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Article

Applications of Differential Form Wu's Method to Determine Symmetries of (Partial) Differential Equations

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Received: 6 July 2018; Accepted: 28 August 2018; Published: 3 September 2018

Abstract: In this paper, we present an application of Wu's method (differential characteristic set (dchar-set) algorithm) for computing the symmetry of (partial) differential equations (PDEs) that provides a direct and systematic procedure to obtain the classical and nonclassical symmetry of the differential equations. The fundamental theory and subalgorithms used in the proposed algorithm consist of a different version of the Lie criterion for the classical symmetry of PDEs and the zero decomposition algorithm of a differential polynomial (d-pol) system (DPS). The version of the Lie criterion yields determining equations (DTEs) of symmetries of differential equations, even those including a nonsolvable equation. The decomposition algorithm is used to solve the DTEs by decomposing the zero set of the DPS associated with the DTEs into a union of a series of zero sets of dchar-sets of the system, which leads to simplification of the computations.

Keywords: Wu's method; symmetry; partial differential equations; computation

1. Introduction

In symmetry analysis of (partial) differential equations (PDEs), the computation of the maximal symmetry admitted by the system is the first step in order to use the symmetry [1–3]. However, the determining equations (DTEs) form a complex PDE system. In particular, for the nonclassical case, it is a nonlinear system. Therefore, the solving of DTEs for the symmetry of PDEs is a challenging problem in terms of mathematical computation. In this article, we present algorithms to deal with the classical and nonclassical symmetry computations based on the differential form Wu's method. The main idea of the algorithms is to decompose the zero set of the differential polynomial (d-pol) system (DPS) from the DTEs into the union of zero sets of the characteristic set for the DPS. The algorithms not only provide a direct and systematic procedure for determining the symmetry of PDEs, but also theoretically remedy the defect existing in the Lie algorithm.

For a good understanding, in this section, we briefly review the symmetry computation problems of PDEs and pose the questions that are the main subjects of the article.

We let $X = (x_1, x_2, \cdots, x_p)$ be independent variables and $U = (u_1, u_2, \cdots, u_q)$ be dependent variables. We use $U^\alpha = \{u_i^\alpha, i = 1, 2, \cdots, q\}$, where $\alpha = \{\alpha_1, \cdots, \alpha_p\} \in N_+^p$ (N_+ is the set of nonnegative integers) and $u_i^\alpha = \frac{\partial^{|\alpha|} u_i}{\partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}$ ($|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_p$), to denote the derivative terms of U with respect to (w.r.t.) X. We let $\partial U = \{U^\alpha, \alpha \in N_+^p\}$. To apply the classical symmetry method to the general PDEs:

$$\Delta(X, \partial U) = 0, \tag{1}$$

we consider the one-parameter infinitesimal transformation in $X \cup U$ given by

$$x_i^* = x + \epsilon\xi_i(X,U) + O(\epsilon^2),$$
$$u_j^* = u_j + \epsilon\phi_j(X,U) + O(\epsilon^2), \tag{2}$$

in which ϵ is the group parameter. The associated infinitesimal vector (InfV) of the transformation is the set of operators of the form

$$\mathcal{X} = \sum_{i=1}^{p} \xi_i \partial_{x_i} + \sum_{j=1}^{q} \phi_j \partial_{u_j}, \tag{3}$$

where $\partial_{x_i} \equiv \partial/\partial_{x_i}$ and so forth. If the PDEs (Equation (1)) are invariant under the transformation given by Equation (2), then the transformation is called a symmetry (Lie group) of the PDEs. This definition yields the following symmetry criterion theorem.

Theorem 1. *(Lie's criterion [2,3]) For the PDEs (Equation (1)) with maximal rank, the operator $\mathcal{X}$ in Equation (3) is the* InfV *of its symmetry (Equation (2)) if and only if the identity*

$$\mathrm{Pr}\mathcal{X}(\mathcal{R})\big|_{\mathcal{R}=0} \equiv 0 \tag{4}$$

holds, where $\mathcal{R} = \Delta(X, \partial U)$ is the left side of Equation (1) and $Pr\mathcal{X}$ the prolongation of $\mathcal{X}$ on the space of $X \times \partial U$.

Consequently, the DTEs, an over-determined and homogeneous linear system of PDEs for the infinitesimals ξ_i, ϕ_j,

$$\mathcal{D} = \mathcal{D}(X, U, \xi_i, \phi_j) = 0, \tag{5}$$

are derived by setting the coefficients of the polynomials given by Equation (4) in ∂U to zero. As a result, the problem of computing the symmetry given by Equation (2) is reduced to the problem of finding the corresponding InfV (Equation (3)), which is equivalent to solving the DTEs of Equation (5) for ξ_i, ϕ_j.

The above procedure was given by Lie [4] and is called the Lie algorithm. To facilitate the algorithm, symbolic manipulation programs by computer algebra systems such as Mathematica and Maple have been developed by many researchers [5–18]. In solving DTEs, Cartan's exterior forms approach [18], Janet–Riquier theory [8,9], the Gröbner base algorithm [11,12], the Rosenfeld–Gröbner algorithm [14], the Ritt–Kolchin differential algebra method [5], formal power series analysis [6], the characteristic set method [19,20], and so on are used to reduce the DTEs into a simpler form, such as the normal, orthonomic, involutive, and passive forms, so that the reduced system is easily manageable. However, from the point of view of the algorithm, producing DTEs (Equation (5)) from Equation (4) for a symmetry is restricted by an imposed requirement on the considered PDEs that the PDEs have to admit a "solved-and-triangular"structure (see a description about the structure of the implementation of the Lie algorithm on pages 372–375 in [16]). Hence, in the general case, Lie's algorithm cannot be directly used without transforming the underlying PDEs to the solved-and-triangular form. On the other hand, there have been several generalizations of the classical Lie symmetry. Bluman and Cole in [21] proposed the nonclassical symmetry. In this method, the original PDEs (Equation (1)) are augmented with the invariant surface conditions:

$$\mathcal{S} = \{\psi_j = \xi_i u_j^{e_i} - \phi_j, j = 1, 2, \cdots, q\}, \tag{6}$$

where e_i denotes the unit vector with the ith coordinate of 1 and the others as 0. By requiring that the set of simultaneous solutions of $\mathcal{R} = \{\Delta, \mathcal{S}\} = 0$ is invariant under the transformation given by

Equation (2), one obtains a nonlinear DTEs for ξ_i and ϕ_j. In [22], Cherniha, R., et al. further generalized the symmetry to more general nonclassical symmetry cases. Despite many researchers having made efforts, the finding of a method for efficiently solving DTEs is still a challenging problem [23]. Particularly, the following two basic questions still remain to be further investigated in the field of computing symmetries.

Question 1: How can DTEs be produced for the symmetry of PDEs that does not satisfy the solved-and-triangular structure constraint?

Question 2: How can DTEs be solved efficiently, particularly for the nonclassical case?

The positive answers for the two questions make the symmetry method more efficient and adaptable for a wider class of PDEs.

In this article, we investigate the computing of symmetries of PDEs from the differential algebra point of view by using the differential form Wu's method [24–26]. Particularly, we give algorithms for questions 1 and 2 mentioned above based on the differential form Wu's method.

Blank assumption: We suppose the PDEs considered in this article are always of polynomial form in terms of the dependent variables and the various derivatives involved in the PDEs.

The rest of this paper is organized as follows. In Section 1, we briefly recall Wu's method, also called the differential characteristic set (dchar-set) algorithm, for a DPS and its basic results; in Section 2, alternative algorithms based on the dchar-set algorithm for a DPS for the computation of the symmetry of a system of PDEs are given; in Section 3, we give several applications of the proposed algorithms to determine the symmetry of mathematical physics equations; in Section 4, we give some concluding remarks.

2. Wu's Method (Characteristic Set Method)

Wu's method, proposed by the Chinese mathematician Wu Wen Tsun in the 1970s, is one of the fundamental algorithmic theories in the algebraic geometry field [11]. The differential form of Wu's method was established in the 1980s in [25]. Compared with other methods such as Gröbner's base [11], Kolchin's [27], and Ritt's [28] methods, Wu's method has advantages for dealing with the reduction of a DPS. The method targets the zero set of a DPS and gives efficient differential elimination without directly involving the concept of algebra ideals.

The notation $\mathcal{K}_X$ represents a differential field of functions of X with derivative operators $\partial_{x_i}, i = 1, 2, \cdots, p$, and $\mathcal{K}_X[\partial U]$ is the d-pol ring with indeterminates ∂U over $\mathcal{K}_X$. We use the notation Z(PS) for the zero points set of a DPS PS $\subset \mathcal{K}_X[\partial U]$ over a universal field of $\mathcal{K}_X$. In fact, it corresponds to the solution set of the PDEs: PS $= 0$. For a d-pol $I \in \mathcal{K}_X[\partial U]$, we denote $\mathrm{Z}(\mathrm{PS}/I) = \mathrm{Z}(\mathrm{PS}) \backslash \mathrm{Z}(I)$ (see these preliminary concepts and notations for abstract differential algebra in [27,28]).

For a given d-pol rank (or order) $\prec$, a d-pol $f \in \mathcal{K}_X[\partial U]$ is written in the order-decreasing form

$$f = I_\alpha (u^\alpha_{k_0})^n + \cdots + I_0$$

in the scheme of Wu's theory. Here, $u^\alpha_{k_0}$ is the leading (highest-order) derivative term in f. I_α and $\partial f / \partial u^\alpha_{k_0}$ are called the initial and the separant of f, respectively, and $n \in Z_+$ is called the leading power of f. We use notation IS to denote the product of initials and separants (ISP) of a DPS ignoring concrete expressions. We use notation IP to denote the integrability polynomials (conditions) obtained from any two d-pols.

Definition 1. *We say a d-pol f is reduced with respect to (w.r.t.) another d-pol g if f does not contain the derivatives of the leading derivative of g and the power of the leading derivative of g in f is less than that of g.*

Definition 2. *A finite DPS:*

$$\mathrm{CS} = \{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_s\} \tag{7}$$

is called a differential chain (d-chain) if it satisfiesthe following two conditions:

(a) $\mathcal{A}_1 \prec \mathcal{A}_2 \prec \cdots \mathcal{A}_s$*;*
(b) $\mathcal{A}_j$ *is reduced w.r.t.* $\mathcal{A}_i$ *for* $i = 1, 2, \cdots, j-1$.

Definition 3. *Under a d-pol rank, the lowest-rank d-chain contained in a DPS is called a basic set of the DPS.*

We suppose f is a d-pol and CS is a d-chain. Then Wu's elimination algorithm yields the pseudo-reduction formula [25]. That is, there exist d-pols $Q_\alpha \in \mathcal{K}_X[\partial U]$ such that

$$\mathrm{IS} \cdot f = \sum_{g_\alpha \in \mathrm{CS}} Q_\alpha D^\alpha g_\alpha + r, \tag{8}$$

where the d-pol r is reduced w.r.t. CS and is called the pseudo-remainder of f w.r.t. the d-chain CS, denoted by Prem(f/CS); that is, r = Prem(f/CS). If CS $= \emptyset$, we have Prem(f/CS) = f. For a DPS PS, we use the notation

$$\mathrm{Prem}(\mathrm{PS}/\mathrm{CS}) = \{\mathrm{Prem}(f/\mathrm{CS}) \text{ for } f \in \mathrm{PS}\}.$$

Definition 4. *A d-chain (Equation (7)) is said to be irreducible if for any* $1 \le i \le s$ *no relation of the form*

$$\mathrm{Prem}(H_i * \mathcal{A}_i/\mathrm{CS}_{i-1}) = B_i * C_i$$

can exist, where CS_{i-1} *is the partial d-chain of Equation (7) consisting of the first* $i-1$ *elements* $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{i-1}$, $CS_0 = \emptyset$, *while* $H_i \in \mathcal{K}_X[\partial U]$ *with a lower leading derivative and is reduced w.r.t.the partial d-chain* CS_{i-1}*. Additionally, for each i,* $B_i, C_i \in \mathcal{K}_X[\partial U]$ *have the same leading derivative as* A_i*.*

The following is the definition of the triangular form and dchar-set of a DPS.

Definition 5. *For a DPS, there exists a d-chain CS satisfying the properties* (a_1) *and* (a_2) *below:*

(a_1) Z(PS) $\subset$ Z(CS)*;*
(a_2) Prem(PS/CS)=0*.*

Then we call the d-chain CS the differential triangular set (dtri-set) of the PS. In addition, if the CS verifies the following property:

(a_3) Prem(IP/CS)=0 *for all IP of CS,*

then we call the d-chain CS the dchar-set of the PS.

The dchar-set has many algebraic properties [25], such as a triangular structure, integrability conditions, and so on, which makes the analysis of the zero set of a DPS more convenient.

Now we show the results found in this article for the dchar-set theory given in [25] first and further discussed in [29–31].

Theorem 2. *There is a dchar-set algorithm that determines a dchar-set CS for a given finite DPS in a finite number of steps and that makes the well-ordering principle:*

$$\begin{aligned}&\mathrm{Z(CS/IS)} \subset \mathrm{Z(PS)} \subset \mathrm{Z(CS)},\\ &\mathrm{Z(PS)}=\mathrm{Z(CS/IS)} \cup \mathrm{Z(PS,IS)},\end{aligned} \tag{9}$$

and zero decomposition:

$$\mathrm{Z(PS)} = \cup_k \mathrm{Z}(\mathrm{CS}_k/\mathrm{IS}_k) = \cup_j \mathrm{Z}(\mathrm{ICS}_j/\mathrm{IS}_j), \tag{10}$$

hold true, where $CS_k(ICS_k)$ are the dchar-sets (the irreducible dchar-sets) of an extension DPS obtained by adding some d-pols in PS and IS and IS_k are ISPs of these dchar-sets.

The dchar-set algorithm is given by the following algorithmic scheme.

The rank of a d-pol plays a key role in Wu's method mentioned above. In practical computations, we use the differential graded lexicographic rank on derivative terms as taken in [25–28,31]. The decomposition given by Equation (10) and Algorithm 1 are the main tools for solving symmetry computation problems.

Algorithm 1: Wu's algorithmfor determining a dchar-set of a DPS

$$\begin{array}{ccccccccc} \mathrm{PS} = & \mathrm{PS}_0 & \subset & \mathrm{PS}_1 & \subset & \cdots & \subset & \mathrm{PS}_s & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & \mathrm{BS}_0 & \succ & \mathrm{BS}_1 & \succ & \cdots & \succ & \mathrm{BS}_s & = \mathrm{CS} \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ & \mathrm{RIS}_0 \uparrow & & \mathrm{RIS}_1 \uparrow & & \cdots \uparrow & & \mathrm{RIS}_s = \emptyset & \end{array}$$

in which

$$\begin{cases} \mathrm{BS}_i \text{ is a base set of } \mathrm{PS}_i \text{ and } \mathrm{BS}_i \succ \mathrm{BS}_{i+1}, \\ R_i = \mathrm{Prem}((\mathrm{PS}_i \backslash \mathrm{BS}_i)/\mathrm{BS}_i) \backslash \{0\}, \\ \mathrm{IT}_i = \mathrm{Prem}(\mathrm{IP}/\mathrm{BS}_i) \backslash \{0\}, \text{ for any IP of } \mathrm{BS}_i, \\ \mathrm{RIS}_i = \mathrm{IT}_i \cup R_i, \\ \mathrm{PS}_i = \mathrm{PS}_0 \cup \mathrm{BS}_{i-1} \cup \mathrm{RIS}_{i-1}, i = 0,1,2,\cdots,s, \end{cases} \tag{W}$$

where $\mathrm{DBS}_{-1} = \mathrm{RIS}_{-1} = \emptyset$ and s is the number of repeating steps. The down-arrow represents that the computation is continuous in this step, and the up-arrow shows the computation progressing into the next loop step.

3. Dchar-Set Algorithm for the Symmetry Computation

In this section, we first give a differential algebraic modification of the Lie criterion (Theorem 1) for the symmetry of PDEs, and then we give our algorithms based on the criterion.

3.1. A Differential Algebra Version of the Lie Criterion

The following theorem is a key step toward using Wu's method to compute the symmetry of PDEs.

Theorem 3. *Let the left side of Equation (1), $\mathcal{R} = \Delta$, be an irreducible d-chain as a DPS and $\mathcal{X}$ be as is given in Equation (3); then*

$$(a)\ \mathrm{Pr}\mathcal{X}(\mathcal{R})|_{\mathcal{R}=0} = 0 \Longrightarrow (b)\ \mathrm{Prem}(\mathrm{Pr}\mathcal{X}(\mathcal{R})/\mathcal{R}) = 0. \tag{11}$$

Additionally, if $\mathrm{IS}(\mathcal{R}) \neq 0$*, then conclusions* (a) *and* (b) *are equivalent to each other.*

Proof. According to the reduction Equation (8), we have

$$\mathrm{IS} \cdot \mathrm{Pr}\mathcal{X}(\mathcal{R}) = \sum_{\mathrm{dqs} \in \mathcal{R}} Q_\alpha \cdot D^\alpha \mathrm{dqs} + \mathrm{Prem}(\mathrm{Pr}\mathcal{X}(\mathcal{R})/\mathcal{R}). \tag{12}$$

This implies $\mathrm{Prem}(\mathrm{Pr}\mathcal{X}(\mathcal{R})/\mathcal{R})\big|_{Z(\mathcal{R})} = 0$ under (a), because $\mathrm{Prem}(\mathrm{Pr}\mathcal{X}(\mathcal{R})/\mathcal{R})$ is reduced w.r.t. the irreducible d-chain $\mathcal{R}$, which leads to (b). Conclusion $(b) \Longrightarrow (a)$ is clearly true from Equation (12) under $\mathrm{IS}(\mathcal{R}) \neq 0$. □

Remark 1. *Theorem 3 indicates someimportant phenomena. Conclusion (a) is the Lie criterion for the classical symmetry of PDEs $\mathcal{R} = 0$; conclusion (b) is the differential algebraic version of the criterion, which is a necessary condition of $\mathcal{X}$ being the InfV of the classical symmetry of the PDEs. Furthermore, we see that if $IS(\mathcal{R}) \neq 0$ (in most application cases, this condition is trivial), then Equation (4) is equivalently replaced by the reduction identity (b) in Equation (11), which is completely constructive. Hence, we obtain DTEs from setting the coefficients of independent monomials in the identity (b) to zero. In this way, we do not need the considered PDEs to admit the solved-and-triangular structure as for the Lie algorithm in order to obtain the DTEs (see question 1).*

Remark 2. *The irreducible restriction of PDEs is not the case in most applications.*

Remark 3. *If we take $\mathcal{R} = \{\Delta, \mathcal{S}\}$, where $\mathcal{S}$ is given in Equation (6), the above theorem is true for the nonclassical symmetry of PDEs. This gives a new proof of the conclusions for computing the nonclassical symmetry of differential equations given in [32].*

3.2. Algorithms

Using Theorems 2–3, we can give algorithms for computing symmetries based on Wu's method. These algorithms are Algorithm 1 and the following Algorithms 2 and 3.

Algorithm 2: Producing determining equations (DTEs)for the classical symmetry of (partial) differential equations (PDEs) $\mathcal{R} = 0$

Input: A differential polynomial system (DPS) $\mathcal{R}$ with a given differential monomial order.
Output: A sequence of DTEs $\mathrm{DTEs_i}$ for symmetries of $\mathcal{R} = 0$.
Begin:
  Step 1: Compute the irreducible dtri-set $\mathrm{ICS_j}$ of $\mathcal{R}$ and obtain $\mathrm{IS_j}$ such that
    $\mathrm{Z}(\mathcal{R}) = \cup_\mathrm{j} \mathrm{Z}(\mathrm{ICS_j}/\mathrm{IS_j})$ (by Algorithm 1 (Theorem 2)).
  Step 2: Repeat for j:
      Apply Theorem 3 to $\mathrm{ICS_j}$ and let
      $DTEs_j = \{$the coefficients of ∂U in $\mathrm{Prem}(\mathrm{Pr}\mathcal{X}(\mathrm{ICS_j})/\mathrm{ICS_j})\}$.
    End of repeat for j.
  Step 3: Return $\mathrm{DTEs_j}$.
End

In general, it is hard to directly obtain closed-form solutions of the DTEs obtained by the above algorithm, even in the classical case [23]. However, we can reduce the DTEs to a well-ordering form of the system in terms of the dchar-set, which helps us to determine the symmetry. In the following, we give an algorithm to transform DTEs to a well-ordering system according to the zero decomposition results (Equations (9) and (10)) in Theorem 2.

To end this, we let $U = \{\xi_j, \phi_k | 1 \leq j \leq p, 1 \leq k \leq q\}$. Hence, the corresponding DPS of the DTEs PS = 0 for symmetries of PDEs with InfV given by Equation (3) is in $\mathcal{K}_X[\partial U]$.

Algorithm 3: Reducingdetermining equations (DTEs): PS $= 0$

Input: PS with a given differential monomial order.
Output: Sequences of dchar-sets CS_i or irreducible dchar-set ICS_j of PS.
Begin
Step 1: Compute decomposition (by Algorithm 1):
$\mathrm{Z(PS)} = \cup_i \mathrm{Z}(\mathrm{CS}_i/\mathrm{IS}_i) = \cup_j \mathrm{Z}(\mathrm{ICS}_j/\mathrm{IS}_j)$.
Step 2: Return CS_i or ICS_j.
End

Subsequently solving the systems $\mathrm{CS}_i = 0$ or $\mathrm{ICS}_j = 0$ (well-ordering equations) in closed form, we obtain Z(PS) and go on to calculate the InfV (see question 2).

Remark 4. *In Algorithm 2, if the given PDE is already a dtri-set and in solved form, then step 1 is not needed. Hence the algorithm reduces to the usual Lie algorithm. In Algorithm 3, if PS is a linear system, then it has only one dchar-set because of its irreducibility. Therefore, in this case, the algorithm reduces to Algorithm 1. These particular cases often occur in the classical symmetry case of scale equations or some systems.*

4. Applications

In the following, we give illustrative examples to show the efficiency of the given algorithms in determining symmetries of PDEs.

4.1. Computing Classical Symmetry

As a first application of the given algorithm, we compute the symmetry of the system

$$v_t = \frac{1}{u^2} u_x + \frac{1}{u}, v_x = u_t. \tag{13}$$

The system is already in irreducible d-chain form under rank $x \prec t \prec u \prec v$ (see Remark 4). Hence, for the InfV $X = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \phi(x,t,u,v)\partial_v$, we use Algorithm 2 and obtain DTEs PS = $\{p_1, p_2, \cdots, p_8\} = 0$. Here

$$\begin{aligned}
p_1 &= \xi_v - \tau_u, \\
p_2 &= \eta_u - \phi_v + \xi_x - \tau_t, \\
p_3 &= \eta_v + u(\eta_t - \phi_x) + \tau_x, \\
p_4 &= u^2 \xi_u - \tau_v, \\
p_5 &= u^2 \phi_u - u\tau_u - \eta_v, \\
p_6 &= u\xi_v + u^2 \xi_t - \tau_x, \\
p_7 &= u(\eta_u - \phi_v - \xi_x + \tau_t) + 2(\tau_v - \eta), \\
p_8 &= u(\phi_v - \tau_t) - (\tau_v + \eta_x - \eta) + u^2 \phi_t,
\end{aligned}$$

which belongs to $\mathcal{K}_X[\partial U]$ with $\mathbf{X} = (x,t,u)$ and $U = (\xi, \phi, \eta, \tau)$. Because PS is a linear system, we have just one dchar-set for the DPS (see Remark 4). Hence, we use Algorithm 3 to calculate its dchar-set. We take the basic rank as $x \prec t \prec u \prec v \prec \xi \prec \phi \prec \eta \prec \tau$. In order for the reader to better understand, here we give some details on the execution of Algorithm 3 (see the scheme of Algorithm 1 and (W)).

Step 1. Select the base $BS_0(i=0)$ of $PS_0 = PS$ by Definition 3. It is easy to determine that $BS_0 = \{p_4, p_5, p_2, p_6\}$, that is,

$$BS_0 = \left\{u^2\xi_u - \tau_v, u^2\phi_u - u\tau_u - \eta_v, \eta_u - \tau_t - \phi_v + \xi_x, u^2\xi_t + u\xi_v - \tau_x\right\}.$$

Step 2. Calculate the nonzero remainders of $PS_0 \backslash BS_0 = \{p_1, p_3, p_7, p_8\}$ w.r.t. BS_0. This means we eliminate the leading terms of p_1, p_3, p_7, and p_8 by using the d-chain BS_0. We should have a total of four remainders. Here, as an example, we give the calculation of the remainder of p_3 w.r.t. BS_0. The leading term of p_3 is τ_x, and $p_6 \in BS_0$ admits this term as the leading term. Thus, we use p_6 to eliminate the term τ_x from p_3 and obtain the remainder as follows:

$$Remd(p_3/p_6) = u\left(\eta_t + u\xi_t + \xi_v - \phi_x\right) + \eta_v.$$

Clearly, the leading term of the remainder is η_v, but no polynomial in BS_0 admits the term as the leading term. Hence, no further reduction occurs for p_3 by BS_0. Hence,

$$Remd(p_3/BS_0) = u\left(\eta_t + u\xi_t + \xi_v - \phi_x\right) + \eta_v.$$

In the same manner, we calculate the rest of the remainders and obtain all nonzero remainders as

$$\begin{aligned} R_0 = \{&u\left(\xi_v - u\phi_u\right) + \eta_v, \eta - u\left(\eta_u + u\xi_u - \phi_v\right), \\ &u\left(\eta_t + u\xi_t + \xi_v - \phi_x\right) + \eta_v, \eta + u\left(u\phi_t - \eta_u - u\xi_u + 2\phi_v - \xi_x\right) - \eta_x\}. \end{aligned}$$

Step 3. Calculate the nonzero integrability polynomials in BS_0 and their remainders w.r.t. BS_0. The leading terms of d-pols in BS_0 are τ_x, τ_t, τ_u, and τ_v, respectively. Therefore, we have integrability polynomials from the six equalities of the cross-differentiations: $\tau_{xt} = \tau_{tx}, \tau_{xu} = \tau_{ux}, \tau_{xv} = \tau_{vx}, \tau_{tu} = \tau_{ut}$, $\tau_{tv} = \tau_{vt}$, and $\tau_{uv} = \tau_{vu}$. After obtaining the integrability polynomials, we reduce them w.r.t. BS_0, as in step 2. Here, we take $\tau_{uv} = \tau_{vu}$ as an example to show the calculation. This integrability polynomial can only comes from the first two terms p_4 and p_5 in BS_0 because they admit τ_v and τ_u as leading terms. This is equivalent to eliminating the term τ_{uv} or τ_{vu} by p_4 and p_5. Thus, we calculate $\partial_v(p_5) - u\partial_u(p_4) = u(u\phi - 2\xi_u - u^2\xi_{uu}) - \eta_{vv}$. The obtained polynomial cannot be reduced further by BS_0 because no polynomial in BS_0 can do so. In the same manner, we obtain the rest of the integrability polynomials and their reductions. As a result, we have all nonzero integrability polynomials:

$$\begin{aligned} IT_0 = \{&u^2\left(u\xi_{uu} - \phi_{uv} + 2\xi_u\right) + \eta_{vv}, u\left(u\left(u\xi_{tu} + \xi_{uv} + 2\xi_t\right) - u\phi_{xu} + \xi_v\right) + \eta_{xv}, \\ &u\left(\xi_{tv} - \xi_{xu}\right) + \xi_{vv}, u\left(\eta_{uu} - u\phi_{tu} - \phi_{uv} + \xi_{xu}\right) + \eta_{tv}, \\ &u^2\xi_{tu} - \eta_{uv} - \xi_{xv} + \phi_{vv}, u^2\xi_{tt} + u\xi_{tv} - \eta_{xu} + \phi_{xv} - \xi_{xx}\}. \end{aligned}$$

Step 4. Set $RIS_0 = R_0 \cup IT_0$ and $PS_1 = PS_0 \cup BS_0 \cup RIS_0$. Because of $RIS_0 \neq \emptyset$, we refer back to step 1 and repeat the above procedure for PS_1, which consists of 18 d-pols.

After finishing the above four step operations, every time, we obtain a new system PS_i, BS_i, RIS_i and then check if $RIS_i = \emptyset$. If it is true, then we stop the calculation and obtain the dchar-set $CS = BS_i$. Otherwise, we repeat the above procedure again for PS_i. Wu's theory guarantees that after a finite number of repetitions, for example, s, of the above procedure, we reach $RIS_s = \emptyset$ [24,25].

In fact, in the example with $s = 9$, we have the dchar-set $CS = BS_9$ of the PS as follows:

$$\text{CS} = \left\{ \begin{array}{lll} \xi_{tv}, \xi_{tt}, \xi_{xt}, & \phi_v, \phi_t, \phi_u + 2\xi_t, & \\ \xi_t + u\xi_{tu}, & \phi_x + 2u\xi_t; & \tau_x - u\xi_v - u^2\xi_t, \\ \xi_v + u\xi_{uv} + u\xi_t - \xi_{xv}, & & \tau_t + u\xi_u - \xi_x - u^{-1}\eta, \\ \xi_{vv} + \xi_x - \xi_{xx}, & \eta_x + u\xi_x, \eta_t + u\xi_t, & \tau_u - \xi_v, \tau_v - u^2\xi_u, \\ \xi_x + u^2\xi_{uu} + 2u\xi_u - \xi_{xx}, & u\eta_u - \phi + u^2\xi_u, & \\ \xi_x + u\xi_{xu} - \xi_{xx}; & \eta_v + u\xi_v + 2u^2\xi_t; & \end{array} \right\}$$

with $\text{IS} = u \not\equiv 0$. Hence, one has

$$\text{Z(PS)} = \text{Z(CS)}$$

by the well-ordering principle in Theorem 2. This shows that solving PS = 0 and CS = 0 is equivalent. The triangular structure (well-ordering) of the dchar-set CS makes the determination of Z(PS) easier by solving Z(CS). Hence, we have infinitesimal functions:

$$\text{Z(PS)} = \left\{ \begin{array}{l} \xi = c_1 t/u + c_2 - c_1 v + e^x A(v, e^x u), \tau = (c_3 - c_1 v)t - c_1 u + c_4 + B(v, ue^x), \\ \eta = c_3 u - c_1(t + uv) - ue^x A(v, e^x u), \phi = -2c_1(\ln u + x), \end{array} \right\}$$

with $A_V - B_U = 0, B_V - U^2 A_U = 0, V = v$, and $U = ue^x$. Consequently, Equation (13) admits four parameters with finite symmetries and an infinite-dimensional symmetry.

4.2. Computing Symmetry of Non-Solved-Form Equation

As second application, we look for symmetry of the PDEs $\mathcal{R} = \{H\} = 0$; here

$$H = u_x^2 + u_x u_y + u_y^2 - u. \tag{14}$$

We observe that this equation does not satisfy the requirement for the implementation of the Lie algorithm as the back-substitutive term u_x or u_y cannot be expressed in polynomial form. However, in our algorithm, this is not the case. For the InfV $\mathcal{X} = \xi(x, y, u)\partial_x + \tau(x, y, u)\partial_y + \eta(x, y, u)\partial_u$, under the basic rank $x \prec y \prec u$, we execute Algorithm 2 and obtain the reduction (b) in Equation (11) as follows:

$$\text{Prem}(\text{Pr}\mathcal{X}(H)/H) = 2h_1 u_x^3 + 2h_1 u_x^2 u_y + h_2 u_x u_y + h_3 u_x^2 + h_4 u_y + h_5 u_x + h_6, \tag{15}$$

with

$$\begin{array}{ll} h_1 = \tau_u, & h_4 = 2\eta_y + \eta_x - 2u\tau_u, \\ h_2 = \tau_y - \tau_x - 2\xi_y - \xi_x, & h_5 = \eta_u + 2\eta_x - 2u\xi_u - 2u\tau_u, \\ h_3 = 2\tau_y + \tau_x - \xi_y - 2\xi_x, & h_6 = 2u\eta_u - \eta - 2u\tau_y - u\tau_x. \end{array}$$

In fact, Equation (15) is obtained by reducing the terms $u_y^\alpha (\alpha \geq 2)$ that appear in $Pr\mathcal{X}(H)$ by the leading term u_y^2 of H. For example, we have a term containing $u_y^3 = u_y u_y^2$ in $Pr\mathcal{X}(H)$; then we eliminate u_y^2 by $u_y^2 = -u_x^2 - u_x u_y + u$ from $H = 0$. Hence, $u_y^3 = u_y(-u_x^2 - u_x u_y + u) = -u_y u_x^2 - u_x u_y^2 + u_y u = -u_y u_x^2 - u_x(-u_x^2 - u_x u_y + u) + u_y u = -u_x^3 - u_x u + u_y u$. This is reduced w.r.t. H (Definition 1). In the same manner, we reduce all such terms in $PrX(H)$ by H, and, finally, we obtain Equation (15).

Hence, by Theorem 3, the DTEs of the classical symmetry of Equation (14) are $\text{PS} = \{h_i, i = 1, 2, \cdots, 6\} = 0$. Moreover, by Algorithm 3, we have the dchar-set of the PS under the order $x \prec y \prec u \prec \xi \prec \tau \prec \eta$:

$$\text{CS} = \left\{ \begin{array}{l} \xi_{xu},\ \xi_{yu},\ \xi_{xy},\ 2\xi_u + 4u\xi_{uu} - 3\xi_{xx},\ \xi_{xx} + \xi_{yy},\ \xi_{xxx}, \\ \tau_u,\ \tau_y - \xi_x - \xi_y,\ \tau_x + \xi_y;\ 3\eta_x - 4u\xi_u,\ 3\eta_y + 2u\xi_u,\ \eta - 2u\eta_u + u\xi_y + 2u\xi_x, \end{array} \right\}$$

and the zero decomposition:

$$\mathrm{Z(PS)} = \mathrm{Z(CS)}.$$

It is easy to solve for

$$\mathrm{Z(CS)} = \left\{ \begin{array}{l} \xi = c_1(x^2 - y^2 + 3u) + c_2 x + c_3 y + 2c_4\sqrt{u} + c_5, \\ \tau = c_1\left(2xy - y^2\right) + c_2 y + c_3(y - x) + c_6, \\ \eta = 2c_1(2x - y)u + 2c_2 u + c_3 u + \frac{2}{3}c_4(2x - y)\sqrt{u} + c_7\sqrt{u}. \end{array} \right\}$$

This shows that Equation (14) admits seven finite-parameter Lie symmetries.

4.3. Computing Nonclassical Symmetry

As a third application, to illustrate the adaptability of our algorithms to the nonlinear case, we find the nonclassical symmetry of the Boussinesq equation: $\Delta = u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = 0$. Although this was done in [33], here we give the results again through the use of our algorithms, which provide a different way to solve the same problem. We let $\mathcal{X} = \partial_t + \xi(x,t,u)\partial_x + \eta(x,t,u)\partial_u$ be the InfV for nonclassical symmetries of the equation. The augmented system $\mathcal{R} = \{\Delta = u_{tt} + uu_{xx} + u_x^2 + u_{xxxx}$, $\mathcal{S} = u_t - \eta(x,t,u) + \xi(x,t,u)u_x\}$ is obtained by combining the original equation $\Delta = 0$ with the invariant surface condition $\mathcal{S}$. First, by eliminating the term u_{tt} from the first d-pol in $\mathcal{R}$ using the invariant surface condition $\mathcal{S}$, we reduce $\mathcal{R}$ into the irreducible d-chain form. Then we use Algorithm 2 on the reduced system, and we have DTEs PS = 0 for $\mathcal{X}$. Here

$$\begin{aligned} \mathrm{PS} \quad = \quad & \Big\{\xi_u, \eta_{uu}, \eta_u + 2\xi_x, 4\eta_{xu} - 6\xi_{xx}, 6\eta_{xxu} - 4\xi_{xxx} + \eta + 2\xi\xi_t + 2u\xi_x + 4\xi^2\xi_x, \\ & \eta_{tt} + 2\eta\left(\eta_{tu} + 2\eta_u\xi_x\right) + u\eta_{xx} + \eta_{xxxx} - 2\xi_t\eta_x + 4\eta_t\xi_x - 4\xi\eta_x\xi_x, \\ & 4\eta_{xxxu} - \xi_{tt} - 2\xi\eta_{tu} + 2u\eta_{xu} - u\xi_{xx} - \xi_{xxxx} - 2\xi_t\eta_u - 2\xi_t\xi_x - 8\xi\eta_u\xi_x + 2\eta_x + 4\xi\xi_x^2.\Big\} \end{aligned}$$

Executing Algorithm 3 on the PS with rank $x \prec t \prec u \prec \xi \prec \eta$, and supposing $\xi \neq 0$ (excluding the trivial case $\xi = 0$), we obtain the dchar-set:

$$\begin{aligned} \mathrm{CS} \quad = \quad & \Big\{\xi_u, \xi_{xx}, \eta + 2\xi\xi_t + 2u\xi_x + 4\xi^2\xi_x, \eta_u + 2\xi_x, \xi^2\eta_{xx} + 2\xi_x\left(\xi_x\left(\eta + 2u\xi_x\right) - \xi\eta_x\right), \\ & \eta^2 + 2\xi^2\eta_t + 8u^2\xi_x^2 + 2\eta\xi_x\left(4\xi^2 + 3u\right) - 2\xi\eta_x\left(2\xi^2 + u\right) + 8u\xi^2\xi_x^2\Big\}. \end{aligned}$$

Because IS $\neq$ 0, we have Z(PS) = Z(CS). Consequently, from the first three equations of CS, we easily obtain the solutions

$$\xi = f(t)x + g(t), \;\; \eta = -2(\xi\xi_t + u\xi_x + 2\xi^2\xi_x)$$

to the nonlinear system $CS = 0$, and the rest of the equations give the conditions

$$g''(t) + 2f(t)g'(t) - 4f(t)^2 g(t) = 0, f''(t) + 2f(t)f'(t) - 4f(t)^3 = 0$$

for arbitrary functions f and g. The above procedure of solving CS = 0 is more automatic than that of directly solving the original system PS = 0. Hence, the results given in [33] are obtained in a different and easy way.

4.4. Computing Symmetry Classification

Another application of our proposed algorithm is to find the potential symmetry classification of the nonlinear equation $u_{tt} + \mu u^{\mu+1} u_t = (f(u)u_x)_x$ through determining the classical symmetry classification of its potential system:

$$v_x = u + u^{\mu},\ v_t = F(u)u_x. \tag{16}$$

In the present case, we have two parameters: $\theta = \{\mu, F\}$, a function F and a constant μ, in the system given by Equation (16). We find all the parameters such that the system given by Equation (16) admits classical symmetry with the InfV $\mathcal{X} = \xi(x,t,u,v)\partial_x + \tau(x,t,u,v)\partial_t + \eta(x,t,u,v)\partial_u + \phi(x,t,u,v)\partial_v$.

Step 1. Producing DTEs. The system given by Equation (16) is already in irreducible chain form under basic rank $x \prec t \prec u \prec v$. Hence, using Algorithm 2, we obtain the DTEs PS = 0 for the classical symmetries of the potential system given by Equation (16). Here,

$$\mathrm{PS} = \left\{ \begin{array}{l} \xi_v - \tau_u, \phi_v - \eta_u - 2u^{\mu}\xi_v - \xi_x + \tau_t, F(u)(u^{\mu}\tau_v + \tau_x - \eta_v) + \phi_u - u^{\mu}\xi_u - \xi_t, \\ \xi_u - F(u)\tau_v, u^{\mu}(F(u)\eta_v + \xi_t) + F(u)\eta_x - \phi_t, F(u)(\phi_v - \eta_u + \xi_x - \tau_t) - F'(u)\eta, \\ u^{\mu}\phi_v + \phi_x - \eta_t - u^{\mu-1}(u^{\mu+1}\xi_v + u\xi_x + \mu\eta), \phi_u - u^{\mu}\xi_u + \xi_t - F(u)(\eta_v + u^{\mu}\tau_v + \tau_x). \end{array} \right\}$$

Because this is a variable-coefficient PDE, it is difficult to obtain all solutions by directly solving.

Step 2. Computing the kernel algebra $\mathcal{X}_{\prime}$. Clearly, for arbitrary values of these parameters, the kernel algebra belongs to

$$\mathrm{CS}_0 = \{\xi_x, \xi_t, \xi_u, \xi_v, \tau_x, \tau_t, \tau_u, \tau_v, \phi_x, \phi_t, \phi_u, \phi_v, \eta\} = 0.$$

As a result, we have

$$\mathrm{Z}(\mathrm{CS}_0) = \{\xi = c_1, \tau = c_2, \phi = c_3, \eta = 0\}.$$

Step 3. Computing the extended algebra $\mathcal{X}_{\theta}$. With the rank $\eta \prec \xi \prec \phi \prec \tau$, we execute Algorithm 3 on the PS and firstly obtain the decomposition

$$\mathrm{Z(PS)} = \mathrm{Z}(\mathrm{CS}_1, \mathrm{I}_0, \mathrm{I}_1/I_3) \cup \mathrm{Z}(\mathrm{PS}, \mathrm{I}_0, \mathrm{I}_3/\mathrm{I}_2) \cup \mathrm{Z}(\mathrm{PS}, \mathrm{I}_0, \mathrm{I}_2) \cup \mathrm{Z}(\mathrm{PS}, \mathrm{I}_4/\mathrm{I}_0), \tag{17}$$

in which

$$\mathrm{CS}_1 = \left\{ \begin{array}{l} \eta_v, \eta_t, \eta_x, \phi_u, \phi_t, \xi_u, \xi_t, \tau_v, \tau_t, \tau_x, \\ \mathrm{I}_2 F(u)\eta_u + (3F'(u)^3 - 4F(u)F'(u)F''(u) + F(u)^2 F^{(3)}(u))\eta, \\ 2\mathrm{I}_2 F(u)\xi_v + (F'(u)^2 F''(u) - 2F(u)F''(u)^2 + F(u)F'(u)F^{(3)}(u))\eta, \\ 2\mathrm{I}_2 F(u)\xi_u + \eta(F'(u)^2 F''(u) - 2F(u)F''(u)^2 + F(u)F'(u)F^{(3)}(u)), \\ 2\mathrm{I}_2 F(u)\xi_x + [F'(u)(3F'(u)^2 - 2F(u)F''(u) - uF'(u)F''(u) - uF(u)F^{(3)}(u)) \\ \qquad +2uF(u)F''(u)^2]\eta, \\ 2\mathrm{I}_2 F(u)\phi_v + \eta(9F'(u)^3 - 10F(u)F'(u)F''(u) + uF'(u)^2 F''(u) - 2u\,F(u)F''(u)^2 \\ \qquad +2F(u)^2 F^{(3)}(u) + uF(u)F'(u)F^{(3)}(u)), \\ 2\mathrm{I}_2 F(u)\phi_x + \eta(6F(u)F'(u)^2 - 6uF'(u)^3 - 4F(u)^2 F''(u) + 8uF(u)F'(u)F''(u) \\ \quad -u^2 F'(u)^2 F''(u) + 2u^2 F(u)F''(u)^2 - 2uF(u)^2 F^{(3)}(u) - u^2\,F(u)F'(u)F^{(3)}(u)), \end{array} \right\}$$

and

$$
\begin{aligned}
&\mathrm{I}_0 = \mu - 1;\\
&\mathrm{I}_1 = \mathrm{I}_2 F(u)F^{(4)}(u) - 3F(u)^2F^{(3)}(u)^2 + 2F'(u)(8F(u)F''(u) - 3F'(u)^2)F^{(3)}(u)\\
&\qquad\qquad + 6F''(u)^2(F'(u)^2 - 2F(u)F''(u)),\\
&\mathrm{I}_2 = 2F(u)F''(u) - 3F'(u)^2,\\
&\mathrm{I}_3 = F'(u)^2\, F''(u) - 2\, F(u)\, F''(u)^2 + F(u)\, F'(u)\, F^{(3)}(u),\\
&\mathrm{I}_4 = -uF'(u)^2 + F(u)F'(u) + uF(u)F''(u).
\end{aligned}
$$

At this point, we first solve these classifying equations $\mathrm{I}_i = 0, i = 1,2,3,4$ and choose representatives of their solutions (specializations) under the equivalence transformations admitted by the system given by Equation (16) to simplify the subsequent computations.

For the case $\mathrm{I}_0 = 0$, the system given by Equation (16) admits the linear equivalence transformations

$$x' = ax + b, t' = t + d, u' = lu + m, v' = alv + amx + p, F' = F/a^2, \tag{18}$$

where $a, b, d, l, m,$ and p are arbitrary constants with $al \neq 0$. Solving the equations $\mathrm{I}_1 = 0$, $\mathrm{I}_2 = 0$, and $\mathrm{I}_3 = 0$, we obtain the specializations of the parameters μ and F. Then we have their solutions under Equation (18) as $F(u) = (u^2+1)^{-1}e^{\alpha \arctan u}$ and $F(u) = (1-u^2)^{-1}e^{\alpha \mathrm{arctanh} u}$ for any real number α, as $F(u) = u^{-2}e^{\frac{1}{u}}$ for $\mathrm{I}_1 = 0$, as $F(u) = u^{-2}$ for $\mathrm{I}_2 = 0$, and as $F(u) = u^{\alpha}$ and $F(u) = e^{u}$ for $\mathrm{I}_3 = 0$, where α is an arbitrary constant.

For the case $\mathrm{I}_0 \neq 0$, the solution of $\mathrm{I}_4 = 0$ is $F(u) = \varrho u^{\sigma}$ for arbitrary constants ϱ and σ.

Therefore, applying Algorithm 3 again on the non-dchar-set parts in Equation (17), we obtain the final zero decomposition as follows:

$$\mathrm{Z(PS)} = \bigcup_{i=1}^{6} \mathrm{Z(CS}_i), \tag{19}$$

under the equivalence transformations of Equation (18), where

$$
\begin{aligned}
&\mathrm{CS}_2 = \{\ \eta_x, \eta_t, \eta_u, \eta_v, \tau_x, \tau_t, \tau_u, \tau_v, \xi_t, \xi_u, \xi_v, \phi_x - \eta, 2\xi_x - \eta, 2\phi_v - \eta,\ \},\\
&\mathrm{CS}_3 = \{\tau_x, \tau_t, \tau_u, \tau_v, \eta_x, \eta_t, \eta_v, \phi_x, \phi_t, \phi_u, u\eta_u - \eta, 2u\phi_v - (\alpha+2)\eta, 2u\xi_x - \alpha\eta\},\\
&\mathrm{CS}_4 = \{\tau_x, \tau_t, \tau_u, \tau_v, \xi_t, \xi_u, \xi_v, \eta_x, \eta_t, \eta_v, \phi_x, \phi_t, \phi_u, u\eta_u - \eta, 3u\phi_v - \eta, 3u\xi_x + 2\eta\},\\
&\mathrm{CS}_5 = \left\{\begin{array}{l}
\phi_x, \phi_v, \xi_{xu}, \xi_{xt}, \xi_{xx}, \eta - u\eta_u - u^2\xi_v,\\
\phi_u - 2u\xi_u + 2\xi_t, \phi_t - 2u^2\xi_u + 2u\xi_t, \eta_v - u^3\xi_u + 2u^2\xi_t,\\
\eta_x - u^4\xi_u + u^3\xi_t, u\xi_{uv} - \xi_{tv}, \xi_{vv} - \xi_t - \xi_{tt},\\
2u\xi_u + u^2\xi_{uu} - \xi_t - \xi_{tt}, u\xi_{tu} - \xi_{tt},\\
\eta + \eta_t + u^2\xi_v + u\xi_x, u^2\xi_u - u\xi_t + \xi_{xv}, u^2\xi_u - \tau_v, \xi_v - \tau_u,\\
\eta + u^2\xi_v + u\xi_x - u\tau_t, u^3\xi_u - u^2\xi_t + \tau_x.
\end{array}\right\},\\
&\mathrm{CS}_6 = \left\{\begin{array}{l}
\eta_x,\ \eta_t,\ \eta_v,\ \eta - u\eta_u,\ \xi_t, \xi_u, \xi_v,\ (2(\mu-1)F(u) - uF')\eta + 2uF\xi_x,\\
\tau_x,\ \tau_u,\ \tau_v,\ (\mu-1)\eta + u\tau_t,\ \phi_x,\ \phi_t,\ \phi_u,\ (2F + uF')\eta - 2uF\phi_v.
\end{array}\right\}.
\end{aligned}
$$

Solving the equations of $\mathrm{CS}_i = 0, i = 1,2,\cdots,6$, we have

$$\mathrm{Z(CS}_1) = \left\{\begin{array}{ll}
\xi = c_2x + c_1v + c_5, \tau = c_1u + c_6, & \text{with } c_1 \neq 0\\
\eta = 2(c_1u + c_2)F(u)/F'(u), &
\end{array}\right\}, \tag{20}$$

with the corresponding functions $F(u) = (u^2+1)^{-1}e^{\alpha \arctan u}$, $F(u) = (1-u^2)^{-1}e^{\alpha \mathrm{arctanh} u}$, and $F(u) = u^{-2}e^{\frac{1}{u}}$;

$$Z(\mathrm{CS}_2) = \{\xi = c_1 x + c_3, \tau = c_4, \eta = 2c_1, \phi = c_1 v + 2c_1 x + c_2\},$$

with the corresponding function $F(u) = e^u$;

$$Z(\mathrm{CS}_3) = \left\{\xi = \frac{\alpha}{2}c_2 x + c_4, \tau = c_1, \eta = c_2 u, \phi = \frac{2+\alpha}{2}c_2 v + c_3\right\},$$

with the corresponding function $F(u) = u^\alpha$, $\alpha \neq -2, -4/3$;

$$Z(\mathrm{CS}_4) = \left\{\xi = -\frac{2}{3}c_2 x + c_4, \tau = c_1, \eta = c_2 u, \phi = \frac{1}{3}c_2 v + c_3\right\},$$

with the corresponding function $F(u) = u^{-4/3}$;

$$Z(\mathrm{CS}_5) = \left\{\begin{array}{l} \xi = \frac{c_1}{u} + (c_2 + c_1 v)x + B\left(v, e^t u\right), \tau = c_4 + c_1(ux - v) + e^{-t}A\left(v, e^t u\right), \\ \eta = -\left(c_2 + c_1(v + ux) + e^{-t}A\left(v, e^t u\right)\right)u, \phi = c_3 - 2c_1(t + \ln u). \end{array}\right\},$$

with the corresponding function $F(u) = u^{-2}$ and with $A_V(V,U) = U^2 B_U(V,U), B_V(V,U) = A_U(V,U), U = e^t u$, and $V = v$; and

$$Z(\mathrm{CS}_6) = \left\{\xi = \frac{\sigma + 2(1-\mu)}{2}c_1 x + c_2, \tau = (1-\mu)c_1 t + c_3, \eta = c_1 u, \phi = \frac{2+\sigma}{2}c_1 v + c_4\right\},$$

with the corresponding function $F(u) = \varrho u^\sigma$.

The cases $Z(\mathrm{CS}_i), i = 1, 2, \cdots, 6$ and the corresponding parameter functions F yield a complete symmetry classification of Equation (16), in which CS_1 and CS_5 extend the classical symmetries of the equations with the corresponding functions $F(u) = (u^2+1)^{-1}e^{\alpha \arctan u}$, $F(u) = (1-u^2)^{-1}e^{\alpha \mathrm{arctanh} u}$, $F(u) = u^{-2}e^{\frac{1}{u}}$, and $F = u^{-2}$. CS_5 corresponds to a linearizable case of the equation.

As comparison, more examples for symmetry classifications are found in [34,35].

5. Concluding Remarks

In this paper, we give an alternative method for computing symmetries of given PDEs by applying Wu's method. Particularly, we first give a differential algebra version of the Lie criterion of the classical symmetry of PDEs to produce DTEs (Theorem 3). The criterion is an efficient way to break the solved-and-triangular structural restrictions forced on considered PDEs as an essential assumption in the Lie algorithm. Then, we regard the DTEs of the symmetries as a DPS and deal with its zero point set (solution set of the DTEs) by using the zero decomposition result of Wu's method (Theorem 2). On the basis of the decomposition of the solution set of the DTEs in terms of a series of zero sets of dchar-sets (Equation (10)) and the new version of the Lie criterion, we design the given algorithms for computing the symmetry of PDEs. Because of the well-ordering property of the dchar-sets, the decomposition makes solving the DTEs easier than directly solving the original DTEs. The algorithms provide a direct and systematic way to solve the DTEs of (nonclassical) symmetries of PDEs. In addition, the given algorithm can be used to solve symmetry classification problems. Consequently, Wu's method provides us an alternative algorithmic theory and algorithm for determining symmetries of PDEs. Hence, it may lead to some explicit applications in physics and engineering.

The innovation of our algorithm is the use of a fundamentally different theory and algorithms. The efficiency of our algorithm shows the prospective applications of Wu's method in symmetry analysis of PDEs.

Undeniably, there may be other ways to achieve a similar algorithm as that given here by some proper substitutions of the theory and the algorithms used in the article. For example, the dchar-set here may be replaced by the Groebner basis [11] or various weak triangular sets, and the reduction algorithm used here in the zero point decomposition takes the place of some different elimination techniques [36].

Author Contributions: T.C. proposed main idea of the article and give theoretical contribution as well as the organization of the whole paper; S.B. make contributions on designing the algorithm, implementing computer program and examples computations.

Funding: This research was funded by the Natural Science Foundation of China under Grant Nos. 1571008 and 11661060.

Acknowledgments: Authors thanks reviewer's valuable comments on the paper and provided references.

Conflicts of Interest: The authorsdeclare no conflict of interest.

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