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On Fitting Ideals of Kahler Differential Module

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Abstract: Let k be an algebraically closed field of characteristic zero, and R/I and S/J be algebras over k. $\Omega_1(R/I)$ and $\Omega_1(S/J)$ denote universal module of first order derivation over k. The main result of this paper asserts that the first nonzero Fitting ideal $\Omega_1(R/I \otimes_k S/J)$ is an invertible ideal, if the first nonzero Fitting ideals $\Omega_1(R/I)$ and $\Omega_1(S/J)$ are invertible ideals. Then using this result, we conclude that the projective dimension of $\Omega_1(R/I \otimes_k S/J)$ is less than or equal to one.

Keywords: universal module; Fitting ideal; invertible ideal

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1. Introduction

Fitting ideals are used in different areas of mathematics to investigate the structure of modules. They appear as invariants which are useful as the annihilator ideal of a module. Fitting ideal or Fitting invariant was introduced by H. Fitting in 1936 [1]. Lipman proves that when R is a quasilocal ring, the first nonzero Fitting ideal is an regular principle ideal if and only if R is a complete intersection and $\Omega_1(R)/T(\Omega_1(R))$ is free [2]. Kunz sets apart a section of his book of Fitting ideals of universal differential modules [3], Olgun and Erdoğan study universal differential modules and their Fitting ideals [4–8]; Olgun also gives examples about Fitting ideals of universal differential modules. Ohm generalizes Lipman's results in a global case [9]. Hadjirezai, Hedayat, and Karimzadeh assert that when a finitely generated module in which the first nonzero Fitting ideal is maximal or regular, they characterize this module [10]. They also study the first nonzero Fitting ideal of the module over unique factorization domain rings [11]. Simis and Ulrich examine the circumstances in which the equation $l(\Lambda^r E) \geq height(F_r(M))$ holds [12]. In [13], it has been shown M is a free R-module if and only if the Fitting ideal $F_T(M)$ is grade unmixed.

Fitting ideals are used in mathematical physics at the same time. They provide a transition between commutative algebra and physics. M. Einsiedler and T. Ward show how the dynamical properties of the system may be deduced from the Fitting ideals. They prove the entropy and expansiveness related with only the first Fitting ideal. This gives an easy computation instead of computing syzygy modules. Also, they show how the dynamical properties and periodic point behavior may be deduced from the determinant of the matrix of relations [14].

In this paper, we will show that if the first nonzero Fitting ideals of $\Omega_1(R/I)$ and $\Omega_1(S/J)$ are invertible ideals, the first nonzero Fitting ideal of $\Omega_1(R/I \otimes_k S/J)$ is an invertible ideal. Then using this result, when the first nonzero Fitting ideals $F_i(\Omega_1(R/I))$ and $F_j(\Omega_1(S/J))$ are invertible ideals, we can conclude that the projective dimension of $\Omega_1(R/I \otimes_k S/J)$ is less than or equal to one.

2. Background Material

Throughout this paper we will suppose that R is a commutative algebra over an algebraically closed field k with characteristic zero. In this section, we will give some results about the Fitting ideal of universal modules. When R is a k-algebra, $\Omega_q(R)$ denotes the universal module of q-th order derivations of R over k, and δ_q denotes the canonical q-th order k-derivative from R to $\Omega_q(R)$ of R. The pair $(\Omega_q(R), \delta_q)$ has the universal mapping property with respect to the q-th order k-differentials of R.

$\Omega_q(R)$ is R-module generated by the set $\{\delta_q(r) : r \epsilon R\}$. If R is finitely generated k-algebra, then $\Omega_q(R)$ will be finitely generated.

Let M be a finitely generated R-module and $\{m_1, m_2, \ldots, m_n\}$ be a system of generators of M. The exact sequence of R-modules

$$0 \to K \to R^n \xrightarrow{\alpha} M \to 0$$

where α maps the i-th canonical basis element e_i onto $m_i (i = 1, 2, \ldots, n)$ and $K = \ker(\alpha)$ is said the presentation of M defined by $\{m_1, m_2, \ldots, m_n\}$.

Let $\{v_\lambda\}_{\lambda \epsilon \Lambda}$ be a system of generators of K with $v_\lambda = (x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_n})_{\lambda \epsilon \Lambda} \epsilon R^n$. Then,

$$A = (x_{\lambda i})_{\lambda \epsilon \Lambda, i=1,2,\ldots n}$$

is called a relation matrix of M with respect to $\{m_1, m_2, \ldots, m_n\}$.

When the matrix A is given, let $F_i(A)$ denote the ideal of R generated by all $(n - i)$ rowed subdeterminants of $A(i = 1, 2, \ldots, n - 1)$, and let $F_i(A) = R$ for $i \geq n$.

Lemma 1. *Let A and B be two relation matrices of M. Then for every $i \in \mathbb{Z}$, $F_i(A) = F_i(B)$ (i.e., the Fitting ideals of M is independent of the special choice of a relation matrix of M) [3].*

Lemma 2. *$F_i(A)(i \in \mathbb{Z})$ is independent of the choice of the generating system $\{m_1, m_2, \ldots, m_n\}$ of M [3].*

Definition 1. *Let A be a relation matrix of M and we set $F_i(A) = F_i(M)$ for $i \in \mathbb{N}$. Then $F_i(M)$ is called the i-th Fitting ideal of M. By construction we have:*

$$F_0(M) \subseteq F_1(M) \subseteq \cdots \subseteq F_i(M) \subseteq \cdots$$

and

$$F_i(M) = R \textit{ for } i \geq \mu(M).$$

Proposition 1. *Suppose that M is a R-module. Fitting ideals of M have the following properties [3].*

(i) *Let M be a finitely generated R-module. Then the $F_i(M)(i \in \mathbb{N})$ are finitely generated ideals of R.*
(ii) *Let $\alpha : M \to M'$ be an epimorphism of R-modules. Then $F_i(M) \subseteq F_i(M')(i \in \mathbb{N})$.*
(iii) *Let S be R-algebra. Then we have $F_i(S \otimes_R M) = S.F_i(M)$ $(i \in \mathbb{N})$.*
(iv) *If $N \subseteq R$ is a multiplicatively closed subset, then Fitting ideal of M_N,*

$$F_i(M_N) = F_i(M)_N (i \in \mathbb{N})$$

(v) *If I is an ideal of R, then the Fitting ideal of quotient module M/IM, $F_i(M/IM) = \overline{F_i(M)}$ where $\overline{F_i(M)}$ points out the image of $F_i(M)$ in R/I.*

Proof. (i) and (ii) are trivial. As for (iii) think the exact sequence:

$$S \otimes_R K \to S^n \xrightarrow{\alpha} S \otimes_R M \to 0$$

derived from exact sequence of R-modules:

$$0 \to K \to R^n \xrightarrow{\alpha} M \to 0.$$

It is obvious that the images of x_{λ_i} in S define a relation matrix of the S-module $S \otimes_R M$ with respect to $1 \otimes m_1, 1 \otimes m_2, \ldots, 1 \otimes m_n$. To prove (iv), we can use the isomorphism $N^{-1}M \simeq M \otimes_R N^{-1}R$ and (iii), then we obtain $F_i(N^{-1}M) \simeq F_i(M \otimes_R N^{-1}R) = N^{-1}RF_i(M) = F_i(M)_N$. Similarly, for (v), we can use the isomorphism $M/IM \simeq M \otimes_R R/I$ and (iii). □

Theorem 1. *Let R be an affine k-algebra and R be a regular ring. Then $\Omega_n(R)$ is a projective R-module [15].*

Theorem 2. *Let R be an affine k-algebra and R be a regular ring if and only if $\Omega_1(R)$ is a projective R-module [16].*

Definition 2. *Let $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial algebra and let $I = (f_1, f_2, \ldots, f_m)$ be an ideal of R and $\Omega_n(R/I)$ is the n-th order universal module of derivatives of R/I.*

$\Omega_n(R/I) \simeq F/N$ where F is a free module which is generated by the set $\{\delta_n(x^\alpha + I) : |\alpha| \leq n\}$ and N is a submodule of F generated by the set $\{\delta_n(x^\alpha f_i + I) : |\alpha| < n, \quad i = 1, 2, \ldots, m\}$. Therefore we have the following exact sequence of R/I-modules:

$$0 \to N \xrightarrow{\theta} F \to \Omega_n(R/I) \to 0.$$

In this sequence θ is a relation matrix of the universal module $\Omega_n(R/I)$. Given such a matrix, $F_i(\Omega_n(R/I))$ denotes the ideal of R/I generated by all rank $F - i$ rowed subdeterminants of θ. $F_i(\Omega_n(R/I))$ is called i-th Fitting ideal of $\Omega_n(R/I)$. We can write the following increasing chain by using properties of Fitting ideals:

$$F_0(\Omega_n(R/I)) \subseteq F_1(\Omega_n(R/I)) \subseteq \cdots \subseteq F_i(\Omega_n(R/I)) \subseteq \cdots$$

and

$$F_i(\Omega_n(R/I)) = R/I \text{ for } i \geq \mu(\Omega_n(R/I)) \text{ for } i \geq rank F.$$

Proposition 2. *Let R be an affine domain with dimension s. If $\Omega_n(R)$ has rank r, then the first nonzero Fitting ideal of $\Omega_n(R)$ is $F_r(\Omega_n(R))$ [3].*

Proof. Let $Q(R)$ be a field of fraction of R. From Proposition 1, we have $F_i(Q(R) \otimes_R \Omega_n(R)) = Q(R)F_i(\Omega_n(R))$. Since $Q(R) \otimes_R \Omega_n(R)$ is a free $Q(R)$-module with rank r, we obtain $F_i(Q(R) \otimes_R \Omega_n(R)) = 0$ for $i = 0, 1, \ldots, r-1$. Hence, $F_i(\Omega_n(R)) = 0$ for $i = 0, 1, \ldots, r-1$. Similarly, since $F_i(Q(R) \otimes_R \Omega_n(R)) = Q(R)F_i(\Omega_n(R)) = Q(R) \neq 0$ for $i \geq r$, $F_i(\Omega_n(R)) \neq 0$. □

Proposition 3. *Let R be an affine local domain with dimension s. $\Omega_n(R)$ is a free R-module with rank r if and only if the first nonzero Fitting ideal is R [3].*

Proof. Let $\Omega_n(R)$ be a free R-module with rank r. Then,

$$0 \to N \xrightarrow{\theta} F \to \Omega_n(R/I) \to 0$$

is an exact sequence of R-modules. The relation matrix of $\Omega_n(R)$ is zero matrix since $\Omega_n(R) \simeq R^r$.

Therefore $F_i(\Omega_n(R)) = 0$ for $i < r$ and $F_i(\Omega_n(R)) = R$ for $i \geq r$.

Conversely, if $F_i(\Omega_n(R)) = 0$ for $i < r$ and $F_i(\Omega_n(R)) = R$ for $i \geq r$ then the relation matrix of $\Omega_n(R)$ is a zero matrix. Hence $\Omega_n(R)$ is a free R-module which has rank r. □

Corollary 1. *Let R be an affine k-algebra with dimension s. $\Omega_n(R)$ is a projective R-module with rank r if and only if the first nonzero Fitting ideal is R [3].*

Theorem 3. *Let R be an affine k-algebra. Then the following are equivalent [6]:*

(i) *R is a regular local ring.*
(ii) *$\Omega_1(R)$ is a free R-module.*
(iii) *The first nonzero Fitting ideal is R.*

Proof. It follows from Proposition 3 and Theorem 2. □

Corollary 2. *Let R be an affine k-algebra. Then the following are equivalent [6]:*

(i) *R is a regular ring.*
(ii) *$\Omega_1(R)$ is a projective R-module.*
(iii) *The first nonzero Fitting ideal is R.*

Theorem 4. *Let R be an affine local k-algebra and $T(\Omega_n(R))$ be torsion module of $\Omega_n(R)$ [2]. Then the following statements are equivalent:*

(i) *The first nonzero Fitting ideal is the principal ideal generated by a nonzero divisor of R*
(ii) *$\frac{\Omega_n(R)}{T(\Omega_n(R))} \simeq R^r$ and there exists an exact sequence*

$$0 \to P_1 \to P_0 \to \Omega_n(R) \to 0$$

with free R-modules P_0, P_1 of finite rank.

3. Results

Definition 3. *Let R be an integral domain, I be the ideal of R and Q be the quotient field of R. If $xI \subseteq R$ for $0 \neq x \in Q$, then I is called a fractional ideal of R. Let I^{-1} denote $\{q \in Q | qI \subseteq R\}$. If $II^{-1} = R$, then I is called an invertible ideal.*

Definition 4. *M is said to be a finite projective dimension if there is a projective resolution of the form $0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$.*

The minimum of the lengths n of such resolutions is called the projective dimension of M. It is denoted by $pd_R M$.

Theorem 5. *Let R be an integral domain and I a be finitely generated ideal of R. Then I is an invertible ideal if and only if I_m is a principal ideal for all maximal ideals of m [17].*

Theorem 6. *Let R be an affine domain. Then the following statements are equivalent [6]:*

(i) *The first nonzero Fitting ideal is an invertible ideal.*
(ii) *$\frac{\Omega_n(R)}{T(\Omega_n(R))}$ is a projective R-module of rank r and $pd_R\Omega_n(R) \leq 1$.*

Proof. Since R is an affine domain, we have $T(\Omega_n(R))_p = T\left(\Omega_n(R)_p\right)$ for each $p \in Spec(R)$. An ideal in a Noetherian ring is an invertible ideal if and only if it is locally generated by a nonzero divisor. It follows from Theorem 4. □

Corollary 3. *Let R be an affine domain and $\Omega_n(R)$ be the universal module of rank r. If $F_r(\Omega_n(R))$ is an invertible ideal then $pd(\Omega_n(R)) \leq 1$ [6].*

Corollary 4. *Let R be an affine domain. If $F_i(\Omega_n(R))$ is not an invertible ideal for all $i \in I$ then R is not a regular ring [6].*

Example 1. *Let $R = k[x,y]/\left(x^2+y^2-4\right)$. Then $2xdx + 2ydy = 0$. Therefore $F_0(\Omega_1(R)) = 0$ and $F_1(\Omega_1(R)) = (x,y) = R$. Hence $\Omega_1(R)$ is a projective R-module of rank 1 and R is a regular ring.*

Example 2. *Let S be the coordinate ring of the $y^2 = xz$. Then $S = k[x,y,z]/(f)$ where $f = y^2 - xz$. It can be found the Fitting ideals of $\Omega_1(S)$ and $\Omega_2(S)$.*

$\Omega_1(S) \simeq F/N$ where F is a free S-module on $\{\delta_1(x), \delta_1(y), \delta_1(z)\}$ and N is a submodule of F generated by $\delta_1(f) = 2y\delta_1(y) - x\delta_1(z) - z\delta_1(x)$. Certainly, N is a free submodule on $\delta_1(f)$. Then $pd(\Omega_1(S)) \leq 1$. Therefore, we have

$$0 \to N \xrightarrow{\phi} F \xrightarrow{\pi} \Omega_1(S) \simeq F/N \to 0$$

a free resolution of $\Omega_1(S)$. In this sequence the homomorphism ϕ is a matrix,

$$\begin{pmatrix} -z \\ 2y \\ -x \end{pmatrix}$$

which is a relation matrix of $\Omega_1(S)$. The Fitting ideals of $\Omega_1(S)$ are $F_0(\Omega_1(S)) = 0 = F_1(\Omega_1(S)) \subseteq F_2(\Omega_1(S)) = (x,y,z) \subseteq F_3(\Omega_1(S)) = S$.

Since $\mathrm{rank}\Omega_1(S) = 2$ and $F_2(\Omega_1(S)) = (x,y,z) \neq S$, $\Omega_1(S)$ is not a projective module and S is not a regular ring by the Corollary 2 and Theorem 1.

As the same argument $\Omega_2(S) \simeq F'/N'$ where F' is a free S-module on:

$$\left\{\delta_2(x), \delta_2(y), \delta_2(z), \delta_2(xy), \delta_2(xz), \delta_2(yz), \delta_2\left(x^2\right), \delta_2\left(y^2\right), \delta_2\left(z^2\right)\right\}$$

N' is generated by $\{\delta_2(f), \delta_2(xf), \delta_2(yf), \delta_2(zf)\}$ which is a submodule of F'.

$$\delta_2(f) = \delta_2\left(y^2\right) - \delta_2(xz).$$

$$\begin{gathered}\delta_2(xf) = \delta_2\left(xy^2 - x^2z\right) = -z\delta_2\left(x^2\right) + x\delta_2\left(y^2\right) + 2y\delta_2(xy) + \\ 2x\delta_2(xz) + xz\delta_2(x) - 2xy\delta_2(y) + x^2\delta_2(z).\end{gathered}$$

$$\begin{gathered}\delta_2(yf) = \delta_2\left(y^3 - xyz\right) = 3y\delta_2\left(y^2\right) - z\delta_2(xy) - y\delta_2(xz) - \\ x\delta_2(yz) + yz\delta_2(x) - 2xz\delta_2(y) + xy\delta_2(z).\end{gathered}$$

$$\begin{gathered}\delta_2(zf) = \delta_2\left(y^2z - xz^2\right) = z\delta_2\left(y^2\right) - x\delta_2\left(z^2\right) - 2z\delta_2(xz) + \\ 2y\delta_2(yz) - z^2\delta_2(x) - 2yz\delta_2(y) + xz\delta_2(z).\end{gathered}$$

Since $\mathrm{rank}\Omega_2(S) = 5$ we have $\mathrm{rank}N' = \mathrm{rank}F' - \mathrm{rank}\Omega_2(S) = 9 - 5 = 4$. So N' is a free S-module. Then $pd(\Omega_2(S)) \leq 1$. Therefore, we have,

$$0 \to N' \xrightarrow{\phi} F' \xrightarrow{\pi} \Omega_2(S) \simeq F'/N' \to 0$$

a free resolution of $\Omega_2(S)$. Here π is the natural surjection and ϕ is given by the following matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -z & x & 0 & 2y & -2x & 0 & xz & -2xy & x^2 \\ 0 & 3y & 0 & -z & -y & -x & yz & -2xz & xy \\ 0 & z & -x & 0 & -2z & 2y & z^2 & -2yz & xz \end{pmatrix}$$

This is a relation matrix of $\Omega_2(S)$. *The Fitting ideals of* $\Omega_2(S)$ *are* $F_0(\Omega_2(S)) = 0 = F_1(\Omega_2(S)) = F_2(\Omega_2(S)) = F_3(\Omega_2(S)) = F_4(\Omega_2(S)) = 0 \subseteq F_5(\Omega_2(S)) \neq 0 \subseteq F_6(\Omega_2(S)) \neq 0 \subseteq F_7(\Omega_2(S)) = (x,y,z) \subseteq F_8(\Omega_2(S)) = S$.

We can say $\Omega_2(S)$ *is not a projective module from Corollary 1. Hence,* S *is not a regular ring by the Theorem 1.*

Example 3. *Let* $S = k[x,y,z]/(f,g,h)$ *where* $f = y^2 - xz$, $g = yz - x^3$ *and* $h = z^2 - x^2y$. *We can find the Fitting ideals of* $\Omega_1(S)$.

$\Omega_1(S) \simeq F/N$ *where* F *is a free* S*-module on* $\{\delta_1(x), \delta_1(y), \delta_1(z)\}$ *and* N *is a submodule of* F *generated by,*

$$\delta_1(f) = \delta_1\left(y^2 - xz\right) = 2y\delta_1(y) - z\delta_1(x) - x\delta_1(z)$$

$$\delta_1(g) = \delta_1\left(yz - x^3\right) = z\delta_1(y) + y\delta_1(z) - 3x^2\delta_1(x)$$

$$\delta_1(h) = \delta_1\left(z^2 - x^2y\right) = 2z\delta_1(z) - 2xy\delta_1(x) - x^2\delta_1(y)$$

$$\begin{pmatrix} -z & -3x^2 & -2xy \\ 2y & z & -x^2 \\ -x & y & 2z \end{pmatrix}$$

This is a relation matrix of $\Omega_1(S)$. *The Fitting ideals of* $\Omega_1(S)$ *are* $F_0(\Omega_1(S)) = 0 = F_1(\Omega_1(S)) \subseteq F_2(\Omega_1(S)) = (x,y,z) \subseteq F_3(\Omega_1(S)) = S$.

Since $rank(\Omega_1(S)) = 2$ *and* $F_2(\Omega_1(S)) = (x,y,z) \neq S$, $\Omega_1(S)$ *is not a projective module, so* S *is not a regular ring by the Corollary 2. Furthermore, we know that* $pd\Omega_1(S) = \infty$ *in* [7], *then from Corollary 3.* $F_2(\Omega_1(S))$ *is not an invertible ideal.*

Example 4. *Let* $R/I = k[x,y]/(y^2 - x^3)$ *and* $S/J = k[z,t]/(z^2 - t^3)$ *be affine k-algebras. Suppose that* $K = I \otimes_k S + R \otimes_k J$. *Let* F *be free* $R \otimes_k S$*-module generated by the set* $\{\delta_1(x \otimes 1), \delta_1(y \otimes 1), \delta_1(1 \otimes z), \delta_1(1 \otimes t)\}$ *and let* N *be the submodule of* F *generated by the set* $\{\delta_1(f \otimes 1), \delta_1(1 \otimes g)\}$.

Since $\Omega_1\left(\frac{R\otimes_k S}{K}\right) \simeq F/N$, *we have the following exact sequence,*

$$0 \to N \to F \to \Omega_1\left(\frac{R \otimes_k S}{K}\right) \to 0$$

$rankN = rankF - rank\Omega_1\left(\frac{R\otimes_k S}{K}\right) = 4 - 2 = 2$. *Then* N *is a free module. Therefore the above sequence is a free resolution of* $\Omega_1\left(\frac{R\otimes_k S}{K}\right)$. *Hence* $pd(\Omega_1\left(\frac{R\otimes_k S}{K}\right)) \leq 1$.

$$\delta_1(f \otimes 1) = \delta_1\left(y^2 \otimes 1\right) - \delta_1\left(x^3 \otimes f\right) = (2y \otimes 1)\delta_1(y \otimes 1) - \left(3x^2 \otimes 1\right)\delta_1(x \otimes 1)$$

$$\delta_1(1 \otimes g) = \delta_1\left(1 \otimes z^2\right) - \delta_1\left(1 \otimes t^3\right) = (1 \otimes 2z)\delta_1(1 \otimes z) - \left(1 \otimes 3t^2\right)\delta_1(1 \otimes t)$$

$$\begin{pmatrix} -(3x^2 \otimes 1) & (2y \otimes 1) & 0 & 0 \\ 0 & 0 & -(1 \otimes 3t^2) & (1 \otimes 2z) \end{pmatrix}$$

The Fitting ideal of $\Omega_1\left(\frac{R\otimes_k S}{K}\right)$ *are,*

$$F_0(\Omega_1\left(\frac{R \otimes_k S}{K}\right)) = 0$$

$$F_1(\Omega_1\left(\frac{R \otimes_k S}{K}\right)) = 0$$

$$F_2(\Omega_1\left(\frac{R\otimes_k S}{K}\right)) = \left(x^2\otimes t^2, x^2\otimes z, y\otimes t^2, y\otimes z\right)$$

$$F_3(\Omega_1\left(\frac{R\otimes_k S}{K}\right)) = \left(x^2\otimes 1, y\otimes 1, 1\otimes t^2, 1\otimes z\right)$$

$$F_4(\Omega_1\left(\frac{R\otimes_k S}{K}\right)) = \frac{R\otimes_k S}{K}$$

Since $rank\Omega_1\left(\frac{R\otimes_k S}{K}\right) = 2$, *then* $\Omega_1\left(\frac{R\otimes_k S}{K}\right)$ *is not projective. So* $\frac{R\otimes_k S}{K}$ *is not a regular ring. Hence,* $pd\Omega_1\left(\frac{R\otimes_k S}{K}\right) = 1$.

Let F' *be the free* $R\otimes_k S$ *module generated by the set:*

$$\{\delta_2(x\otimes 1), \delta_2(y\otimes 1), \delta_2(1\otimes z), \delta_2(1\otimes t), \delta_2(x\otimes z), \delta_2(x\otimes t), \delta_2(y\otimes z), \delta_2(y\otimes t),$$
$$\delta_2(x^2\otimes 1), \delta_2(y^2\otimes 1), \delta_2(1\otimes z^2), \delta_2(1\otimes t^2), \delta_2(xy\otimes 1), \delta_2(1\otimes zt)\}$$

And let N' *be submodule of* F' *generated by:*

$$\{\delta_2(f\otimes 1), \delta_2(1\otimes g), \delta_2(fx\otimes 1), \delta_2(fy\otimes 1), \delta_2(1\otimes zg), \delta_2(1\otimes tg), \delta_2(f\otimes z), \delta_2(f\otimes t),$$
$$\delta_2(x\otimes g), \delta_2(y\otimes g)\}.$$

Since $\Omega_2\left(\frac{R\otimes_k S}{K}\right) \simeq F'/N'$, *we have the following exact sequence:*

$$0\to N'\to F'\to \Omega_2\left(\frac{R\otimes_k S}{K}\right)\otimes 0$$

Since $rankN' = rankF' - rank\Omega_2\left(\frac{R\otimes_k S}{K}\right) = 14 - 5 = 9$, *the generating set of* N' *is not a basis:*

$$\delta_2(f\otimes 1) = \delta_2\left(y^2\otimes 1\right) - (3x\otimes 1)\delta_2\left(x^2\otimes 1\right) - \left(3x^2\otimes 1\right)\delta_2(x\otimes 1)$$

$$\delta_2(1\otimes g) = \delta_2\left(1\otimes t^2\right) - (1\otimes 3z)\delta_2\left(1\otimes z^2\right) - \left(1\otimes 3z^2\right)\delta_2(1\otimes z)$$

$$\delta_2(xf\otimes 1) = (x\otimes 1)\delta_2(y^2\otimes 1) - (2y\otimes 1)\delta_2(xy\otimes 1) + (7x^3\otimes 1)\delta_2(x\otimes 1) + (2xy\otimes 1)\delta_2(y\otimes 1)$$
$$-(6x^2\otimes 1)\delta_2(x^2\otimes 1)$$

$$\delta_2(yf\otimes 1) = (3y\otimes 1)\delta_2(y^2\otimes 1) - (x^3\otimes 1)\delta_2(y\otimes 1) - (3xy\otimes 1)\delta_2(x^2\otimes 1) - (3x^2\otimes 1)\delta_2(y\otimes 1)$$
$$-(6x^2y\otimes 1)\delta_2(x\otimes 1)$$

$$\delta_2(f\otimes z) = (1\otimes z)\delta_2(y^2\otimes 1) + (2y\otimes 1)\delta_2(y\otimes z) - (2y\otimes z)\delta_2(y\otimes 1) - (3x\otimes z)\delta_2(x^2\otimes 1)$$
$$+(x^3\otimes 1)\delta_2(1\otimes z) + (6x^2\otimes z)\delta_2(x\otimes 1)$$

$$\delta_2(f\otimes t) = (1\otimes t)\delta_2(y^2\otimes 1) + (2y\otimes 1)\delta_2(y\otimes t) + (x^3\otimes 1)\delta_2(1\otimes t) - (2y\otimes t)\delta_2(y\otimes 1)$$
$$-(3x\otimes t)\delta_2(x^2\otimes 1) - (3x^2\otimes 1)\delta_2(x\otimes t) + (6x^2\otimes t)\delta_2(x\otimes 1)$$

$$\delta_2(x\otimes g) = (x\otimes 1)\delta_2(1\otimes t^2) + (2\otimes t)\delta_2(x\otimes t) - (2x\otimes t)\delta_2(1\otimes t) - (1\otimes z^3)\delta_2(x\otimes 1)$$
$$-(3x\otimes z)\delta_2(1\otimes z^2) - (1\otimes 3z^2)\delta_2(x\otimes z) + (6x\otimes z^2)\delta_2(1\otimes z)$$

$$\delta_2(y\otimes g) = (y\otimes 1)\delta_2(1\otimes t^2) + (1\otimes 2t)\delta_2(y\otimes t) - (2y\otimes t)\delta_2(1\otimes t) - (1\otimes z^3)\delta_2(y\otimes 1)$$
$$-(3y\otimes z)\delta_2(1\otimes z^2) - (3\otimes z^2)\delta_2(y\otimes z) + (6y\otimes z^2)\delta_2(1\otimes z)$$

$$\delta_2(z\otimes g) = (1\otimes z)\delta_2(1\otimes t^2) + (2\otimes t)\delta_2(1\otimes zt) + (7\otimes z^3)\delta_2(1\otimes z) - (2z\otimes t)\delta_2(1\otimes t)$$
$$-(6\otimes z^2)\delta_2(1\otimes z^2)$$

$$\delta_2(t\otimes g) = (3\otimes t)\delta_2(1\otimes t^2) - (1\otimes z^3)\delta_2(1\otimes t) - (1\otimes zt)\delta_2(1\otimes z^2) - (3\otimes z^2)\delta_2(1\otimes zt)$$
$$+(1\otimes z^2t)\delta_2(1\otimes z)$$

The Fitting ideals of $\Omega_2\left(\frac{R\otimes_k S}{K}\right)$ are:

$$F_0(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = F_1(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = F_2(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = 0$$

$$F_3(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = F_4(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = 0$$

$$F_5(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = (xy\otimes z) \subseteq F_6(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = \frac{R\otimes_k S}{K}$$

We know that $pd\Omega_2\frac{R\otimes_k S}{K} \leq 2$. *Since rank of* $\Omega_2\left(\frac{R\otimes_k S}{K}\right)$ *is* 5 *and* $F_5(\Omega_2\left(\frac{R\otimes_k S}{K}\right)) = (xy\otimes z) \neq \frac{R\otimes_k S}{K}$, *then* $\Omega_2\left(\frac{R\otimes_k S}{K}\right)$ *is not a projective module (i.e.,* $\Omega_2\frac{R\otimes_k S}{K} \neq 0$ *), so* $\frac{R\otimes_k S}{K}$ *is not a regular ring by the Corollary 2. Hence,* $pd\Omega_2\left(\frac{R\otimes_k S}{K}\right)$ *must be* 1 and 2.

Theorem 7. *If* S_1 *and* S_2 *are R-algebras such that* $\Omega_1(S_1)$ *and* $\Omega_1(S_2)$ *are finitely generated [3], then,*

$$F_i(\Omega_1(S_1\otimes_R S_2)) = \sum_{p+q=i} F_p(\Omega_1(S_1))\otimes_R F_q(\Omega_1(S_2)).$$

In particular,

$$F_0(\Omega_1(S_1\otimes_R S_2)) = F_0(\Omega_1(S_1))\otimes_R F_0(\Omega_1(S_2))$$

Proof. We know that from [16], $\Omega_1(S_1\otimes_R S_2) = (S_2\otimes_R \Omega_1(S_1))\otimes(S_1\otimes_R \Omega_1(S_2))$

Using properties of the Fitting ideal of direct product of modules, then we obtain the following equality:

$$\begin{aligned} F_i(\Omega_1(S_1\otimes_R S_2)) &= \sum_{p+q=i} F_p(S_2\otimes_R \Omega_1(S_1))\otimes_R F_q(S_1\otimes_R \Omega_1(S_2)) \\ &= \sum_{p+q=i} S_2F_p(\Omega_1(S_1))\otimes_R S_1F_p(\Omega_1(S_2)) \\ &= \sum_{p+q=i} S_1F_p(\Omega_1(S_1))\otimes_R S_2F_p(\Omega_1(S_2)) \\ &= \sum_{p+q=i} F_p(\Omega_1(S_1))\otimes_R F_p(\Omega_1(S_2)) \end{aligned}$$

□

Now, we can give our important result as follows.

Theorem 8. *Let* $R = k[x_1, x_2, \ldots, x_s]$ *and* $S = k[y_1, y_2, \ldots, y_t]$ *be polynomial algebras,* $R/I = \frac{k[x_1,x_2,\ldots,x_s]}{(f_1,f_2,\ldots,f_m)}$ *and* $S/J = \frac{k[y_1,y_2,\ldots,y_t]}{(g_1,g_2,\ldots,g_n)}$ *be affine k-algebras. If* $rank\Omega_1(R/I) = i$, $rank\Omega_1(S/J) = j$,$F_i(\Omega_1(R/I))$ *and* $F_j(\Omega_1(S/J))$ *are invertible ideals, then* $F_{i+j}(\Omega_1(R/I\otimes_k S/J))$ is an invertible ideal.

Proof. We have from Theorem 4:

$$F_{i+j}(\Omega_1(R/I\otimes_k S/J)) = \sum_{p+q=i+j} F_p(\Omega_1(S_1))\otimes_R F_p(\Omega_1(S_2)).$$

Since $rank\Omega_1(R/I) = i$ and $rank\Omega_1(S/J) = j$, respectively, we obtain $F_k(\Omega_1(R/I)) = 0$ for $k < i$ and $F_l(\Omega_1(S/J)) = 0$ for $l < j$ (Proposition 2). Therefore, we have:

$$F_{i+j}(\Omega_1(R/I \otimes_k S/J)) = \sum_{p+q=i+j} F_p(\Omega_1(S_1)) \otimes_R F_p(\Omega_1(S_2)).$$

$$\begin{gathered} F_{i+j}(\Omega_1(R/I \otimes_k S/J)) = F_0(\Omega_1(R/I)) \otimes_k F_{i+j}(\Omega_1(S/J)) + F_1(\Omega_1(R/I)) \otimes_k F_{i+j-1}(\Omega_1(S/J)) + \ldots + \\ F_{i-1}(\Omega_1(R/I)) \otimes_k F_{j+1}(\Omega_1(S/J)) + F_i(\Omega_1(R/I)) \otimes_k F_j(\Omega_1(S/J)) + \\ F_{i+1}(\Omega_1(R/I)) \otimes_k F_{j-1}(\Omega_1(S/J)) + \ldots + F_{i+j}(\Omega_1(R/I)) \otimes_k F_0(\Omega_1(S/J)) \\ = 0 \otimes_k F_{i+j}(\Omega_1(S/J)) + 0 \otimes_k F_{i+j-1}(\Omega_1(S/J)) + \ldots + 0 \otimes_k F_{j+1}(\Omega_1(S/J)) + \\ F_i(\Omega_1(R/I)) \otimes_k F_j(\Omega_1(S/J)) + F_{i+1}(\Omega_1(RI)) \otimes_k 0) + \ldots + F_{i+j}(\Omega_1(R/I)) \otimes_k 0 \end{gathered}$$

$$F_{i+j}(\Omega_1(R/I \otimes_k S/J)) = F_i(\Omega_1(R/I)) \otimes_k F_j(\Omega_1(S/J)) \neq 0.$$

Now we localize these ideals at maximal ideal:

$$[F_{i+j}(\Omega_1(R/I \otimes_k S/J))]_m = (F_i(\Omega_1(R/I)) \otimes_k F_j(\Omega_1(S/J)))_m \neq 0.$$

$$[F_{i+j}(\Omega_1(R/I \otimes_k S/J))]_m = [F_i(\Omega_1(R/I))]_m \otimes_{k_m} [F_j(\Omega_1(S/J))]_m$$

We know that $[F_i(\Omega_1(R/I))]_m$ and $[F_j(\Omega_1(S/J))]_m$ are principal ideals (Theorem 5). Therefore, we obtain that $[F_{i+j}(\Omega_1(R/I \otimes_k S/J))]_m$ is a principal ideal. Thus $[F_{i+j}(\Omega_1(R/I \otimes_k S/J))]_m$ is an invertible ideal. □

Theorem 8 can be expressed as follows:

Theorem 9. *Suppose that R/I and S/J be as Theorem 8. If the first nonzero Fitting ideals of $\Omega_1(R/I)$ and $\Omega_1(S/J)$ are invertible ideals, then the first nonzero Fitting ideal of $\Omega_1(R/I \otimes_k S/J)$ is an invertible ideal.*

We can obtain the following result by using above the theorem.

Corollary 5. *If the first nonzero Fitting ideals of $\Omega_1(R/I)$ and $\Omega_1(S/J)$ are invertible ideals, then $pd\Omega_1(R/I \otimes_k S/J) \leq 1$.*

Proof. Suppose that $F_i(\Omega_1(R/I))$ and $F_j(\Omega_1(S/J))$ are invertible ideals for some integers i and j, then $F_{i+j}(\Omega_1(R/I \otimes_k S/J))$ is an invertible ideal from Theorem 9. Therefore $pd\Omega_1(R/I \otimes_k S/J)$ is zero or one from Corollary 3. □

4. Discussion

Fitting ideals are important tools to determine the projective dimension of the modules. In this study, we obtain a result for the projective dimension of Kahler modules $\Omega_1(R/I \otimes_k S/J)$ using the invertibility of Fitting ideals. At this point, the following questions arise:

1. Can we generalize these results to the universal module of nth order derivations of $R/I \otimes_k S/J$?
2. Using the invertibility of Fitting ideals of modules, what else can we say about other properties of modules?
3. We know that the first Fitting ideal of a module is important for the dynamical properties of the system. Is there any relation between the invertible Fitting ideal and the dynamical properties of the system?

5. Conclusions

Fitting ideals are important tools to characterize modules and determine regularity of ring. However, there are few studies about the Fitting ideal of universal modules. Here, we try to determine the projective dimension of the Kahler module $\Omega_1(R/I \otimes_k S/J)$ by using the invertibility of Fitting ideals of $\Omega_1(R/I)$ and $\Omega_1(S/J)$. So, we use another way to determine the projective dimension of Kahler modules. Finally, we give examples for our conclusions.

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