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Multivalued Fixed Point Results in Dislocated *b*-Metric Spaces with Application to the System of Nonlinear Integral Equations

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Abstract: The purpose of this paper is to find out fixed point results for a pair of semi α_*-dominated multivalued mappings fulfilling a generalized locally F-dominated multivalued contractive condition on a closed ball in complete dislocated b-metric space. Some new fixed point results with graphic contractions on closed ball for a pair of multi graph dominated mappings on dislocated b-metric space have been established. An application to the existence of unique common solution of a system of integral equations is presented. 2010 Mathematics Subject Classification: 46Txx, 47H04, 47H10; 54H25.

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1. Introduction and Preliminaries

Fixed point theory plays a foundational role in functional analysis. Banach [1] proved significant result for contraction mappings. Due to its significance, a large number of authors have proved many interesting multiplications of his result (see [1–34]). Recently, Kumari et al. [22] discussed some fixed point theorem in b-dislocated metric space and proved efficient soloution for a non-linear integral equations and non-linear fractional differential equations. In this paper, we have obtained common fixed point for a pair of multivalued mappings satisfying generalized rational type F-dominated contractive conditions on a closed ball in complete dislocated b-metric space. We have used weaker class of strictly increasing mappings F rather than class of mappings F used by Wardowski [34]. Moreover, we investigate our results in a better framework of dislocated b-metric space. Additionally, some new fixed point results with graphic contractions on closed ball for multi graph dominated mappings on dislocated b-metric space have been established. New results in ordered spaces, partial b-metric space, dislocated metric space, partial metric space, b-metric space, and metric space can be obtained as corollaries of our results. We give the following concepts which will be helpful to understand the paper.

Definition 1 ([16])**.** *Let Z be a nonempty set and $d_l : Z \times Z \to [0, \infty)$ be a function, called a dislocated b-metric (or simply d_l-metric), if there exists $b \geq 1$ such that for any $g, p, q \in Z$, the following conditions hold:*

(i) If $d_l(g, p) = 0$, then $g = p$;

(ii) $d_l(g, p) = d_l(p, g)$;

(iii) $d_l(g,p) \leq b[d_l(g,q) + d_l(q,p)]$.

The pair (Z, d_l) *is called a dislocated b-metric space. It should be noted that every dislocated metric is a dislocated b-metric with* $b = 1$.

It is clear that if $d_l(g,p) = 0$, *then from (i),* $g = p$. *But if* $g = p$, $d_l(g,p)$ *may not be* 0. *For* $g \in Z$ *and* $\varepsilon > 0$, $\overline{B(g,\varepsilon)} = \{p \in Z : d_l(g,p) \leq \varepsilon\}$ *is a closed ball in* (Z, d_l). *We use D.B.M.S instead dislocated b-metric space. Let* $Z = \mathbb{Q}^+ \cup \{0\}$. *Define* $d_l(g,p) = (g+p)^2$ *for all* $g, p \in Z$. *Then* (Z, d_l) *is a D.B.M.S with constant* $b = 2$.

Definition 2 ([16]). *Let* (Z, d_l) *be a D.B.M.S .*

(i) A sequence $\{g_n\}$ *in* (Z, d_l) *is called Cauchy sequence if given* $\varepsilon > 0$, *there corresponds* $n_0 \in N$ *such that for all* $n, m \geq n_0$ *we have* $d_l(g_m, g_n) < \varepsilon$ *or* $\lim_{n,m\to\infty} d_l(g_n, g_m) = 0$.

(ii) A sequence $\{g_n\}$ *dislocated b-converges (for short* d_l *-converges) to* g *if* $\lim_{n\to\infty} d_l(g_n, g) = 0$. *In this case* g *is called a* d_l*-limit of* $\{g_n\}$.

(iii) (Z, d_l) *is called complete if every Cauchy sequence in* Z *converges to a point* $g \in Z$ *such that* $d_l(g,g) = 0$.

Definition 3. *Let* Q *be a nonempty subset of D.B.M.S of* Z *and let* $g \in Z$. *An element* $p_0 \in Q$ *is called a best approximation in* Q *if*

$$d_l(g, Q) = d_l(g, p_0), \text{ where } d_l(g, Q) = \inf_{p\in Q} d_l(g,p).$$

We denote $P(Z)$ *be the set of all closed proximinal subsets of* Z. *Let* $X = \mathbb{R}^+ \cup \{0\}$ *and* $d_l(g,p) = (g+p)^2$. *Define a set* $A = [3,5]$, *then for each* $x \in X$

$$d_l(x, A) = d_l(x, [3,5]) = \inf_{u\in[3,5]} d_l(x,u) = d_l(x,3).$$

Hence 3 *is a best approximation in* A *for each* $x \in X$. *Also,* $[3,5]$ *is a proximinal set.*

Definition 4 ([32]). *The function* $H_{d_l} : P(Z) \times P(Z) \to R^+$, *defined by*

$$H_{d_l}(N, M) = \max\{\sup_{n\in N} d_l(n, M), \sup_{m\in M} d_l(N, m)\}$$

is called dislocated Hausdorff b-metric on $P(Z)$. *Let* $X = \mathbb{R}^+ \cup \{0\}$ *and* $d_l(x,y) = (x+y)^2$. *If* $N = [3,5]$, $R = [7,8]$, *then* $H_{d_l}(N, R) = 144$.

Definition 5 ([32]). *Let* $S : Z \to P(Z)$ *be a multivalued mapping and* $\alpha : Z \times Z \to [0, +\infty)$. *Let* $K \subseteq Z$, *we say that* S *is semi* α_**-admissible on* K, *whenever* $\alpha(i,j) \geq 1$ *implies that* $\alpha_*(Si, Sj) \geq 1$, *for all i,j* $\in K$, *where* $\alpha_*(Si, Sj) = inf\{\alpha(u,v) : u \in Si, v \in Sj\}$. *If* $K = Z$, *then we say that* S *is* α_**-admissible.*

Definition 6. *Let* (Z, d_l) *be a D.B.M.S. Let* $S : Z \to P(Z)$ *be multivalued mapping and* $\alpha : Z \times Z \to [0, +\infty)$. *Let* $A \subseteq Z$, *we say that the* S *is semi* α_**-dominated on* H, *whenever* $\alpha_*(i, Si) \geq 1$ *for all* $i \in H$, *where* $\alpha_*(i, Si) = \inf\{\alpha(i,l) : l \in Si\}$. *If* $H = Z$, *then we say that the* S *is* α_**-dominated. If* $S : Z \to Z$ *be a self mapping, then* S *is semi* α*-dominated on* H, *whenever* $\alpha(i, Si) \geq 1$ *for all* $i \in H$.

Definition 7 ([34]). *Let* (Z, d) *be a metric space. A mapping* $H : Z \to Z$ *is said to be an F-contraction if there exists* $\tau > 0$ *such that*

$$\forall j, k \in Z, \ d(Hj, Hk) > 0 \Rightarrow \tau + F\left(d(Hj, Hk)\right) \leq F\left(d(j,k)\right)$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ *is a mapping satisfying the following conditions:*

(F1) F *is strictly increasing, i.e. for all* $j, k \in \mathbb{R}_+$ *such that* $j < k$, $F(j) < F(k)$;

(F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ *of positive numbers,* $\lim_{n\to\infty} \alpha_n = 0$ *if and only if*

$\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) There exists $k \in (0,1)$ *such that* $\lim \alpha \to 0^+ \alpha^k F(\alpha) = 0$.

Lemma 1. *Let* (Z, d_l) *be a D.B.M.S. Let* $(P(Z), H_{d_l})$ *be a dislocated Hausdorff b-metric space on* $P(Z)$. *Then, for all* $G, H \in P(Z)$ *and for each* $g \in G$ *there exist* $h_g \in H$ *satisfies* $d_l(g, H) = d_l(g, h_g)$, *then* $H_{d_l}(G, H) \geq d_l(g, h_g)$.

Proof. If $H_{d_l}(G, H) = \sup\limits_{g\in G} d_l(g, H)$, then $H_{d_l}(G, H) \geq d_l(g, H)$ for each $g \in G$. As H is a proximinal set, so for each $g \in Z$, there exists at least one best approximination $h_g \in H$ satisfies $d_l(g, H) = d_l(g, h_g)$. Now we have, $H_{d_l}(G, H) \geq d_l(g, h_g)$. Now, if

$$H_{d_l}(G, H) = \sup_{h\in H} d_l(G, h) \geq \sup_{g\in G} d_l(g, H) \geq d_l(g, h_g).$$

Hence proved. □

Example 1. *Let* $Z = \mathbb{R}$. *Define the mapping* $\alpha : Z \times Z \to [0, \infty)$ *by*

$$\alpha(j, k) = \left\{ \begin{array}{c} 1 \text{ if } j > k \\ \frac{1}{2} \text{ otherwise} \end{array} \right\}.$$

Define $S, T : Z \to P(Z)$ *by*

$$Sj = [j - 4, j - 3] \text{ and } Tk = [k - 2, k - 1].$$

Suppose $j = 3$ *and* $k = 2.5$. *As* $3 > 2.5$, *then* $\alpha(3, 2.5) \geq 1$.*Now,* $\alpha_*(S3, T2.5) = \inf\{\alpha(a, b) : a \in S3, b \in T2.5\} = \frac{1}{2} \ngeq 1$, *this means the pair* (S, T) *is not* α_**-admissible. Also,* $\alpha_*(S3, S2) \ngeq 1$ *and* $\alpha_*(T3, T2) \ngeq 1$. *This implies S and T are not* α_**-admissible individually. Now,* $\alpha_*(j, Sj) = \inf\{\alpha(j, b) : b \in Sj\} \geq 1$, *for all* $j \in Z$. *Hence S is* α_**-dominated mapping. Similarly* $\alpha_*(k, Tk) = \inf\{\alpha(k, b) : b \in Tk\} \geq 1$. *Hence it is clear that S and Tare* α_**-dominated but not* α_**-admissible.*

2. Main Result

Let (Z, d_l) be a *D.B.M.S*, $g_0 \in Z$ and $S, T : Z \to P(Z)$ be the multifunctions on Z. Let $g_1 \in Sg_0$ be an element such that $d_l(g_0, Sg_0) = d_l(g_0, g_1)$. Let $g_2 \in Tg_1$ be such that $d_l(g_1, Tg_1) = d_l(g_1, g_2)$. Let $g_3 \in Sg_2$ be such that $d_l(g_2, Sg_2) = d_l(g_2, g_3)$. Continuing this method, we get a sequence g_n of points in Z such that $g_{2n+1} \in Sg_{2n}$ and $g_{2n+2} \in Tg_{2n+1}$, where $n = 0, 1, 2, ...$ Also, $d_l(g_{2n}, Sg_{2n}) = d_l(g_{2n}, g_{2n+1})$, $d_l(g_{2n+1}, Tg_{2n+1}) = d_l(g_{2n+1}, g_{2n+2})$. We denote this iterative sequence by $\{TS(g_n)\}$. We say that $\{TS(g_n)\}$ is a sequence in Z generated by g_0.

Theorem 1. *Let* (Z, d_l) *be a complete D.B.M.S with constant* $b \geq 1$. *Let* $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ *and* $S, T : Z \to P(Z)$ *be the semi* α_**-dominated mappings on* $\overline{B_{d_l}(g_0, r)}$. *Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ *satisfying* $b\eta_1 + b\eta_2 + (1 + b)b\eta_3 + \eta_4 < 1$ *and a strictly increasing mapping F such that*

$$\tau + F(H_{d_l}(Se, Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Ty) + \eta_4 \frac{d_l^2(e,Se)\cdot d_l(y,Ty)}{1+d_l^2(e,y)} \end{array} \right), \tag{1}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ *and* $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3-\eta_4}$, *then*

$$d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r. \tag{2}$$

Then $\{TS(g_n)\}$ *is a sequence in* $\overline{B_{d_l}(g_0,r)}$, $\alpha(g_n,g_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0,r)}$. *Also, if the inequality (1) holds for* $e, y \in \{u\}$ *and either* $\alpha(g_n,u) \geq 1$ *or* $\alpha(u,g_n) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$, *then* S *and* T *have common fixed point* u *in* $\overline{B_{d_l}(g_0,r)}$.

Proof. Consider a sequence $\{TS(g_n)\}$. From (2), we get

$$d_l(g_0,g_1) = d_l(g_0,Sg_0) \leq \lambda(1-b\lambda)r < r.$$

It follows that,

$$g_1 \in \overline{B_{d_l}(g_0,r)}.$$

Let $g_2,\cdots,g_j \in \overline{B_{d_l}(g_0,r)}$ for some $j \in \mathbb{N}$. If j is odd, then $j = 2\imath + 1$ for some $\imath \in \mathbb{N}$. Since $S,T : Z \to P(Z)$ be a semi α_*-dominated mappings on $\overline{B_{d_l}(g_0,r)}$, so $\alpha_*(g_{2\imath},Sg_{2\imath}) \geq 1$ and $\alpha_*(g_{2\imath+1},Tg_{2\imath+1}) \geq 1$. As $\alpha_*(g_{2\imath},Sg_{2\imath}) \geq 1$, this implies $\inf\{\alpha(g_{2\imath},b) : b \in Sg_{2\imath}\} \geq 1$. Also, $g_{2\imath+1} \in Sg_{2\imath}$, so $\alpha(g_{2\imath},g_{2\imath+1}) \geq 1$. Now, by using Lemma 1, we have

$$\tau + F(d_l(g_{2\imath+1},g_{2\imath+2})) \leq \tau + F(H_{d_l}(Sg_{2\imath},Tg_{2\imath+1}))$$

Now, by using inequality (2.1), we have

$$\begin{aligned}
\tau + F(d_l(g_{2\imath+1},g_{2\imath+2})) &\leq F[\eta_1 d_l(g_{2\imath},g_{2\imath+1}) + \eta_2 d_l(g_{2\imath},Sg_{2\imath}) + \eta_3 d_l(g_{2\imath},Tg_{2\imath+1}) \\
&\quad + \eta_4 \frac{d_l^2(g_{2\imath},Sg_{2\imath}) \cdot d_l(g_{2\imath+1},Tg_{2\imath+1})}{1 + d_l^2(g_{2\imath},g_{2\imath+1})}] \\
&= F[\eta_1 d_l(g_{2\imath},g_{2\imath+1}) + \eta_2 d_l(g_{2\imath},g_{2\imath+1}) + \eta_3 d_l(g_{2\imath},g_{2\imath+2}) \\
&\quad + \eta_4 \frac{d_l^2(g_{2\imath},g_{2\imath+1}) \cdot d_l(g_{2\imath+1},g_{2\imath+2})}{1 + d_l^2(g_{2\imath},g_{2\imath+1})}]
\end{aligned}$$

$$\begin{aligned}
&\leq F[\eta_1 d_l(g_{2\imath},g_{2\imath+1}) + \eta_2 d_l(g_{2\imath},g_{2\imath+1}) + b\eta_3 d_l(g_{2\imath},g_{2\imath+1}) \\
&\quad + b\eta_3 d_l(g_{2\imath+1},g_{2\imath+2}) + \eta_4 \frac{d_l^2(g_{2\imath},g_{2\imath+1}) \cdot d_l(g_{2\imath+1},g_{2\imath+2})}{1 + d_l^2(g_{2\imath},g_{2\imath+1})}] \\
&\leq F((\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath},g_{2\imath+1}) + (b\eta_3 + \eta_4) d_l(g_{2\imath+1},g_{2\imath+2})).
\end{aligned}$$

This implies

$$\begin{aligned}
F(d_l(g_{2\imath+1},g_{2\imath+2})) &< F((\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath},g_{2\imath+1}) \\
&\quad + (b\eta_3 + \eta_4) d_l(g_{2\imath+1},g_{2\imath+2})).
\end{aligned}$$

As F is strictly increasing. So, we have

$$\begin{aligned}
d_l(g_{2\imath+1},g_{2\imath+2}) &< (\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath},g_{2\imath+1}) \\
&\quad + (b\eta_3 + \eta_4) d_l(g_{2\imath+1},g_{2\imath+2}).
\end{aligned}$$

Which implies

$$\begin{aligned}
(1 - b\eta_3 - \eta_4) d_l(g_{2\imath+1},g_{2\imath+2}) &< (\eta_1 + \eta_2 + b\eta_3) d_l(g_{2\imath},g_{2\imath+1}) \\
d_l(g_{2\imath+1},g_{2\imath+2}) &< \left(\frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3 - \eta_4}\right) d_l(g_{2\imath},g_{2\imath+1}).
\end{aligned}$$

As $\lambda = \frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3-\eta_4} < 1$. Hence

$$d_l(g_{2i+1}, g_{2i+2}) < \lambda d_l\left(g_{2i}, g_{2i+1}\right) < \lambda^2 d_l\left(g_{2i-1}, g_{2i}\right) < \cdots < \lambda^{2i+1} d_l\left(g_0, g_1\right).$$

Similarly, if j is even, we have

$$d_l(g_{2i+2}, g_{2i+3}) < \lambda^{2i+2} d_l\left(g_0, g_1\right).$$

Now, we have

$$d_l(g_j, g_{j+1}) < \lambda^j d_l\left(g_0, g_1\right) \text{ for some } j \in \mathbb{N}. \tag{3}$$

Now,

$$\begin{aligned} d_l(x_0, g_{j+1}) &\leq b d_l(g_0, g_1) + b^2 d_l(g_1, g_2) + \cdots + b^{j+1} d_l(g_j, g_{j+1}) \\ &\leq b d_l(g_0, g_1) + b^2 \lambda(d_l(g_0, g_1)) + \cdots \\ &\quad + b^{j+1}\lambda^{j+1}(d_l(g_0, g_1)), \qquad \text{(by (3))} \\ d_l(g_0, g_{j+1}) &\leq \frac{b(1-(b\lambda)^{j+1})}{1-b\lambda}\lambda(1-b\lambda)r < r, \end{aligned}$$

which implies $g_{j+1} \in \overline{B_{d_l}(g_0, r)}$. Hence, by induction $g_n \in \overline{B_{d_l}(g_0, r)}$ for all $n \in N$. Also, $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now,

$$d_l(g_n, g_{n+1}) < \lambda^n d_l\left(g_0, g_1\right) \text{ for all } n \in \mathbb{N}. \tag{4}$$

Now, for any positive integers m, n $(n > m)$, we have

$$\begin{aligned} d_l(g_m, g_n) &\leq b(d_l(g_m, g_{m+1})) + b^2(d_l(g_{m+1}, g_{m+2})) + \cdots \\ &\quad + b^{n-m}(d_l(g_{n-1}, g_n)), \\ &< b\lambda^m d_l(g_0, g_1) + b^2\lambda^{m+1} d_l(g_0, g_1) + \cdots \\ &\quad + b^{n-m}\lambda^{n-1} d_l(g_0, g_1), \qquad \text{(by (4))} \\ &< b\lambda^m(1 + b\lambda + \cdots) d_l(g_0, g_1) \end{aligned}$$

As $\eta_1, \eta_2, \eta_3, \eta_4 > 0$, $b \geq 1$ and $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$, so $|b\lambda| < 1$. Then, we have

$$d_l(g_m, g_n) < \frac{b\lambda^m}{1-b\lambda} d_l(g_0, g_1) \to 0 \text{ as } m \to \infty.$$

Hence $\{TS(g_n)\}$ is a Cauchy sequence in $\overline{B_{d_l}(g_0, r)}$. Since $(\overline{B_{d_l}(g_0, r)}, d_l)$ is a complete metric space, so there exist $u \in \overline{B_{d_l}(g_0, r)}$ such that $\{TS(g_n)\} \to u$ as $n \to \infty$, then

$$\lim_{n\to\infty} d_l(g_n, u) = 0. \tag{5}$$

By assumption, $\alpha(g_n, u) \geq 1$. Suppose that $d_l(u, Tg) > 0$, then there exist positive integer k such that $d_l(g_n, Tu) > 0$ for all $n \geq k$. For $n \geq k$, we have

$$\begin{aligned} d_l(u, Tu) &\leq d_l(u, g_{2n+1}) + d_l(g_{2n+1}, Tu) \\ &\leq d_l(u, g_{2n+1}) + H_{d_l}(Sg_{2n}, Tu) \\ &< d_l(u, g_{2n+1}) + \eta_1 d_l(g_{2n}, u) + \eta_2 d_l(g_{2n}, Sg_{2n}) \\ &\quad + \eta_3 d_l(g_{2n}, Tu) + \eta_4 \frac{d_l^2(g_{2n}, Sg_{2n}) \cdot d_l(u, Tu)}{1 + d_l^2(g_{2n}, u)}. \end{aligned}$$

Letting $n \to \infty$, and by using (5) we get

$$d_l(u, Tu) < \eta_3 d_l(u, Tu) < d_l(u, Tu),$$

which is a contradiction. So our supposition is wrong. Hence $d_l(u, Tu) = 0$ or $u \in Tu$. Similarly, by using Lemma 1, inequality (1), we can show that $d_l(u, Su) = 0$ or $u \in Su$. Hence the S and T have a common fixed point u in $\overline{B_{d_l}(g_0, r)}$. Now,

$$d_l(u, u) \leq b d_l(u, Tu) + b d_l(Tu, u) \leq 0.$$

This implies that $d_l(u, u) = 0$. □

Example 2. *Let $Z = Q^+ \cup \{0\}$ and let $d_l : Z \times Z \to Z$ be the complete D.B.M.S defined by*

$$d_l(s, o) = (s + o)^2 \text{ for all } s, o \in Z.$$

with $b = 2$. Define the multivalued mapping, $S, T : Z \times Z \to P(Z)$ by,

$$Sg = \begin{cases} [\frac{g}{3}, \frac{2}{3}g] \text{ if } g \in [0, 14] \cap Z \\ [g, g+1] \text{ if } g \in (14, \infty) \cap Z \end{cases}$$

and,

$$Tp = \begin{cases} [\frac{z}{4}, \frac{3}{4}z] \text{ if } p \in [0, 14] \cap Z \\ [p+1, p+3] \text{ if } p \in (14, \infty) \cap Z. \end{cases}$$

Suppose that, $g_0 = 1$, $r = 225$, then $\overline{B_{d_l}(g_0, r)} = [0, 14] \cap Z$ and $\{TS(g_n)\} = \{1, \frac{1}{3}, \frac{1}{12}, ...\}$. Take $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$, $\eta_3 = \frac{1}{60}$, $\eta_4 = \frac{1}{30}$, then $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$ and $\lambda = \frac{11}{56}$. Now

$$d_l(g_0, Sg_0) = \frac{16}{9} < \frac{11}{56}(1 - \frac{22}{56})225 = \lambda(1 - b\lambda)r$$

Consider the mapping $\alpha : Z \times Z \to [0, \infty)$ by

$$\alpha(g, p) = \begin{Bmatrix} 1 \text{ if } g > p \\ \frac{1}{2} \text{ otherwise} \end{Bmatrix}.$$

Now, if $g, p \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$ with $\alpha(g, p) \geq 1$, we have

$$\begin{aligned} H_{d_l}(Sg, Tp) &= \max\{\sup_{a \in Sg} d_l(a, Tp), \sup_{b \in Tp} d_l(Sg, b)\} \\ &= \max\{\sup_{a \in Sg} d_l(a, [\frac{p}{4}, \frac{3p}{4}]), \sup_{b \in Tp} d_l([\frac{g}{3}, \frac{2g}{3}], b)\} \\ &= \max\{d_l(\frac{2g}{3}, [\frac{p}{4}, \frac{3p}{4}]), d_l([\frac{g}{3}, \frac{2g}{3}], \frac{3p}{4})\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{d_l(\frac{2g}{3},\frac{p}{4}), d_l(\frac{g}{3},\frac{3p}{4})\}\\
&= \max\left\{\left(\frac{2g}{3}+\frac{p}{4}\right)^2, \left(\frac{g}{3}+\frac{3p}{4}\right)^2\right\}\\
&< \frac{1}{10}(g+p)^2+\frac{4g^2}{45}+\frac{(4g+p)^2}{960}+\frac{40g^4p^2}{243\{1+(g+p)^4\}}\\
&= \frac{1}{10}d_l(g,p)+\frac{1}{20}d_l(g,[\frac{g}{3},\frac{2}{3}g])+\frac{1}{60}d_l(g,[\frac{p}{4},\frac{3}{4}p])\\
&\quad +\frac{1}{30}\frac{d_l^2(g,[\frac{g}{3},\frac{2}{3}g])\cdot d_l(p,[\frac{p}{4},\frac{3}{4}p])}{1+d_l^2(g,p)}.
\end{aligned}
$$

Thus,

$$H_{d_l}(Sg,Tp) < \eta_1 d_l(g,p)+\eta_2 d_l(g,Sg)+\eta_3 d_l(g,Tp)+\eta_4\frac{d_l^2(g,Sg)\cdot d_l(p,Tp)}{1+d_l^2(g,p)},$$

which implies that, for any $\tau\in(0,\frac{12}{95}]$ and for a strictly increasing mapping $F(s)=\ln s$, we have

$$\tau+F(H_{d_l}(Sg,Tp))\le F\left(\begin{array}{c}\eta_1 d_l(g,p)+\eta_2 d_l(g,Sg)+\eta_3 d_l(g,Tp)\\ +\eta_4\frac{d_l^2(g,Sg)\cdot d_l(p,Tp)}{1+d_l^2(g,p)}\end{array}\right).$$

Note that, for $16,15\in X$, then $\alpha(16,15)\ge 1$. But, we have

$$\tau+F(H_{d_l}(S16,T15))> F\left(\begin{array}{c}\eta_1 d_l(16,15)+\eta_2 d_l(16,S16)+\eta_3 d_l(16,T15)\\ +\eta_4\frac{d_l^2(16,S16)\cdot(15,T15)}{1+d_l^2(16,15)}\end{array}\right).$$

So condition (1) does not hold on Z. Thus the mappings S and T are satisfying all the conditions of Theorem 1 only for $g,p\in\overline{B_{d_l}(g_0,r)}\cap\{TS(g_n)\}$ with $\alpha(g,p)\ge 1$. Hence S and T have a common fixed point.

If, we take $S=T$ in Theorem 1, then we are left with the result.

Corollary 1. *Let (Z,d_l) be a complete D.B.M.S with constant $b\ge 1$. Let $r>0$, $g_0\in\overline{B_{d_l}(g_0,r)}\subseteq Z$, $\alpha: Z\times Z\to[0,\infty)$ and $S: Z\to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0,r)}$. Suppose that the following satisfy:*

(i) There exist $\tau,\eta_1,\eta_2,\eta_3,\eta_4>0$ satisfying $b\eta_1+b\eta_2+(1+b)b\eta_3+\eta_4<1$ and a strictly increasing mapping F such that

$$\tau+F(H_{d_l}(Se,Sy))\le F\left(\begin{array}{c}\eta_1 d_l(e,y)+\eta_2 d_l(e,Se)\\ +\eta_3 d_l(e,Sy)+\eta_4\frac{d_l^2(e,Se)\cdot d_l(y,Sy)}{1+d_l^2(x,y)}\end{array}\right), \tag{6}$$

whenever $e,y\in\overline{B_{d_l}(g_0,r)}\cap\{SS(g_n)\}$, $\alpha(e,y)\ge 1$ and $H_{d_l}(Se,Sy)>0$.

(ii) If $\lambda=\frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3-\eta_4}$, then

$$d_l(g_0,Sg_0)\le\lambda(1-b\lambda)r.$$

Then $\{SS(g_n)\}$ is a sequence in $\overline{B_{d_l}(g_0,r)}$, $\alpha(g_n,g_{n+1})\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$ and $\{SS(g_n)\}\to u\in\overline{B_{d_l}(g_0,r)}$. Also, if the inequality (6) holds for $e,y\in\{u\}$ and either $\alpha(g_n,u)\ge 1$ or $\alpha(u,g_n)\ge 1$ for all $n\in\mathbb{N}\cup\{0\}$, then S has a fixed point u in $\overline{B_{d_l}(g_0,r)}$.

If, we take $\eta_2=0$ in Theorem 1, then we are left with the result.

Corollary 2. *Let (Z, d_l) be a complete D.B.M.S with constant $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0, r)}$. Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + (1+b)b\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Se, Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_3 d_l(e, Ty) \\ +\eta_4 \frac{d_l^2(e, Se) \cdot d_l(y, Ty)}{1 + d_l^2(e, y)} \end{array}\right), \tag{7}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + b\eta_3}{1 - b\eta_3 - \eta_4}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_l}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0, r)}$. Also, if the inequality (7) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in $\overline{B_{d_l}(g_0, r)}$.

If, we take $\eta_3 = 0$ in Theorem 1, then we are left with the result.

Corollary 3. *Let (Z, d_l) be a complete D.B.M.S with constant $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0, r)}$. Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Se, Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_4 \frac{d_l^2(e, Se) \cdot d_l(y, Ty)}{1 + d_l^2(e, y)} \end{array}\right), \tag{8}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_4}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_l}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0, r)}$. Also, if the inequality (8) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in $\overline{B_{d_l}(g_0, r)}$.

If, we take $\eta_4 = 0$ in Theorem 1, then we are left only with the result.

Corollary 4. *Let (Z, d_l) be a complete D.B.M.S with constant $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to P(Z)$ be the semi α_*-dominated mappings on $\overline{B_{d_l}(g_0, r)}$. Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_3 > 0$ satisfying $b\eta_1 + b\eta_2 + (1+b)b\eta_3 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_3 d_l(e, Ty)\right), \tag{9}$$

whenever $e, y \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2 + b\eta_3}{1 - b\eta_3}$, then

$$d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_l}(g_0, r)}$, $\alpha(g_n, g_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{TS(g_n)\} \to u \in \overline{B_{d_l}(g_0, r)}$. Also, if the inequality (9) holds for $e, y \in \{u\}$ and either $\alpha(g_n, u) \geq 1$ or $\alpha(u, g_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in $\overline{B_{d_l}(g_0, r)}$.

3. Fixed Point Results For Graphic Contractions

In this section we presents an application of Theorem 1 in graph theory. Jachymski [20] proved the result concerning for contraction mappings on metric space with a graph. Hussain et al. [14] introduced the fixed points theorem for graphic contraction and gave an application. Furtheremore, avoiding sets condition is closed related to fixed point and is applied to the study of multi-agent systems (see [30]).

Definition 8. *Let Z be a nonempty set and $Q = (V(Q), W(Q))$ be a graph such that $V(Q) = Z$, $A \subseteq Z$. A mapping $S : Z \to P(Z)$ is said to be multi graph dominated on A if $(p, q) \in W(Q)$, for all $q \in Sp$ and $q \in A$.*

Theorem 2. *Let (Z, d_l) be a complete D.B.M.S endowed with a graph Q with constant $b \geq 1$. Let $r > 0$, $g_0 \in \overline{B_{d_l}(g_0, r)}$ and $S, T : Z \to P(Z)$. Suppose that the following satisfy:*

(i) S and T are multi graph dominated on $\overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$.

(ii) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1 + b)b\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(H_{d_l}(Sp, Tq)) \leq F\begin{pmatrix} \eta_1 d_l(p, q) + \eta_2 d_l(p, Sp) \\ +\eta_3 d_l(p, Tq) + \eta_4 \frac{d_l^2(p,Sp).d_l(q,Tq)}{1+d_l^2(p,q)} \end{pmatrix}, \tag{10}$$

whenever $p, q \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $(p, q) \in W(Q)$ and $H_{d_l}(Sp, Tq) > 0$.

(iii) $d_l(g_0, Sg_0) \leq \lambda(1 - b\lambda)r$, where $\lambda = \frac{\eta_1+\eta_2+b\eta_3}{1-b\eta_3-\eta_4}$.

Then, $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_l}(g_0, r)}$, $\{TS(g_n)\} \to m^*$ and $(g_n, g_{n+1}) \in W(Q)$, where $g_n, g_{n+1} \in \{TS(g_n)\}$. Also, if the inequality (10) holds for $p, q \in \{m^*\}$ and $(g_n, m^*) \in W(Q)$ or $(m^*, g_n) \in W(Q)$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point m^* in $\overline{B_{d_l}(g_0, r)}$.

Proof. Define, $\alpha : Z \times Z \to [0, \infty)$ by

$$\alpha(p, q) = \begin{cases} 1, & \text{if } p \in \overline{B_{d_l}(g_0, r)},\ (p, q) \in W(Q) \\ 0, & \text{otherwise.} \end{cases}$$

As S and T are semi graph dominated on $\overline{B_{d_l}(g_0, r)}$, then for $p \in \overline{B_{d_l}(g_0, r)}$, $(p, q) \in W(Q)$ for all $q \in Sp$ and $(p, q) \in W(Q)$ for all $q \in Tp$. So, $\alpha(p, q) = 1$ for all $q \in Sp$ and $\alpha(p, q) = 1$ for all $q \in Tp$. This implies that $\inf\{\alpha(p, q) : q \in Sp\} = 1$ and $\inf\{\alpha(p, q) : q \in Tp\} = 1$. Hence $\alpha_*(p, Sp) = 1$, $\alpha_*(p, Tp) = 1$ for all $p \in \overline{B_{d_l}(g_0, r)}$. So, $S, T : Z \to P(Z)$ are the semi α_*-dominated mapping on $\overline{B_{d_l}(g_0, r)}$. Moreover, inequality (10) can be written as

$$\tau + F(H_{d_l}(Sp, Tq)) \leq F\begin{pmatrix} \eta_1 d_l(p, q) + \eta_2 d_l(p, Sp) \\ +\eta_3 d_l(p, Tq) + \eta_4 \frac{d_l^2(p,Sp).d_l(q,Tq)}{1+d_l^2(p,q)} \end{pmatrix}$$

whenever $p, q \in \overline{B_{d_l}(g_0, r)} \cap \{TS(g_n)\}$, $\alpha(p, q) \geq 1$ and $H_{d_l}(Sp, Tq) > 0$. Also, (iii) holds. Then, by Theorem 1, we have $\{TS(g_n)\}$ is a sequence in $\overline{B_{d_l}(g_0, r)}$ and $\{TS(g_n)\} \to m^* \in \overline{B_{d_l}(g_0, r)}$. Now, $g_n, m^* \in \overline{B_{d_l}(g_0, r)}$ and either $(g_n, m^*) \in W(Q)$ or $(m^*, g_n) \in W(Q)$ implies that either $\alpha(g_n, m^*) \geq 1$ or $\alpha(m^*, g_n) \geq 1$. So, all the conditions of Theorem 1 are satisfied. Hence, by Theorem 1, S and T have a common fixed point m^* in $\overline{B_{d_l}(g_0, r)}$ and $d_l(m^*, m^*) = 0$. □

4. Fixed Point Results for Single Valued Mapping

In this section, we discussed some new fixed point results for single valued mapping in complete $D.B.M.S.$ Let (Z, d_l) be a $D.B.M.S$, $c_0 \in Z$ and $S, T : Z \to Z$ be the mappings. Let $c_1 = Sc_0$, $c_2 = Tc_1$, $c_3 = Sc_2$. Continuing in this way, we get a sequence c_n of points in Z such that $c_{2n+1} = Sc_{2n}$ and $c_{2n+2} = Tc_{2n+1}$, where $n = 0, 1, 2, ...$ We denote this iterative sequence by $\{TS(c_n)\}$. We say that $\{TS(c_n)\}$ is a sequence in Z generated by c_0.

Theorem 3. *Let (Z, d_l) be a complete D.B.M.S with constant $t \geq 1$. Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated mappings on $\overline{B_{d_l}(c_0, r)}$. Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + (1+t)t\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Ty) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{array}\right), \tag{11}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+t\eta_3}{1-t\eta_3-\eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{TS(c_n)\}$ is a sequence in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also, if the inequality (4.1) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in $\overline{B_{d_l}(c_0, r)}$.

Proof. The proof of the above Theorem is similar as Theorem 1.

If, we take $S = T$ in Theorem 3, then we are left with the result. □

Corollary 5. *Let (Z, d_l) be a complete D.B.M.S with constant $t \geq 1$. Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S : Z \to Z$ be the semi α-dominated mappings on $\overline{B_{d_l}(c_0, r)}$. Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + (1+t)t\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(d_l(Se, Sy)) \leq F\left(\begin{array}{c} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Sy) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Sy)}{1+d_l^2(e,y)} \end{array}\right), \tag{12}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{SS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Sy) > 0$.

(ii) If $\lambda = \frac{\eta_1+\eta_2+t\eta_3}{1-t\eta_3-\eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{SS(c_n)\}$ is a sequence in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{SS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also, if the inequality (12) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S has a fixed point u in $\overline{B_{d_l}(c_0, r)}$.

If, we take $\eta_2 = 0$ in Theorem 3, then we are left with the result.

Corollary 6. *Let (Z, d_l) be a complete D.B.M.S with constant $b \geq 1$. Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated mappings on $\overline{B_{d_l}(c_0, r)}$. Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_3, \eta_4 > 0$ satisfying $t\eta_1 + (1+t)t\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_3 d_l(e, Ty) + \eta_4 \frac{d_l^2(e, Se) \cdot d_l(y, Ty)}{1 + d_l^2(e, y)}\right), \tag{13}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + t\eta_3}{1 - t\eta_3 - \eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{TS(c_n)\}$ is a sequence in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also, if the inequality (13) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in $\overline{B_{d_l}(c_0, r)}$.

If, we take $\eta_3 = 0$ in Theorem 3, then we are left with the result.

Corollary 7. *Let (Z, d_l) be a complete D.B.M.S with constant $t \geq 1$. Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated mappings on $\overline{B_{d_l}(c_0, r)}$. Suppose that the following satisfy:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_4 > 0$ satisfying $t\eta_1 + t\eta_2 + \eta_4 < 1$ and a strictly increasing mapping F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_4 \frac{d_l^2(e, Se) \cdot d_l(y, Ty)}{1 + d_l^2(e, y)}\right), \tag{14}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2}{1 - \eta_4}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - b\lambda)r.$$

Then $\{TS(c_n)\}$ is a sequence in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also, if the inequality (14) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in $\overline{B_{d_l}(c_0, r)}$.

If, we take $\eta_4 = 0$ in Theorem 3, then we are left with the result.

Corollary 8. *Let (Z, d_l) be a complete D.B.M.S with constant $b \geq 1$. Let $r > 0$, $c_0 \in \overline{B_{d_l}(c_0, r)} \subseteq Z$, $\alpha : Z \times Z \to [0, \infty)$ and $S, T : Z \to Z$ be the semi α-dominated mappings on $\overline{B_{d_l}(c_0, r)}$. Assume that the following hold:*

(i) There exist $\tau, \eta_1, \eta_2, \eta_3 > 0$ satisfying $t\eta_1 + t\eta_2 + (1+t)t\eta_3 < 1$ and a strictly increasing mapping F such that

$$\tau + F(d_l(Se, Ty)) \leq F\left(\eta_1 d_l(e, y) + \eta_2 d_l(e, Se) + \eta_3 d_l(e, Ty)\right), \tag{15}$$

whenever $e, y \in \overline{B_{d_l}(c_0, r)} \cap \{TS(c_n)\}$, $\alpha(e, y) \geq 1$ and $H_{d_l}(Se, Ty) > 0$.

(ii) If $\lambda = \frac{\eta_1 + \eta_2 + t\eta_3}{1 - t\eta_3}$, then

$$d_l(c_0, Sc_0) \leq \lambda(1 - t\lambda)r.$$

Then $\{TS(c_n)\}$ is a sequence in $\overline{B_{d_l}(c_0, r)}$, $\alpha(c_n, c_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{TS(c_n)\} \to u \in \overline{B_{d_l}(c_0, r)}$. Also, if the inequality (15) holds for $e, y \in \{u\}$ and either $\alpha(c_n, u) \geq 1$ or $\alpha(u, c_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then S and T have common fixed point u in $\overline{B_{d_l}(c_0, r)}$.

5. Application to the Systems of Integral Equations

Theorem 4. *Let (Z, d_l) be a complete D.B.M.S with constant $b \geq 1$. Let $c_0 \in Z$ and $S, T : Z \to Z$. Assume that, There exist $\tau, \eta_1, \eta_2, \eta_3, \eta_4 > 0$ satisfying $b\eta_1 + b\eta_2 + (1+b)b\eta_3 + \eta_4 < 1$ and a strictly increasing mapping F such that the following satisfy:*

$$\tau + F(d_l(Se, Ty)) \leq F\begin{pmatrix} \eta_1 d_l(e, y) + \eta_2 d_l(e, Se) \\ +\eta_3 d_l(e, Ty) + \eta_4 \frac{d_l^2(e,Se).d_l(y,Ty)}{1+d_l^2(e,y)} \end{pmatrix}, \tag{16}$$

whenever $e, y \in \{TS(c_n)\}$ and $d(Se, Ty) > 0$. Then $\{TS(c_n)\} \to g \in Z$. Also, if the inequality (16) holds for g, then S and T have unique common fixed point g in Z.

Proof. The proof of this Theorem is similar as Theorem 1. We have to prove the uniqueness only. Let p be another common fixed point of S and T. Suppose $d_l(Sg, Tp) > 0$. Then, we have

$$\tau + F(d_l(Sg, Tp)) \leq F\left(\eta_1 d_l(g, p) + \eta_2 d_l(g, Sg) + \eta_3 d_l(g, Tp) + \eta_4 \frac{d_l^2(g, Sg) \cdot d_l(p, Tp)}{1 + d_l^2(g, p)}\right)$$

This implies that

$$d_l(g, p) < \eta_1 d_l(g, p) + \eta_3 d_l(g, p) < d_l(g, p),$$

which is a contradiction. So $d_l(Sg, Tp) = 0$. Hence $g = p$.

In this section, we discuss the application of fixed point Theorem 4 in form of Volterra type integral equations.

$$g(k) = \int_0^k H_1(k, h, g(h))dh, \tag{17}$$

$$p(k) = \int_0^k H_2(k, h, p(h))dh \tag{18}$$

for all $k \in [0, 1]$. We find the solution of (17) and (18). Let $Z = C([0, 1], \mathbb{R}_+)$ be the set of all continuous functions on $[0, 1]$, endowed with the complete dislocated b-metric. For $g \in C([0, 1], \mathbb{R}_+)$, define supremum norm as: $\|g\|_\tau = \sup_{k \in [0,1]} \{|g(k)| \, e^{-\tau k}\}$, where $\tau > 0$ is taken arbitrary. Then define

$$d_\tau(g, p) = \left[\sup_{k \in [0,1]} \{|g(k) + p(k)| \, e^{-\tau k}\}\right]^2 = \|g + p\|_\tau^2$$

for all $g, p \in C([0, 1], \mathbb{R}_+)$, with these settings, $(C([0, 1], \mathbb{R}_+), d_\tau)$ becomes a complete $D.B.M.S.$ with constant $b = 2$.

Now we prove the following theorem to ensure the existence of solution of integral equations. □

Theorem 5. *Assume the following conditions are satisfied:*

(i) $H_1, H_2 : [0, 1] \times [0, 1] \times C([0, 1], \mathbb{R}_+) \to \mathbb{R}$;

(ii) Define

$$\begin{aligned} Sg(k) &= \int_0^k H_1(k,h,g(h))dh, \\ Tp(k) &= \int_0^k H_2(k,h,p(h))dh. \end{aligned}$$

Suppose there exist $\tau > 0$*, such that*

$$|H_1(k,h,g) + H_2(k,h,p)| \le \frac{\tau M(g,p)}{\tau \|M(g,p)\|_\tau + 1}$$

for all $k,h \in [0,1]$ *and* $g,p \in C([0,1],\mathbb{R})$*, where*

$$\begin{aligned} M(g(h),p(h)) &= \eta_1[|g(h)+p(h)|]^2 + \eta_2[|g(h)+Sg(h)|]^2 + \eta_3[|g(h)+Tp(h)|]^2 \\ &\quad + \eta_4 \frac{[|g(h)+Sg(h)|]^4 \cdot [|p(h)+Tp(h)|]^2}{1+[|g(h)+p(h)|]^4}, \end{aligned}$$

where $\eta_1, \eta_2, \eta_3, \eta_4 \ge 0$*, and* $2\eta_1 + 2\eta_2 + 6\eta_3 + \eta_4 < 1$*. Then integral Equations* (17) *and* (18) *has a solution.*

Proof. By assumption (ii)

$$\begin{aligned} |Sg(k) + Tp(k)| &= \int_0^k |H_1(k,h,g(h) + H_2(k,h,p(h)))|\, dh, \\ &\le \int_0^k \frac{\tau}{\tau\|M(g,p)\|_\tau + 1}([M(g,p)]e^{-\tau h})e^{\tau h} dh \\ &\le \int_0^k \frac{\tau}{\tau\|M(g,p)\|_\tau + 1}\|M(g,p)\|_\tau e^{\tau h} dh \end{aligned}$$

$$\begin{aligned} &\le \frac{\tau\|M(g,p)\|_\tau}{\tau\|M(g,p)\|_\tau + 1}\int_0^k e^{\tau h} dh, \\ &\le \frac{\|M(g,p)\|_\tau}{\tau\|M(g,p)\|_\tau + 1} e^{\tau k}. \end{aligned}$$

This implies

$$|Sg(k) + Tp(k)|\, e^{-\tau k} \le \frac{\|M(g,p)\|_\tau}{\tau\|M(g,p)\|_\tau + 1}.$$

$$\|Sg(k) + Tp(k)\|_\tau \le \frac{\|M(g,p)\|_\tau}{\tau\|M(g,p)\|_\tau + 1}.$$

$$\frac{\tau\|M(g,p)\|_\tau + 1}{\|M(g,p)\|_\tau} \le \frac{1}{\|Sg(k) + Tp(k)\|_\tau}.$$

$$\tau + \frac{1}{\|M(g,p)\|_\tau} \le \frac{1}{\|Sg(k) + Tp(k)\|_\tau}.$$

which further implies

$$\tau - \frac{1}{\|Sg(k) + Tp(k)\|_\tau} \leq \frac{-1}{\|M(g,p)\|_\tau}.$$

So all the conditions of Theorem 4 are satisfied for $F(p) = \frac{-1}{\sqrt{p}}; p > 0$ and $d_\tau(g,p) = \|g+p\|_\tau^2$, $b = 2$. Hence, the integral equations given in (17) and (18) has a unique common solution. □

Example 3. *Consider the integral equations*

$$g(k) = \frac{1}{3}\int_0^k g(h)dh, \quad p(k) = \frac{1}{4}\int_0^k p(h))dh, \text{ where } k \in [0,1].$$

Define $H_1, H_2 : [0,1] \times [0,1] \times C([0,1], \mathbb{R}_+) \to \mathbb{R}$ *by* $H_1 = \frac{1}{3}g(h)$, $H_2 = \frac{1}{4}p(h)$. *Now,*

$$Sg(k) = \frac{1}{3}\int_0^k g(h)dh, \quad Tp(k) = \frac{1}{4}\int_0^k p(h))dh$$

Take $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$, $\eta_3 = \frac{1}{60}$, $\eta_4 = \frac{1}{30}$, $\tau = \frac{12}{95}$, *then* $2\eta_1 + 2\eta_2 + 6\eta_3 + \eta_4 < 1$. *Moreover, all conditions of Theorem 5 are satisfied and* $g(k) = p(k) = 0$ *for all* k, *is a unique common solution to the above equations.*

6. Conclusions

In the present paper, we have achieved fixed point results for new generalized F-contraction on an intersection of a closed ball and a sequence for a more general class of semi α_*-dominated mappings rather than α_*-admissible mappings, and for a weaker class of strictly increasing mappings F rather than a class of mappings F used by Wardowski [34]. The notion of multi graph dominated mapping is introduced. Fixed point results with graphic contractions on a closed ball for such mappings are established. Examples are given to demonstrate the variety of our results. An application is given to approximate the unique common solution of nonlinear integral equations. Moreover, we investigate our results in a better, new framework. New results in ordered spaces, partial b-metric space, dislocated metric space, partial metric space, b-metric space, and metric space can be obtained as corollaries of our results. One can further extend our results to fuzzy mappings, bipolar fuzzy mappings, and fuzzy neutrosophic soft mappings. More applications on delayed scaled consensus problems can be investigated (see [31]).

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