frac{2(\kappa^2-1)}{(\kappa-1)(\kappa^2-3\kappa-2)} \leq 1$. To prove our result, we have to show that $\kappa^3 - 6\kappa^2 + 7\kappa + 6 \geq 0$. The inequality is satisfied for $\kappa \geq 3.5615528$. Hence, $E_{\alpha,\kappa}(z)$ is convex in $\mathcal{D}$. $\square$

Consider the integral operator $\mathcal{F}_\gamma : \mathcal{D} \to \mathbb{C}$, where $\gamma \in \mathbb{C}$, $\gamma \neq 0$,

$$\mathcal{F}_\gamma(z) = \left\{ \gamma \int_0^z t^{\gamma-2} E_{\alpha,\kappa}(t)\, dt \right\}, \quad z \in \mathcal{D}.$$

Here, $\mathcal{F}_\gamma \in \mathcal{A}$. We prove that $\mathcal{F}_\gamma \in \mathcal{S}$ in $\mathcal{D}$.

Theorem 8. *Let $M \in \mathbb{R}^+$ such that $|E_{\alpha,\kappa}(z)| \leq M$ in $\mathcal{D}$. If*

$$|\gamma - 1| + \frac{\kappa(\kappa+1)}{(\kappa-1)(\kappa^2-\kappa-1)} + \frac{M}{|\gamma|} \leq 1,$$

then $\mathcal{F}_\gamma \in \mathcal{S}$ in $\mathcal{D}$.

Proof. A calculation gives

$$\frac{z\mathcal{F}_\gamma''(z)}{\mathcal{F}_\gamma'(z)} = \frac{zE_{\alpha,\kappa}'(z)}{E_{\alpha,\kappa}(z)} + \frac{z^{\gamma-1}}{\gamma} E_{\alpha,\kappa}(z) + \gamma - 2, \quad z \in \mathcal{D}.$$

Since $E_{\alpha,\kappa}(z) \in \mathcal{A}$, then by Schwarz Lemma, triangle inequality and (6), we obtain

$$\begin{aligned} \left(1-|z|^2\right) \frac{z\mathcal{F}_\gamma''(z)}{\mathcal{F}_\gamma'(z)} &\leq \left(1-|z|^2\right)\left[|\gamma-1| + \left|\frac{zE_{\alpha,\kappa}'(z)}{E_{\alpha,\kappa}(z)} - 1\right| + \frac{|z|^{\gamma-1}}{|\gamma|}\left|\frac{E_{\alpha,\kappa}(z)}{z}\right|\right] \\ &\leq \left(1-|z|^2\right)\left[|\gamma-1| + \frac{\kappa(\kappa+1)}{(\kappa-1)(\kappa^2-\kappa-1)} + \frac{M}{|\gamma|}\right]. \end{aligned}$$

By using Lemma 2, $\mathcal{F}_\gamma \in \mathcal{S}$ in $\mathcal{D}$. $\square$

Theorem 9. *Let $\alpha \geq 1$, $\mu \in [0,1)$ and $z \in \mathcal{D}$.*
(i) If $\kappa > \frac{(11-3\mu)+\sqrt{\mu^2-12\mu+17}}{2(1-\mu)}$, then $E_{\alpha,\kappa}(z) \in \mathcal{K}\left(\frac{1+\mu}{2}\right)$.
(ii) If $\preceq < 1 - \frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1)+(\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)}$, then $\frac{E_{\alpha,\kappa}(z)}{z} \in \mathcal{P}(\mu)$.
(iii) If $(1-\mu)\kappa^3 + (2\mu-3)\kappa^2 - \kappa + (1-\mu) > 0$, then $E_{\alpha,\kappa}(z) \in \mathcal{S}^*(\mu)$.
(iv) If $(1-\mu)\kappa^3 + (6\mu-8)\kappa^2 + (7-7\mu)\kappa + (8-6\mu) > 0$, then $E_{\alpha,\kappa}(z) \in \mathcal{C}(\mu)$.

Proof. (i) Using (7) and Lemma 1, we get

$$\left|zE_{\alpha,\kappa}''(z)\right| \leq \frac{2(\kappa+1)}{\kappa(\kappa-3)} < \frac{1-\mu}{4},$$

where $0 \leq \mu < 1 - \frac{8(\kappa+1)}{\kappa(\kappa-3)}$ and $\kappa > \frac{(11-3\mu)+\sqrt{\mu^2-12\mu+17}}{2(1-\mu)}$. This shows that $E_{\alpha,\kappa}(z) \in \mathcal{K}\left(\frac{1+\mu}{2}\right)$.

(ii) To prove $\frac{E_{\alpha,\kappa}(z)}{z} \in \mathcal{P}(\mu)$, we have to show that $|g(z)-1| < 1$, where $g(z) = \frac{\{E_{\alpha,\kappa}(z)/z\}-\mu}{(1-\mu)}$. By using triangle inequality with

$$\begin{aligned} \frac{\Gamma(\kappa)}{\Gamma(\alpha m+\kappa)} &\leq \frac{1}{(\kappa)_m}, \quad m \in \mathbb{N}, \\ (\kappa)_m &> \kappa(\kappa+\alpha_0)^{m-1}, \quad (\kappa > 0;\ m \in \mathbb{N}\backslash\{1,2\}), \end{aligned}$$

(see [16]), where

$$\alpha_0 \approx 1.302775637...$$

is the largest root of the equation

$$\alpha^2 + \alpha - 3 = 0,$$

we have

$$\begin{aligned} |g(z) - 1| &= \left| \frac{1}{(1-\mu)} \sum_{m=1}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\alpha m + \kappa)} z^m \right| \\ &\leq \frac{1}{(1-\mu)} \sum_{m=1}^{\infty} \frac{1}{(\kappa)_m} \\ &\leq \frac{1}{(1-\mu)} \left\{ \frac{1}{\kappa} + \frac{1}{\kappa(\kappa+1)} + \sum_{m=3}^{\infty} \frac{1}{\kappa(\kappa+\alpha_0)^{m-1}} \right\} \\ &= \frac{1}{(1-\mu)} \frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1) + (\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)}. \end{aligned}$$

This implies that $\frac{E_{\alpha,\kappa}(z)}{z} \in \mathcal{P}(\mu)$, for $0 < \mu < 1 - \frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1)+(\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)}$.

(iii) We use the inequality $\left| \frac{zE'_{\alpha,\kappa}(z)}{E_{\alpha,\kappa}(z)} - 1 \right| < 1 - \mu$ to show the starlikeness of order μ for the function $E_{\alpha,\kappa}(z)$. By using (4) and (5), we have

$$\left| \frac{zE'_{\alpha,\kappa}(z)}{E_{\alpha,\kappa}(z)} - 1 \right| \leq \frac{\kappa(\kappa+1)}{(\kappa-1)(\kappa^2-\kappa-1)} < 1 - \mu.$$

This implies that

$$\mu < 1 - \frac{\kappa(\kappa+1)}{(\kappa-1)(\kappa^2-\kappa-1)}.$$

This completes the proof.

(iv) We use the inequality $\left| \frac{zE''_{\alpha,\kappa}(z)}{E'_{\alpha,\kappa}(z)} \right| < 1 - \mu$ to show that $E_{\alpha,\kappa}(z) \in \mathcal{C}(\mu)$. By using (7) and (8), we have

$$\left| \frac{zE''_{\alpha,\kappa}(z)}{E'_{\alpha,\kappa}(z)} \right| \leq \frac{2(\kappa^2-1)}{(\kappa-3)(\kappa^2-3\kappa-2)} < 1 - \mu.$$

This implies that

$$\mu < 1 - \frac{2(\kappa^2-1)}{(\kappa-3)(\kappa^2-3\kappa-2)},$$

hence the result. □

Substituting $\mu = 0$ in Theorem 9, we obtained the following results.

Corollary 1. *Let* $\alpha \geq 1, z \in \mathcal{D}$.

(i) If $\kappa > \frac{11+\sqrt{17}}{2}$, *then* $E_{\alpha,\kappa}(z) \in \mathcal{K}\left(\frac{1}{2}\right)$.

(ii) If $\frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1)+(\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)} < 1$, *then* $\frac{E_{\alpha,\kappa}(z)}{z} \in \mathcal{P}$.

(iii) If $\kappa^3 - 3\kappa^2 - \kappa + 1 > 0$, *then* $E_{\alpha,\kappa}(z) \in \mathcal{S}^*$.

(iv) If $\kappa^3 - 8\kappa^2 + 7\kappa + 8 > 0$, *then* $E_{\alpha,\kappa}(z) \in \mathcal{C}$.

Remark 2. *It is clear that* $E_{\alpha,\kappa}(z) \in \mathcal{K}\left(\frac{1}{2}\right)$ *when* $\alpha \geq 1$, $\kappa > 7.56155$ *and* $E_{\alpha,\kappa}(z) \in \mathcal{C}$ *when* $\alpha \geq 1$, $\kappa > 6.796963$. *It concludes that our results improve the results of ([17], corollary 2.1).*

3. Hardy Space of Mittag–Leffler Function

Consider the class $\mathcal{H}$ of analytic functions in $\mathcal{D} = \{z : |z| < 1\}$ and $\mathcal{H}^\infty$ denote the space bounded functions on $\mathcal{H}$. Let $g \in \mathcal{H}$, set

$$M_q(r,g) = \begin{cases} \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \left| g\left(re^{i\theta}\right)\right|^q d\theta \right)^{1/q}, & 0 < q < \infty, \\ \max\{|g(z)| : |z| \le r\}, & q = \infty. \end{cases}$$

If $M_q(r,g)$ is bounded for $r \in [0,1)$, then $g \in \mathcal{H}^q$. It is clear that

$$\mathcal{H}^\infty \subset \mathcal{H}^p \subset \mathcal{H}^q,\ 0 < p < q < \infty.$$

For some details, see [18]. It is also known [18] that, if $Re\left(g'(z)\right) > 0$ in $\mathcal{D}$, then

$$\begin{cases} g' \in \mathcal{H}^p, & p < 1, \\ g \in \mathcal{H}^{p/(1-p)}, & 0 < p < 1. \end{cases}$$

Hardy spaces of certain special functions are studied in [10,19,20].

Lemma 6 ([21])**.** *$\mathcal{P}_0(\mu) * \mathcal{P}_0(\eta) \subset \mathcal{P}_0(\gamma)$, where $\gamma = 1 - 2(1-\mu)(1-\eta)$ and $\mu,\ \eta < 1$. The value γ can not be improved.*

Lemma 7 ([22])**.** *For $\mu,\ \eta < 1$ and $\gamma = 1 - 2(1-\mu)(1-\eta)$, we have $\mathcal{R}_0(\mu) * \mathcal{R}_0(\eta) \subset \mathcal{R}_0(\gamma)$, or equivalently $\mathcal{P}_0(\mu) * \mathcal{P}_0(\eta) \subset \mathcal{P}_0(\gamma)$.*

Lemma 8 ([23])**.** *If the function g, convex of order μ, where $\mu \in [0,1)$, is not of the form*

$$g(z) = \begin{cases} l + dz\left(1 - ze^{i\varsigma}\right)^{2\mu-1}, & \mu \ne 1/2, \\ l + d\log\left(1 - ze^{i\varsigma}\right), & \mu = 1/2, \end{cases}$$

for $d,\ l \in \mathbb{C}$, and $\varsigma \in \mathbb{R}$, then the following statements are true:
(i) There exist $\delta = \delta(g) > 0$ such that $g' \in \mathcal{H}^{\delta+1/[2(1-\mu)]}$.
(ii) If $\mu \in [0,1/2)$, then there exists $\tau = \tau(g) > 0$ such that $g \in \mathcal{H}^{\tau+1/(1-2\mu)}$.
(iii) If $\mu \ge 1/2$, then $g \in \mathcal{H}^\infty$.

Theorem 10. *Let $\mu \in [0,1)$, $(1-\mu)\kappa^3 + (6\mu - 8)\kappa^2 + (7 - 7\mu)\kappa + (8 - 6\mu) > 0$.*
(i) If $\mu \in [0,1/2)$, then $E_{\alpha,\kappa}(z) \in \mathcal{H}^{1/(1-2\mu)}$.
(ii) If $\mu \ge 1/2$, then $E_{\alpha,\kappa}(z) \in \mathcal{H}^\infty$.

Proof. By using the definition of the hypergeometric function

$${}_2F_1(a,b,c;z) = \sum_{m=0}^{\infty} \frac{(a)^m (b)^m}{(c)^m} \frac{z^m}{m!},$$

we have

$$\begin{aligned} l + \frac{dz}{\left(1 - ze^{i\varsigma}\right)^{1-2\mu}} &= l + dz{}_2F_1\left(1, 1-2\alpha, 1; ze^{i\varsigma}\right), \\ &= l + d\sum_{m=0}^{\infty} \frac{(1-2\alpha)_m}{m!} e^{i\varsigma m} z^{m+1}, \end{aligned}$$

for $l,\ d \in \mathbb{C}$, $\mu \neq 1/2$ and for real ς. On the other hand,

$$\begin{aligned} l + d\log\left(1 - ze^{i\gamma}\right) &= l - dz{}_2F_1\left(1,1,2;ze^{i\varsigma}\right), \\ &= l - d\sum_{m=0}^{\infty}\frac{1}{m+1}e^{i\varsigma m}z^{m+1}. \end{aligned}$$

Therefore, the function $E_{\alpha,\kappa}(z)$ is not of the form of $l + dz\left(1 - ze^{i\gamma}\right)^{2\mu-1}$ (for $\mu \neq 1/2$) and $l + d\log\left(1 - ze^{i\gamma}\right)$ (for $\mu = 1/2$). We know that, by part (iv) of Theorem 9, $E_{\alpha,\kappa}(z) \in \mathcal{C}(\mu)$. Therefore, by using Lemma 8, we have the required result. □

Theorem 11. *Let* $\frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1)+(\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)} < 1$ *and* $f \in \mathcal{D}$*. Then, convolution* $E_{\alpha,\kappa} * f$ *is in* $\mathcal{H}^{\infty} \cap \mathcal{R}$*.*

Proof. Let $h(z) = E_{\alpha,\kappa}(z) * g(z)$. Then, $h'(z) = \frac{E_{\alpha,\kappa}(z)}{z} * g'(z)$. Using the Corollary 1 part *ii*, we have $\frac{E_{\alpha,\kappa}(z)}{z} \in \mathcal{P}$. As $g \in \mathcal{R}$; therefore, by using Lemma 6 $h \in \mathcal{R}$. Now, the function $\frac{E_{\alpha,\kappa}(z)}{z}$ is complete; therefore, $h(z)$ is complete. This implies that $h(z)$ is bounded. Thus, we have the required result. □

Theorem 12. *Let* $\preceq < 1 - \frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1)+(\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)}$, $\mu \in [0,1)$ *and* $z \in \mathcal{D}$*. If* $g \in \mathcal{P}(\eta)$*, then* $E_{\alpha,\kappa}(z) * g \in \mathcal{R}(\gamma)$*, where* $\gamma = 1 - 2(1-\mu)(1-\eta)$*.*

Proof. Let $h(z) = E_{\alpha,\kappa}(z) * g(z)$. Then, it is clear that $h'(z) = \frac{E_{\alpha,\kappa}(z)}{z} * g'(z)$. Using Theorem 9 part (ii), we have $\frac{E_{\alpha,\kappa}(z)}{z} \in \mathcal{P}(\mu)$. As $g \in \mathcal{R}$, therefore, by using Lemma 6 and the fact that $g' \in \mathcal{P}(\eta)$, we have $h'(z) \in \mathcal{P}(\gamma)$, where $.\gamma = 1 - 2(1-\mu)(1-\eta)$. Consequently, $h \in \mathcal{R}(\gamma)$. □

Corollary 2. *Let* $\mu \in [0,1)$ *and* $\preceq < 1 - \frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1)+(\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)}$*. If* $g \in \mathcal{R}(\eta)$, $\eta = (1-2\mu)/(2-2\mu)$*, then* $E_{\alpha,\kappa}(z) * g \in \mathcal{R}(0)$*.*

Corollary 3. *Let* $\mu \in [0,1)$ *and* $\frac{\{(\kappa+2)(\kappa+\alpha_0)(\kappa+\alpha_0-1)+(\kappa+1)\}}{\kappa(\kappa+1)(\kappa+\alpha_0)(\kappa+\alpha_0-1)} < 1$*. If* $g \in \mathcal{R}(1/2)$*, then* $E_{\alpha,\kappa}(z) * g \in \mathcal{R}(0)$*.*

4. Applications

Now, we present some applications of the above theorems. It is clear that

$$E_{1,2}(z) = e^z - 1,\ E_{1,3}(z) = \frac{2e^z - z - 1}{z},\ \ E_{1,4}(z) = \frac{6e^z - 3z^2, -6z - 6}{z^2}.$$

From Theorem 9, we get the following:

Corollary 4. (i) *If* $0 \le \mu < \mu_0$*, where* $\mu_0 \approx 0.26759$*, then* $E_{1,2}(z) \in \mathcal{P}(\mu)$*.*
(ii) *If* $0 \le \mu < \mu_1$*, where* $\mu_1 \approx 0.55988$*, then* $E_{1,3}(z) \in \mathcal{P}(\mu)$*.*
(iii) *If* $0 \le \mu < \mu_2$*, where* $\mu_2 \approx 0.68904$*, then* $E_{1,4}(z) \in \mathcal{P}(\mu)$*.*

Corollary 5. *If* $0 \le \mu < \mu_3$*, where* $\mu_3 \approx 0.39393$*, then* $E_{1,4}(z) \in \mathcal{S}^*(\mu)$*.*

Corollary 6. (i) *Let* $0 \le \mu < \mu_4$*, where* $\mu_4 \approx 0.2675930$*. If* $g \in \mathcal{R}(\eta)$*,* $\eta = (1-2\mu)/(2-2\mu)$*, then* $E_{1,2}(z) * g \in \mathcal{R}(0)$*.*
(ii) *Let* $0 \le \mu < \mu_5$*, where* $\mu_5 \approx 0.55987780$*. If* $g \in \mathcal{R}(\eta)$*,* $\eta = (1-2\mu)/(2-2\mu)$*, then* $E_{1,3}(z) * g \in \mathcal{R}(0)$*.*
(iii) *Let* $0 \le \mu < \mu_6$*, where* $\mu_6 \approx 0.68904320$*. If* $g \in \mathcal{R}(\eta)$*,* $\eta = (1-2\mu)/(2-2\mu)$*, then* $E_{1,4}(z) * g \in \mathcal{R}(0)$*.*

5. Conclusions

In this paper, we have studied certain geometric properties of Mittag-Leffler functions such as starlikeness, convexity and close-to-convexity. We have also found the Hardy spaces of Mittag-Leffler functions. Further, we have improved some results in the literature.

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