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The Absolute Ruin Insurance Risk Model with a Threshold Dividend Strategy

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Abstract: The absolute ruin insurance risk model is modified by including some valuable market economic information factors, such as credit interest, debit interest and dividend payments. Such information is especially important for insurance companies to control risks. We further assume that the insurance company is able to finance and continue to operate when its reserve is negative. We investigate the integro-differential equations for some interest actuarial diagnostics. We also provide numerical examples to explain the effects of relevant parameters on actuarial diagnostics.

Keywords: absolute ruin; integro-differential equations; moment generating function; threshold dividend strategy; debit interest

MSC: 39A14; 39A50; 60H30; 91B30

1. Introduction

Consider the classical insurance risk model, the cash flow of company is modeled by the risk reserve process $\{R_t^u; t \geq 0\}$, with

$$R_t^u = u + pt - S(t), \quad t \geq 0. \tag{1}$$

Here, $R_0^u = u \geq 0$ denotes the initial reserve, and $p > 0$ denotes the premium density which is assumed to be constant. $S(t) = \sum_{k=1}^{N_t} Y_k$, representing the aggregate claim process, is a compound Poisson process, given by a Poisson rate $\lambda > 0$, $\{Y_k; k = 1, 2, \ldots\}$ that represents the claim size process and is independent of the Poisson process $\{N_t; t \geq 0\}$ which are i.i.d. random variables with the distribution function $G(y)$ and mean $\mu > 0$. In the present paper, in order to make the risk model closer to the actual operating situation, we added three other properties related to the risk reserve process (1), namely, debit interest, credit interest and dividend payments. In particular, we made a distinction between ruin and absolute ruin. That is, the insurance company can borrow money and continue to operate when the company's reserve is negative.

It should be mentioned that many authors have studied the problems of absolute ruin, for example, Cai [1] studied the Gerber-Shiu function in the case of absolute ruin. Wang and Yin [2] investigated the absolute ruin model with barrier strategy. Wang et al. [3], Yuan et al. [4] and Peng et al. [5] extended the work of Wang and Yin [2], and studied interest income with the barrier strategy. Wang et al. [6] further considered a threshold dividend barrier under the absolute ruin risk model. Li and Lu [7] further explored the case of the Markov-dependent risk model under absolute ruin. The advantage of this Markov-dependent risk model is that different economic conditions can be expressed by different states of Markov chains. Such models can better cope with changes in the economic environment. For more recent studies about absolute ruin problems, see Huu et al. [8], Luo and Taksar [9], Yang et al. [10,11], Yu [12], Cai and Yang [13], Zhu [14,15], Bi and Zhang [16], Liu and Yang [17], Zeng and Li [18], Zeng et al. [19], Peng and Wang [20,21] and Avram et al. [22].

Motivated by the above literature, in our model, we divided the risk reserve process into three cases according to the size of the reserve. When the risk reserve is between zero and a fixed level ($b > 0$), the premium income rate is p_1, at which interest is not earnt and no dividends are paid. When the risk reserve attains the level b, its interest at a credit interest rate of $\gamma > 0$ and moreover dividends are paid to shareholders continuously at a certain rate $\varepsilon(0 < \varepsilon \leq p_1)$. The premium income rate at this time is $p_2 = p_1 - \varepsilon$. When the risk reserve is at a negative value, the company is able to finance at a debit interest rate $\beta > 0$ and carry on their business operations.

By incorporating the above-mentioned three features into the reserve process $\{R_t^u; t \geq 0\}$ of (1), the new resulting risk reserve process $\{R_t^{u,b}; t \geq 0\}$ is given by the following equations

$$dR_t^{u,b} = \begin{cases} (p_2 + \gamma R_t^{u,b})dt - dS(t), & R_t^{u,b} > b, \\ p_1 dt - dS(t), & 0 \leq R_t^{u,b} \leq b, \\ (p_1 + \beta R_t^{u,b})dt - dS(t), & -p_1/R_t^{u,b} \leq 0. \end{cases} \tag{2}$$

Here, $R_0^{u,b} = u$, $S(t) = \sum_{k=1}^{N_t} Y_k$ is defined in model (1).

Let us denote the set $T_b = \inf\{t \geq 0 | R_t^{u,b} \leq -p_1/\beta\}$ by T_b with $T_b = \infty$ if $R_t^{u,b} > -p_1/\beta$ for all $t \geq 0$, and name it the time of absolute ruin. $\alpha(\alpha > 0)$ is defined as the force of interest, and $D(t)$ is the accumulated value of all dividends payable until t time. Then, the present value of all dividends until absolute ruin time is given by

$$D_{u,b} = \int_0^{T_b} e^{-\alpha t} dD(t) = \varepsilon \int_0^{T_b} e^{-\alpha t} I(R_t^{u,b} > b)dt. \tag{3}$$

Here, $I(\cdot)$ denotes the indicator function. It is worth noting that $D_{u,b}$ satisfies $0 < D_{u,b} \leq \varepsilon \int_0^{+\infty} e^{-\alpha t} dt = \varepsilon/\alpha$.

Next, we focused on the following four related actuarial functions of $D_{u,b}$.

The moment generating function of $D_{u,b}$ is

$$Q(u, z; b) = E[e^{zD_{u,b}}], \tag{4}$$

for some values of z where it exists.

The nth moment function of $D_{u,b}$ is

$$W_n(u; b) = E\{[D_{u,b}]^n]\}, \ \ n \in N, \tag{5}$$

with $W_0(u; b) = 1$.

The Laplace transform of absolute ruin time (ρ is a positive constant) is

$$\varphi(u; b) = E[e^{-\rho T_b} I(T_b < \infty) | R_0^{u,b} = u]. \tag{6}$$

The Gerber-Shiu expected discounted penalty function is

$$\Phi(u; b) = E[e^{-\alpha T_b} \omega(R_{T_b-}^{u,b}, |R_{T_b}^{u,b}|) I(T_b < \infty) | R_0^{u,b} = u], \tag{7}$$

where, $R_{T_b-}^{u,b}$ is the instantaneous reserve before absolute ruin time. $|R_{T_b}^{u,b}|$ is the deficit at absolute ruin time. $\omega(x_1, x_2)$ is a measurable function defined on $(-p_1/\beta, +\infty) \times (p_1/\beta, +\infty)$ that can be interpreted as a penalty at the time of absolute ruin.

2. Integro-Differential Equations for $Q(u,z;b)$ and $W_n(u;b)$

In the sections below, we first give a system of partial integro-differential equations satisfied by $Q(u,z;b)$, through which we can further analyze the $W_n(u;b)$. Note that $Q(u,z;b)$ has different expressions according to the different values of u. Hence, we discuss it for three cases by writing $Q(u,z;b) = Q_1(u,z;b)$ for $0 \le u \le b$, $Q(u,z;b) = Q_2(u,z;b)$ for $u > b$, and $Q(u,z;b) = Q_3(u,z;b)$ for $-p_1/\beta < u < 0$. For convenience of the following proof, we set

$$h_1(u,t) = ue^{\beta t} + p_1(e^{\beta t} - 1)/\beta, \quad h_2(u,t) = ue^{\gamma t} + p_2(e^{\gamma t} - 1)/\gamma. \tag{8}$$

Theorem 1. *When $0 \le u \le b$,*

$$\begin{aligned} p_1 \frac{\partial Q_1(u,z;b)}{\partial u} =& \lambda Q_1(u,z;b) + \alpha z \frac{\partial Q_1(u,z;b)}{\partial z} - \lambda \left[\int_0^u Q_1(u-y,z;b) dG(y) \right. \\ &\left. + \int_u^{u+p_1/\beta} Q_3(u-y,z;b) dG(y) + \bar{G}(u + \frac{p_1}{\beta}) \right], \end{aligned} \tag{9}$$

and, when $u > b$,

$$\begin{aligned} (\gamma u + p_2) \frac{\partial Q_2(u,z;b)}{\partial u} =& \lambda Q_2(u,z;b) + \alpha z \frac{\partial Q_2(u,z;b)}{\partial z} - \lambda \left[\int_0^{u-b} Q_2(u-y,z;b) dG(y) \right. \\ & + \int_{u-b}^u Q_1(u-y,z;b) dG(y) + \int_u^{u+p_1/\beta} Q_3(u-y,z;b) dG(y), \\ & \left. + \bar{G}(u + \frac{p_1}{\beta}) \right] \end{aligned} \tag{10}$$

and, when $-p_1/\beta < u < 0$,

$$\begin{aligned} (\beta u + p_1) \frac{\partial Q_3(u,z;b)}{\partial u} =& \lambda Q_3(u,z;b) + \alpha z \frac{\partial Q_3(u,z;b)}{\partial z} \\ & - \lambda \left[\int_0^{u+p_1/\beta} Q_3(u-y,z;b) dG(y) + \bar{G}(u + \frac{p_1}{\beta}) \right]. \end{aligned} \tag{11}$$

Proof. (1) For $0 \le u \le b$, as discussed in Albrecher et al. [23], and using the strong Markov property of the risk reserve process $\{R_t^{u,b}, t \ge 0\}$, we obtain

$$\begin{aligned} Q_1(u,z;b) =& (1 - \lambda t) Q_1(u + p_1 t, ze^{-\alpha t}; b) \\ & + \lambda t \cdot \left[\int_0^{u+p_1 t} Q_1(u + p_1 t - y, ze^{-\alpha t}; b) dG(y) \right. \\ & \left. + \int_{u+p_1 t}^{u+p_1 t + \frac{p_1}{\beta}} Q_3(u + p_1 t - y, ze^{-\alpha t}; b) dG(y) + \bar{G}(u + p_1 t + \frac{p_1}{\beta}) \right] + o(t), \end{aligned} \tag{12}$$

where, $O(t)$ is the high order infinitesimal of t when $t \to 0$, i.e., $\lim_{t \to 0} \frac{O(t)}{t} = 0$.

By Taylor expansion,

$$Q_1(u + p_1 t, ze^{-\alpha t}; b) = Q_1(u,z;b) + p_1 t \frac{\partial Q_1(u,z;b)}{\partial u} - \alpha z t \frac{\partial Q_1(u,z;b)}{\partial z} + o(t). \tag{13}$$

By plugging (13) into (12), we can obtain (9).

(2) The above method is applied to $Q_2(u,z;b)$ when $u > b$, and we have

$$
\begin{aligned}
Q_2(u,z;b) =&(1-\lambda t)\cdot Q_2(h_2(u,t),ze^{-\alpha t};b)+\lambda t\cdot\left[\int_0^{h_2(u,t)-b} Q_2(h_2(u,t)-y,ze^{-\alpha t};b)dG(y)\right.\\
&+\int_{h_2(u,t)-b}^{h_2(u,t)} Q_1(h_2(u,t)-y,ze^{-\alpha t};b)dG(y)\\
&+\int_{h_2(u,t)}^{h_2(u,t)+\frac{p_1}{\beta}} Q_3(h_2(u,t)-y,ze^{-\alpha t};b)dG(y)\\
&\left.+\bar{G}(h_2(u,t)+\frac{p_1}{\beta})\right]+o(t).
\end{aligned} \tag{14}
$$

By Taylor expansion,

$$
\begin{aligned}
Q_2(h_2(u,t),ze^{-\alpha t};b) =&Q_2(u,z;b)+(\gamma u+p_2)t\frac{\partial Q_2(u,z;b)}{\partial u}\\
&-\alpha z t\frac{\partial Q_2(u,z;b)}{\partial z}+o(t).
\end{aligned} \tag{15}
$$

By plugging (15) into (14), we have (10).

(3) For $-p_1/\beta<u<0$, the same argument as in the proof of (10) gives

$$
\begin{aligned}
Q_3(u,z;b) =&(1-\lambda t)\cdot Q_3(h_1(u,t),ze^{-\alpha t};b)\\
&+\lambda t\cdot\left[\int_0^{h_1(u,t)+\frac{p_1}{\beta}} Q_3(h_1(u,t)-y,ze^{-\alpha t};b)dG(y)\right.\\
&\left.+\bar{G}(h_1(u,t)+\frac{p_1}{\beta})\right]+o(t).
\end{aligned} \tag{16}
$$

By Taylor expansion, we have (11).

□

Theorem 2. *$Q_1(u,z;b)$, $Q_2(u,z;b)$ and $Q_3(u,z;b)$ satisfy*

$$Q_3(-\frac{p_1}{\beta},z;b)=1, \tag{17}$$

$$\frac{\partial Q_1(u,z;b)}{\partial u}|_{u=b}=yQ_1(b,z;b), \tag{18}$$

$$Q_1(b-,z;b)=Q_2(b+,z;b), \tag{19}$$

$$Q_1(0+,z;b)=Q_3(0-,z;b). \tag{20}$$

Proof. (1) For (17), if $u=-\frac{p_1}{\beta}$, it is obvious that the absolute ruin will happen immediately, and no dividend is paid, which implies (17).

(2) For (18), when $u=b$, we have

$$
\begin{aligned}
Q_1(b,z;b) =&(1-\lambda t)e^{p_1 t}Q_1(b,ze^{-\alpha t};b)+\lambda t\cdot\left[\int_0^b Q_1(b-y,ze^{-\alpha t};b)dG(y)\right.\\
&\left.+\int_b^{b+\frac{p_1}{\beta}} Q_2(b-y,ze^{-\alpha t};b)dG(y)+\bar{G}(b+\frac{p_1}{\beta})\right]+o(t).
\end{aligned} \tag{21}
$$

By plugging $u=b$ into (9) and using (21), we obtain (18).

(3) For (19) and (20), the method is analogous to Wan [24], so we leave it out here.

Let us now consider the problem of $W_n(u,b)$. Following the same argument as above according to the different initial reserves, $W_n(u,b)$ is a piecewise function. We denote

$$W_n(u;b) = \begin{cases} W_{n2}(u;b), & u > b, \\ W_{n1}(u;b), & 0 \leq u \leq b, \\ W_{n3}(u;b), & -p_1/\beta < u < 0, \end{cases} \tag{22}$$

where $W_{01}(b;b) = 1$.

According to the representation theorem, we have

$$Q_i(u,z;b) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} W_{ni}(u;b), \;\; i = 1,2,3, \; n \in N^+, \tag{23}$$

and equating the coefficients of z^n in (9)–(11), we can show that $W_{ni}(u;b)$ $(i = 1,2,3.)$ satisfies the following integro-differential equations and corresponding boundary conditions. □

Theorem 3. *When* $0 \leq u \leq b$,

$$\begin{aligned} p_1 W'_{n1}(u;b) =& (\lambda + n\alpha) W_{n1}(u;b) - \lambda \left[\int_0^u W_{n1}(u-y;b) dG(y) \right. \\ & \left. + \int_u^{u+p_1/\beta} W_{n3}(u-y;b) dG(y) \right], \end{aligned} \tag{24}$$

and when $u > b$,

$$\begin{aligned} (\gamma u + p_2) W'_{n2}(u;b) =& (\lambda + n\alpha) W_{n2}(u;b) - \lambda \left[\int_0^{u-b} W_{n2}(u-y;b) dG(y) \right. \\ & \left. + \int_{u-b}^{u} W_{n1}(u-y;b) dG(y) + \int_u^{u+p_1/\beta} W_{n3}(u-y;b) dG(y) \right], \end{aligned} \tag{25}$$

and when $-p_1/\beta < u < 0$,

$$(\beta u + p_1) W'_{n3}(u;b) = (\lambda + n\alpha) W_{n3}(u;b) - \lambda \int_0^{u+p_1/\beta} W_{n3}(u-y;b) dG(y), \tag{26}$$

with boundary conditions

$$W_{n3}(-p_1/\beta;b) = 0, \tag{27}$$

$$W'_{n1}(u;b)|_{u=b} = n W_{n-1,1}(b;b), \tag{28}$$

$$W_{n1}(0+;b) = W_{n3}(0-;b), \tag{29}$$

$$W_{n1}(b-;b) = W_{n2}(b+;b), \tag{30}$$

$$W'_{n1}(0+;b) = W'_{n3}(0-;b), \tag{31}$$

$$p_1 W'_{n1}(b-;b) = (\gamma b + p_2) W'_{n2}(b+;b), \tag{32}$$

where N^+ *represents non-negative integers.*

3. Explicit Expressions for Exponential Claims to $W_n(u;b)$ and Numerical Examples

We suppose that claim sizes obey an exponential distribution with mean $\mu > 0$. Then, Equations (24)–(26) are reduced to

$$p_1 W'_{n1}(u;b) = (\lambda+n\alpha)W_{n1}(u;b) - \frac{\lambda}{\mu}e^{-\frac{u}{\mu}}\left[\int_0^u W_{n1}(y;b)e^{\frac{y}{\mu}}dy + \int_{-p_1/\beta}^0 W_{n3}(y;b)e^{\frac{y}{\mu}}dy\right], \quad 0 \le u \le b, \tag{33}$$

$$(\gamma u + p_2)W'_{n2}(u;b) = (\lambda+n\alpha)W_{n2}(u;b) - \frac{\lambda}{\mu}e^{-\frac{u}{\mu}}\left[\int_0^u W_{n2}(y;b)e^{\frac{y}{\mu}}dy + \int_0^b W_{n1}(y;b)e^{\frac{y}{\mu}}dy + \int_{-p_1/\beta}^0 W_{n3}(y;b)e^{\frac{y}{\mu}}dy\right], \quad u > b, \tag{34}$$

$$(\beta u + p_1)W'_{n3}(u;b) = (\lambda+n\alpha)W_{n3}(u;b) - \frac{\lambda}{\mu}e^{-\frac{u}{\mu}}\int_{-p_1/\beta}^u W_{n3}(y;b)e^{\frac{y}{\mu}}dy, \quad -p_1/\beta < u \le 0. \tag{35}$$

By applying the operator $\left(\frac{d}{du}+\frac{1}{\mu}\right)$ on (33)–(35), respectively, and then rearranging them, we yield

$$W''_{n1}(u;b) + \left(\frac{1}{\mu} - \frac{\lambda+n\alpha}{p_1}\right)W'_{n1}(u;b) - \frac{n\alpha}{\mu p_1}W_{n1}(u;b) = 0, \quad 0 \le u \le b, \tag{36}$$

$$(\gamma u + p_2)W''_{n2}(u;b) + \left(\frac{\gamma u + p_2}{\mu} + \gamma - (\lambda+n\alpha)\right)W'_{n2}(u;b) - \frac{n\alpha}{\mu}W_{n2}(u;b) = 0, \quad u > b, \tag{37}$$

$$(\beta u + p_1)W''_{n3}(u;b) + \left(\frac{\beta u + p_1}{\mu} + \beta - (\lambda+n\alpha)\right)W'_{n3}(u;b) - \frac{n\alpha}{\mu}W_{n3}(u;b) = 0, \quad -p_1/\beta < u \le 0. \tag{38}$$

Obviously, the general solution of Equation (36) can be expressed as

$$W_{n1}(u;b) = \xi_{n1}e^{\delta_{n1}u} + \xi_{n2}e^{\delta_{n2}u}, \quad 0 \le u \le b, \tag{39}$$

where ξ_{n1} and ξ_{n3} are arbitrary constants, and δ_{n1} and δ_{n2} are the two real roots of the following equation

$$\delta^2 + \varsigma_{1n}\delta + \varsigma_{2n} = 0, \tag{40}$$

with $\varsigma_{1n} = \frac{1}{\mu} - \frac{\lambda+n\alpha}{p_1}$, $\varsigma_{2n} = \frac{-n\alpha}{\mu p_1}$ satisfying $\varsigma_{1n}^2 - 4\varsigma_{2n} > 0$, i.e.,

$$\delta_{n1} = \frac{-\varsigma_{1n} + \sqrt{\varsigma_{1n}^2 - 4\varsigma_{2n}}}{2}, \quad \delta_{n2} = \frac{-\varsigma_{1n} - \sqrt{\varsigma_{1n}^2 - 4\varsigma_{2n}}}{2}. \tag{41}$$

Equations similar to (37) and (38) can be found in Paulsen and Gjessing [25] and Cai and Yang [26]. By introducing the new variables, $x = -\frac{\gamma u + p_2}{\gamma\mu}$ for $u > b$ and $z = -\frac{\beta u + p_1}{\beta\mu}$ for $-p_1/\beta < u \le 0$, and letting $W_{n2}(u,b) = g_n(x)$ and $W_{n3}(u,b) = f_n(z)$, Equations (37) and (38) can be converted into Kummer's confluent hypergeometric equation (see Salter [27] and Seaborn [28]) for functions $g_n(x)$ and $f_n(z)$:

$$xg''_n(x) + \left(1 - \frac{\lambda+n\alpha}{\gamma} - y\right)g'_n(x) - \frac{n\alpha}{\gamma}g_n(x) = 0, \quad -\frac{p_2}{\gamma\mu} < x < 0, \tag{42}$$

$$zf''_n(z) + \left(1 - \frac{\lambda+n\alpha}{\beta} - z\right)f'_n(z) - \frac{n\alpha}{\beta}f_n(z) = 0, \quad -\frac{p_1}{\beta\mu} < z < 0. \tag{43}$$

Using the solutions of (42) and (43), we conclude that

$$W_{n2}(u;b) = g_n(x) = \xi_{n3}\eta_{n3}(u) + \xi_{n4}\eta_{n4}(u), \quad u > b, \tag{44}$$

$$W_{n3}(u;b) = f_n(x) = \xi_{n5}\eta_{n5}(u) + \xi_{n6}\eta_{n6}(u), \quad -p_1/\beta < u \le 0, \tag{45}$$

where, ξ_{n3}, ξ_{n4}, ξ_{n5} and ξ_{n6} are arbitrary constants, and

$$\eta_{n3}(u) = \exp\{-\frac{\gamma u + p_2}{\gamma\mu}\} \cdot U\left(1 - \frac{\lambda}{\gamma}, 1 - \frac{\lambda + n\alpha}{\gamma}; \frac{\gamma u + p_2}{\gamma\mu}\right),$$
$$\eta_{n4}(u) = \left(\frac{\gamma u + p_2}{\gamma\mu}\right)^{(\lambda+n\alpha)/\gamma} \cdot \exp\{-\frac{\gamma u + p_2}{\gamma\mu}\} \cdot M\left(1 + \frac{n\alpha}{\gamma}, 1 + \frac{\lambda + n\alpha}{\gamma}; \frac{\gamma u + p_2}{\gamma\mu}\right),$$
$$\eta_{n5}(u) = \exp\{-\frac{\beta u + p_1}{\beta\mu}\} \cdot U\left(1 - \frac{\lambda}{\beta}, 1 - \frac{\lambda + n\alpha}{\beta}; \frac{\beta u + p_1}{\beta\mu}\right),$$
$$\eta_{n6}(u) = \left(\frac{\beta u + p_1}{\beta\mu}\right)^{(\lambda+n\alpha)/\beta} \cdot \exp\{-\frac{\beta u + p_1}{\beta\mu}\} \cdot M\left(1 + \frac{n\alpha}{\beta}, 1 + \frac{\lambda + n\alpha}{\beta}; \frac{\beta u + p_1}{\beta\mu}\right),$$
$$M(a_1, a_2; x) = \frac{\Gamma(a_2)}{\Gamma(a_2 - a_1)\Gamma(a_1)} \int_0^1 e^{xt} t^{a_1-1}(1-t)^{a_2-a_1-1} dt, \quad a_2 > a_1 > 0,$$
$$U(a_1, a_2; x) = \frac{1}{\Gamma(a_1)} \int_0^\infty e^{-xt} t^{a_1-1}(1+t)^{a_2-a_1-1} dt, \quad y > 0, a_1 > 0.$$

Using the confluent hypergeometric function property, if $\beta \neq \lambda + n\alpha$, we get

$$\lim_{u \downarrow -\frac{p_1}{\beta}} \eta_{n5}(u) = \Gamma\left(\frac{\lambda + n\alpha}{\beta}\right) / \Gamma\left(\frac{\beta + n\alpha}{\beta}\right), \qquad \lim_{u \downarrow -\frac{p_1}{\beta}} \eta_{n6}(u) = 0, \tag{46}$$

where $\downarrow$ denotes a decreasing approach. Letting $u \downarrow -\frac{p_1}{\beta}$ in (45) on both sides and together with (45), (46) and (27), we see that $\xi_{n5} = 0$, which means that for $-\frac{p_1}{\beta} < u < 0$,

$$V_{n3}(u; b) = \xi_{n6}\eta_{n6}(u). \tag{47}$$

Next, we give the explicit values of $\xi_{n1}, \xi_{n2}, \xi_{n3}, \xi_{n4}, \xi_{n6}$ $(\xi_{n5} = 0)$ for $n = 1$ and $n \geq 2$. When $n = 1$, according to (28)–(32), (39), (44) and (47), we have

$$\begin{cases} \xi_{11}\delta_{11}e^{\delta_{11}b} + \xi_{12}\delta_{12}e^{\delta_{12}b} = 1, \\ \xi_{11} + \xi_{12} = \xi_{16}\eta_{16}(0), \\ \xi_{11}\delta_{11} + \xi_{12}\delta_{12} = \xi_{16}\eta'_{16}(0), \\ \xi_{11}e^{\delta_{11}b} + \xi_{12}e^{\delta_{12}b} = \xi_{13}\eta_{13}(b) + \xi_{14}\eta_{14}(b), \\ p_1(\xi_{11}\delta_{11}e^{\delta_{11}b} + \xi_{12}\delta_{12}e^{\delta_{12}b}) = (\gamma b + p_2)[\xi_{13}\eta'_{13}(b) + \xi_{14}\eta'_{14}(b)]. \end{cases} \tag{48}$$

By solving the above equations (48), we obtain

$$\begin{cases} \xi_{11} &= \dfrac{\eta'_{16}(0) - \delta_{12}\eta_{16}(0)}{\delta_{11}e^{\delta_{11}b}[\eta'_{16}(0) - \delta_{12}\eta_{16}(0)] + \delta_{12}e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta'_{16}(b)]}, \\ \xi_{12} &= \dfrac{\delta_{11}\eta_{16}(0) - \eta'_{16}(0)}{\delta_{11}e^{\delta_{11}b}[\eta'_{16}(0) - \delta_{12}\eta_{16}(0)] + \delta_{12}e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta'_{16}(b)]}, \\ \xi_{13} &= \dfrac{1}{\eta_{13}(b)}\{\theta_1 - \eta_{14}(b)\dfrac{(\gamma b + p_2)\eta'_{13}(b))\theta_1 - \eta_{13}(b)p_1}{(\gamma b + p_2)[\eta'_{13}(b)\eta_{14}(b) - \eta_{13}(b)\eta'_{14}(b)]}\}, \\ \xi_{14} &= \dfrac{(\gamma b + p_2)\eta'_{13}(b)\theta_1 - \eta_{13}(b)p_1}{(\gamma b + p_2)[\eta'_{13}(b)\eta_{14}(b) - \eta_{13}(b)\eta'_{14}(b)]}, \\ \xi_{16} &= \dfrac{\delta_{11} - \delta_{12}}{\delta_{11}e^{\delta_{11}b}[\eta'_{16}(0) - \delta_{12}\eta_{16}(0)] + \delta_{12}e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta'_{16}(b)]}, \end{cases}$$

where δ_{11} and δ_{12} are given by (41) in the case of $n = 1$, and

$$\theta_1 = \frac{e^{\delta_{11}b}[\eta_{16}'(0) - \delta_{12}\eta_{16}(0)] + e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta_{16}'(0)]}{\delta_{11}e^{\delta_{11}b}[\eta_{16}'(0) - \delta_{12}\eta_{16}(0)] + \delta_{12}e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta_{16}'(0)]}.$$

Therefore, we arrive at the explicit expressions for $W_{11}(u;b)$,$W_{12}(u;b)$ and $W_{13}(u;b)$, namely,

$$W_{11}(u;b) = \frac{[\eta_{16}'(0) - \delta_{12}\eta_{16}(0)]e^{\delta_{11}u} + [\delta_{11}\eta_{16}(0) - \eta_{16}'(0)]e^{\delta_{12}u}}{\delta_{11}e^{\delta_{11}b}[\eta_{16}'(0) - \delta_{12}\eta_{16}(0)] + \delta_{12}e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta_{16}'(0)]}, \quad 0 \le u \le b, \tag{49}$$

$$\begin{aligned} W_{12}(u;b) = &\frac{\eta_{13}(u)}{\eta_{13}(b)}\{W_{11}(b;b) - \eta_{14}(b)\frac{(\gamma b + p_2)\eta_{13}'(b)W_{11}(b;b) - \eta_{13}(b)p_1}{(\gamma b + p_2)[\eta_{13}'(b)\eta_{14}(b) - \eta_{13}(b)\eta_{14}'(b)]}\} \\ &+ \frac{\eta_{14}(u)[(\gamma b + p_2)\eta_{13}'(b)\theta_2 - \eta_{13}(b)p_1]}{(\gamma b + p_2)[\eta_{13}'(b)\eta_{14}(b) - \eta_{13}(b)\eta_{14}'(b)]}, \quad u > b, \end{aligned} \tag{50}$$

$$W_{13}(u;b) = \frac{\eta_{16}(u)(\delta_{11} - \delta_{12})}{\delta_{11}e^{\delta_{11}b}[\eta_{16}'(0) - \delta_{12}\eta_{16}(0)] + \delta_{12}e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta_{n6}'(0)]}, \quad -p_1/\beta \le u \le 0. \tag{51}$$

When $n \ge 2$, we provide the explicit expressions of $\xi_{n1}, \xi_{n2}, \xi_{n3}, \xi_{n4}$ and ξ_{n6} ($\xi_{n5} = 0$) by recursive formulas. It follows from (28)–(32), (39) (44), and (47) that

$$\begin{cases} \xi_{n1}\delta_{n1}e^{\delta_{n1}b} + \xi_{n2}\delta_{n2}e^{\delta_{n2}b} = nW_{n-1,n}(b;b), \\ \xi_{n1} + \xi_{n2} = \xi_{n6}\eta_{n6}(0), \\ \xi_{n1}\delta_{n1} + \xi_{n2}\delta_{n2} = \xi_{n6}\eta_{n6}'(0), \\ \xi_{n1}e^{\delta_{n1}b} + \xi_{n2}e^{\delta_{n2}b} = \xi_{n3}\eta_{n3}(b) + \xi_{n4}\eta_{n4}(b), \\ p_1(\xi_{n1}\delta_{n1}e^{\delta_{n1}b} + \xi_{n2}\delta_{n2}e^{\delta_{n2}b}) = (\gamma b + p_2)[\xi_{n3}\eta_{n3}'(b) + \xi_{n4}\eta_{n4}'(b)]. \end{cases} \tag{52}$$

By solving the equations above, one finds

$$\begin{cases} \xi_{n1} &= \dfrac{nW_{n-1,n}(b;b)[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)]}{\delta_{n1}e^{\delta_{n1}b}[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)] + \delta_{n2}e^{\delta_{n2}b}[\delta_{n1}\eta_{n6}(0) - \eta_{n6}'(b)]}, \\ \xi_{n2} &= \dfrac{nW_{n-1,n}(b;b)[\delta_{n1}\eta_{n6}(0) - \eta_{n6}'(0)]}{\delta_{n1}e^{\delta_{n1}b}[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)] + \delta_{n2}e^{\delta_{n2}b}[\delta_{n1}\eta_{16}(0) - \eta_{n6}'(b)]}, \\ \xi_{n3} &= \dfrac{1}{\eta_{n3}(b)}\{\theta_2 - \eta_{n4}(b)\dfrac{(\gamma b + p_2)\eta_{n3}'(b))\theta_2 - nW_{n-1,n}(b;b)\eta_{n3}(b)p_1}{(\gamma b + p_2)[\eta_{n3}'(b)\eta_{n4}(b) - \eta_{n3}(b)\eta_{n4}'(b)]}\}, \\ \xi_{n4} &= \dfrac{(\gamma b + p_2)\eta_{n3}'(b)\theta_2 - nW_{n-1,n}(b;b)\eta_{n3}(b)p_1}{(\gamma b + p_2)[\eta_{n3}'(b)\eta_{n4}(b) - \eta_{n3}(b)\eta_{n4}'(b)]}, \\ \xi_{n6} &= \dfrac{nW_{n-1,n}(b;b)(\delta_{n1} - \delta_{n2})}{\delta_{n1}e^{\delta_{n1}b}[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)] + \delta_{n2}e^{\delta_{n2}b}[\delta_{n1}\eta_{n6}(0) - \eta_{n6}'(b)]}, \end{cases} \tag{53}$$

where δ_{n1} and δ_{n2} are given by (41), and

$$\theta_2 = \frac{nW_{n-1,n}(b;b)\{e^{\delta_{n1}b}[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)] + e^{\delta_{n2}b}[\delta_{11}\eta_{n6}(0) - \eta_{n6}'(0)]\}}{\delta_{n1}e^{\delta_{n1}b}[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)] + \delta_{n2}e^{\delta_{n2}b}[\delta_{n1}\eta_{n6}(0) - \eta_{n6}'(0)]}.$$

Thus, we get the recursive formula for $W_{n1}(u;b)$, $W_{n2}(u;b)$ and $W_{n3}(u;b)$ as being

$$W_{n1}(u;b) = \frac{nW_{n-1,1}(b;b)\{[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)]e^{\delta_{n1}u} + [\delta_{n1}\eta_{n6}(0) - \eta_{n6}'(0)]e^{\delta_{n2}u}\}}{\delta_{n1}e^{\delta_{n1}b}[\eta_{n6}'(0) - \delta_{n2}\eta_{n6}(0)] + \delta_{n2}e^{\delta_{n2}b}[\delta_{n1}\eta_{n6}(0) - \eta_{n6}'(0)]}, \quad 0 \le u \le b, \tag{54}$$

$$W_{n2}(u;b) = \frac{\eta_{n3}(u)}{\eta_{n3}(b)}\{\theta_2 - \eta_{n4}(b)\frac{(\gamma b+p_2)\eta'_{n3}(b)\theta_2 - nW_{n-1,1}(b;b)\eta_{n3}(b)p_1}{(\gamma b+p_2)[\eta'_{n3}(b)\eta_{n4}(b) - \eta_{n3}(b)\eta'_{n4}(b)]}\}$$
$$+\frac{\eta_{n4}(u)[(\gamma b+p_2)\eta'_{n3}(b)\theta_2 - nW_{n-1,1}(b;b)\eta_{n3}(b)p_1]}{(\gamma b+p_2)[\eta'_{n3}(b)\eta_{n4}(b) - \eta_{n3}(b)\eta'_{n4}(b)]}, \quad u > b, \tag{55}$$

$$W_{n3}(u;b) = \frac{\eta_{n6}(u)nW_{n-1,1}(b;b)(\delta_{n1}-\delta_{n2})}{\delta_{n1}e^{\delta_{n1}b}[\eta'_{n6}(0) - \delta_{n2}\eta_{n6}(0)] + \delta_{n2}e^{\delta_{n2}b}[\delta_{n1}\eta_{n6}(0) - \eta'_{n6}(0)]}, \quad -p_1/\beta \le u \le 0, \tag{56}$$

with an initial value of

$$W_{11}(b;b) = \frac{[\eta'_{16}(0) - \delta_{12}\eta_{16}(0)]e^{\delta_{11}b} + [\delta_{11}\eta_{16}(0) - \eta'_{16}(0)]e^{\delta_{12}b}}{\delta_{11}e^{\delta_{11}b}[\eta'_{16}(0) - \delta_{12}\eta_{16}(0)] + \delta_{12}e^{\delta_{12}b}[\delta_{11}\eta_{16}(0) - \eta'_{16}(0)]}.$$

In the following examples, $n = 1$, and we illustrate the influences of relevant parameters on $W_1(u;b)$.

Example 1. *Suppose $\lambda = 0.02$, $\mu = 0.5$, $p_1 = 0.2$, $p_2 = 0.1$, $\gamma = 0.08$, $\alpha = 0.02$. Figure 1 shows the curves of $W_{11}(u,b)$ using the formulas derived above for dividend barriers of $b = 6$, $b = 8$ and $b = 10$, respectively. Figure 2 shows the curves of $W_{11}(u,b)$ for $\alpha = 0.02$, $\alpha = 0.03$ and $\alpha = 0.04$ $(b = 10)$, respectively. Figure 3 shows the surface of $W_{11}(u,b)$ with respect to two variables u and b. From the figures, we see that $W_{11}(u,b)$ decreases when b and α increase, respectively.*

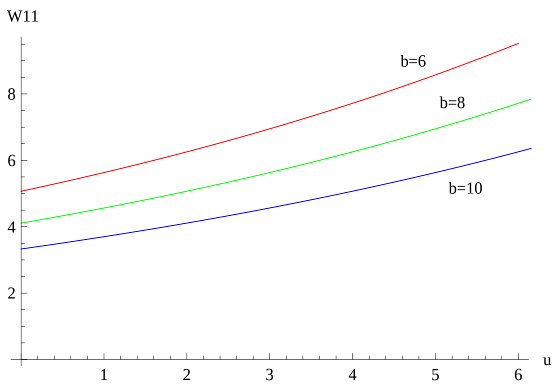

Figure 1. The curves of $W_{11}(u,b)$ for dividend barriers of $b = 6$, $b = 8$ and $b = 10$.

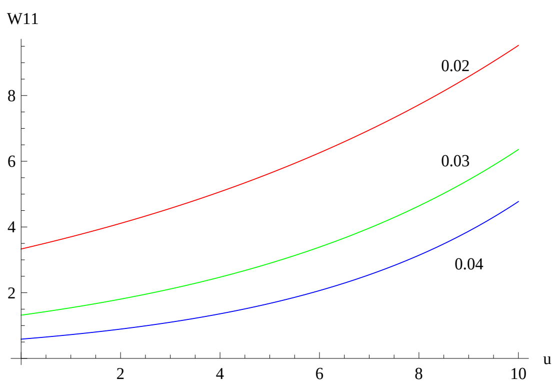

Figure 2. The curves of $W_{11}(u,b)$ for $\alpha = 0.02$, $\alpha = 0.03$ and $\alpha = 0.04$ $(b = 10)$.

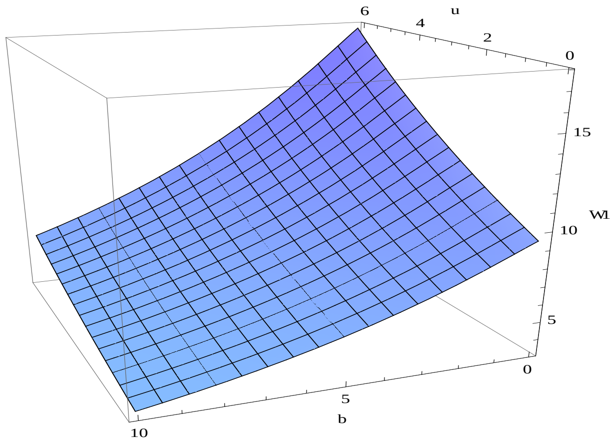

Figure 3. The surface of $W_{11}(u, b)$ with respect to two variables u and b.

Example 2. *Suppose* $\lambda = 0.02$, $\beta = 0.05$, $p_1 = 0.2$, $p_2 = 0.1$, *and* $\mu = 0.5$. *Figure 4 shows the curves of* $W_{12}(u, b)$ *for dividend barriers of* $b = 6$, $b = 8$ *and* $b = 10$. *Figure 5 shows the curves of* $W_{12}(u, b)$ *for* $\alpha = 0.02$, $\alpha = 0.03$ *and* $\alpha = 0.04$ $(b = 10, \gamma = 0.08)$. *Figure 6 shows the curves of* $W_{12}(u, b)$ *for* $\gamma = 0.05$, $\gamma = 0.07$ *and* $\gamma = 0.09$ $(b = 10, \alpha = 0.02)$. *Figure 7 shows the surface of* $W_{12}(u, b)$ *with respect to variables* u *and* b.

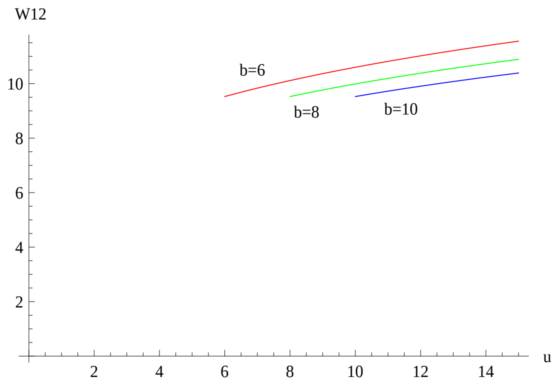

Figure 4. The curves of $W_{12}(u, b)$ for dividend barriers of $b = 6$, $b = 8$ and $b = 10$.

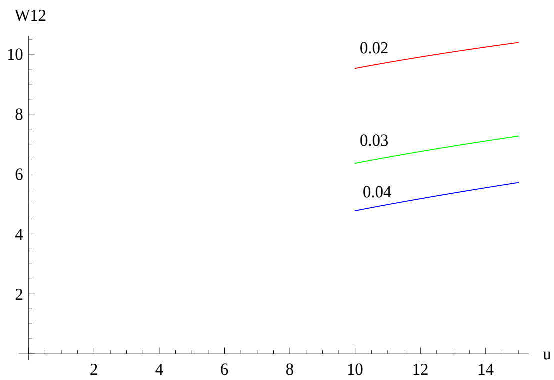

Figure 5. The curves of $W_{12}(u, b)$ for $\alpha = 0.02$, $\alpha = 0.03$ and $\alpha = 0.04$ $(b = 10, \gamma = 0.08)$.

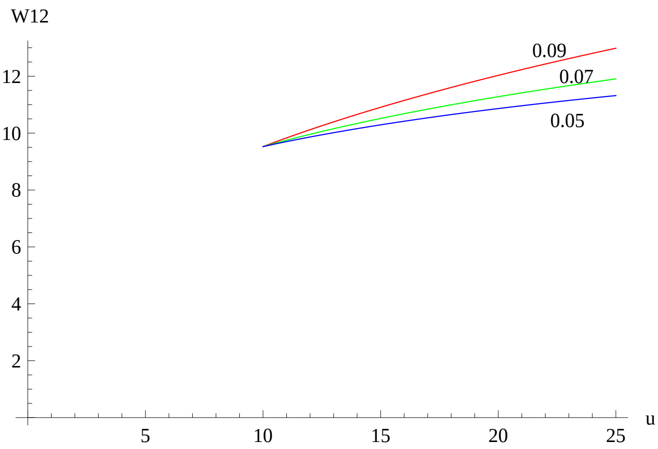

Figure 6. The curves of $W_{12}(u, b)$ for $\gamma = 0.05$, $\gamma = 0.07$ and $\gamma = 0.09$ $(b = 10, \alpha = 0.02)$.

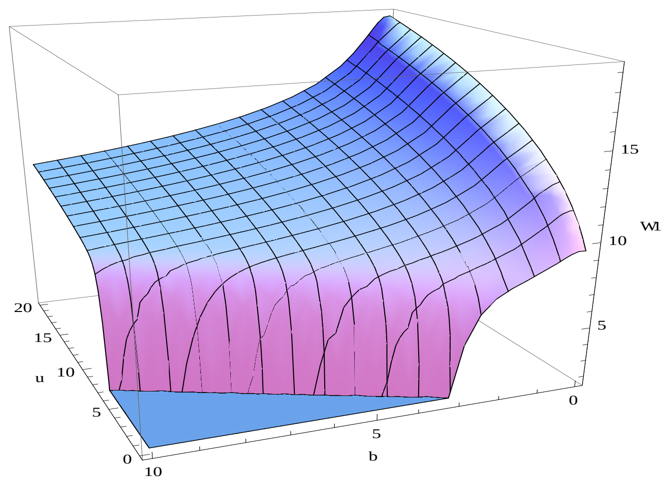

Figure 7. The surface of $W_{12}(u, b)$ with respect to variables u and b.

From Figures 4–7, we can see that $W_{12}(u, b)$ is a decreasing function of b, α, and γ, respectively. The results can be compared to the results of Peng et al. [5] who considered a compound Poisson risk model with a constant dividend barrier and liquid reserves in the case of absolute ruin. From the comparative results, we obtained the conclusion that the influence of parameter b on the moment of the present value of all dividends until absolute ruin is the same, regardless of whether a constant dividend barrier or the threshold dividend strategy is used. In addition, the effect of parameter γ is the opposite. This is consistent with the actual situation.

Example 3. *The parameters used were as follows:* $\lambda = 0.02$, $\beta = 0.05$, $\alpha = 0.02$, $\gamma = 0.08$, $p_1 = 0.2$, $p_2 = 0.1$, $\mu = 0.5$. *Figure 8 shows the curves of* $W_{13}(u, b)$ *for dividend barriers of* $b = 6$, $b = 8$ *and* $b = 10$. *Figure 9 shows the curves of* $V_{13}(u, b)$ *for* $\alpha = 0.01$, $\alpha = 0.015$ *and* $\alpha = 0.02$ $(b = 10, \gamma = 0.08)$. *Figure 10 shows the curves of* $W_{13}(u, b)$ *for* $\beta = 0.05$, $\beta = 0.07$ *and* $\beta = 0.09$ $(b = 10, \alpha = 0.02)$. *Figure 11 shows the surface of* $W_{13}(u, b)$ *with respect to variables u and b. The results show that* $W_{13}(u, b)$ *decreases as b,* α*, and* β *increase but increases as u increases.*

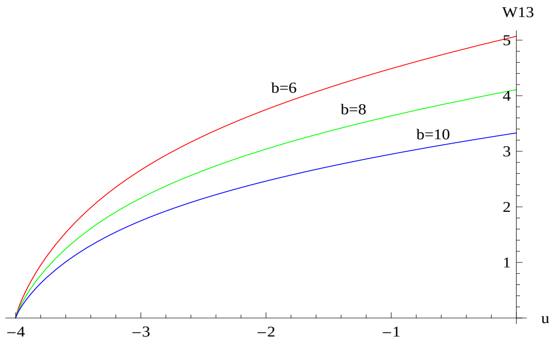

Figure 8. The curves of $W_{13}(u,b)$ for dividend barriers of $b = 6$, $b = 8$ and $b = 10$.

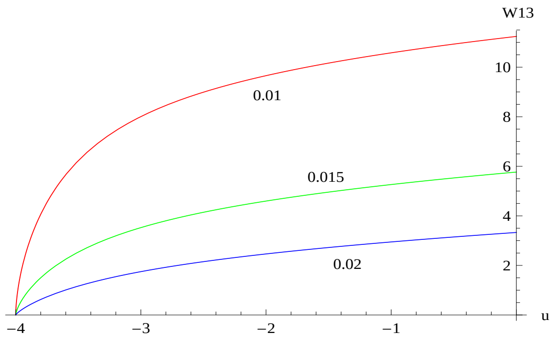

Figure 9. The curves of $V_{13}(u,b)$ for $\alpha = 0.01$, $\alpha = 0.015$ and $\alpha = 0.02$ $(b = 10, \gamma = 0.08)$.

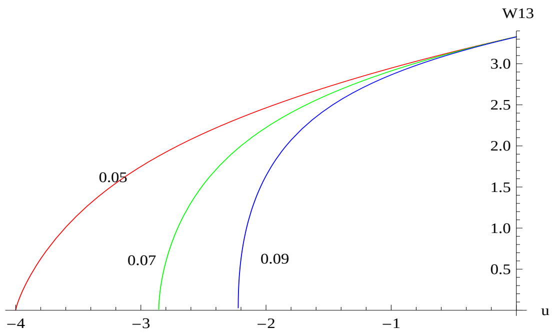

Figure 10. The curves of $W_{13}(u,b)$ for $\beta = 0.05$, $\beta = 0.07$ and $\beta = 0.09$ $(b = 10, \alpha = 0.02)$.

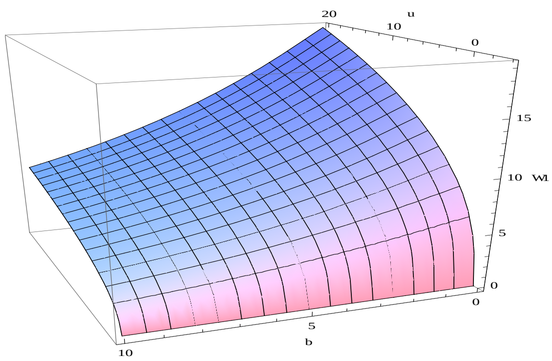

Figure 11. The surface of $W_{13}(u,b)$ with respect to variables u and b.

4. The Gerber-Shiu Expected Discounted Penalty Function

Similarly, $\Phi(u;b)$ can be expressed as

$$\Phi(u;b) = \begin{cases} \Phi_2(u;b), & u > b, \\ \Phi_1(u;b), & 0 \le u \le b, \\ \Phi_3(u;b), & -p_1/\beta < u < 0. \end{cases} \tag{57}$$

Similar to the arguments in Theorem 1, we can easily show that the Gerber-Shiu expected discounted penalty function satisfies the following integro-differential equations:

Theorem 4. *When* $0 \le u \le b$*,*

$$\begin{aligned} p_1\Phi_1'(u;b) =& (\lambda+\alpha)\Phi_1(u;b) - \lambda\left[\int_0^u \Phi_1(u-y;b)dG(y)\right. \\ &\left. + \int_u^{u+p_1/\beta} \Phi_3(u-y;b)dG(y) + B(u)\right], \end{aligned} \tag{58}$$

and when $u > b$*,*

$$\begin{aligned} (\gamma u + p_2)\Phi_2'(u;b) =& (\lambda+\alpha)\Phi_2(u;b) - \lambda\left[\int_0^{u-b} \Phi_2(u-y;b)dG(y)\right. \\ &\left. + \int_{u-b}^{u} \Phi_1(u-y;b)dG(y) + \int_u^{u+p_1/\beta} \Phi_3(u-y;b)dG(y) + B(u)\right], \end{aligned} \tag{59}$$

and when $-p_1/\beta < u < 0$*,*

$$(\beta u + p_1)\Phi_3'(u;b) = (\lambda+\alpha)\Phi_3(u;b) - \lambda\int_0^{u+p_1/\beta} \Phi_3(u-y;b)dG(y) + B(u), \tag{60}$$

with boundary conditions

$$\Phi_1(0+;b) = \Phi_3(0-;b), \tag{61}$$

$$\Phi_1(b-;b) = \Phi_2(b+;b), \tag{62}$$

$$p_1\Phi_1'(b-;b) = (\gamma b + p_2)\Phi_2'(b+;b), \tag{63}$$

$$\Phi_1'(0+;b) = \Phi_3'(0-;b), \tag{64}$$

where, $B(u) = \int_{u+p_1/\beta}^{\infty} \omega(u, y-u)dG(y)$*.*

Theorem 5. *The integro-differential Equations (58)–(60) can be expressed by the Volterra equations*

$$\Phi_1(u;b) = \int_0^u k_1(u,x)\Phi_1(x;b)dx + \psi_1(u), \quad 0 \le u \le b, \tag{65}$$

$$\Phi_2(u;b) = \int_b^u k_2(u,x)\Phi_2(x;b)dx + \psi_2(u), \quad u > b, \tag{66}$$

$$\Phi_3(u;b) = \int_{-p_1/\beta}^u k_3(u,x)\Phi_3(x;b)dx + \psi_3(u), \quad -p_1/\beta < u < 0, \tag{67}$$

where

$$
\begin{aligned}
k_1(u,x) =& \frac{\lambda+\alpha}{p_1} - \frac{\lambda}{p_1} G(u-x), \\
\psi_1(u) =& \Phi_1(0;b) - \frac{\lambda}{p_1} \int_{-p_1/\beta}^{0} \Phi_3(x;b)[G(u-x)-G(-x)]dx + \frac{\lambda}{p_1}\int_0^u B(x)dx, \\
k_2(u,x) =& \frac{\lambda+\alpha+\gamma}{\gamma u + p_2} - \frac{\lambda}{\gamma u+p_2} G(u-x), \\
\psi_2(u) =& \frac{\gamma b + p_2}{\gamma u + p_2}\Phi_2(b;b) - \frac{\lambda}{\gamma u + p_2}\int_0^b \Phi_1(x;b)[G(u-x)-G(b-x)]dx \\
& - \frac{\lambda}{\gamma u + p_2}\int_{-p_1/\beta}^{0} \Phi_3(x;b))[G(u-x)-G(b-x)]dx \\
& - \frac{\lambda}{\gamma u + p_2}\int_0^u B(x)dx, \\
k_3(u,x) =& \frac{\lambda+\beta+\gamma}{\beta u + p_1} - \frac{\lambda}{\beta u + p_1} G(u-x), \\
\psi_3(u) =& \frac{\lambda}{\beta u + p_1}\int_0^u B(x)dx.
\end{aligned}
$$

Proof. In (58), integrating (58) over $(0,u)$ yields

$$
\begin{aligned}
p_1\Phi_1(u;b) =& p_1\Phi_1(0;b) + \int_0^u \Phi_1(x;b)(\lambda+\alpha-\lambda G(u-x))dx \\
& - \lambda\int_{-p_1/\beta}^{0} [G(u-x)-G(-x)]dx - \lambda\int_0^u B(x)dx.
\end{aligned} \tag{68}
$$

In (68), integrating (68) over $(0,u)$, one concludes

$$
\begin{aligned}
p_1\int_0^u \Phi_1(y;b)dy =& p_1\Phi_1(0;b)u + \int_0^u \left[\int_0^y \Phi_1(x;b)(\lambda+\alpha-\lambda G(y-x))dx\right]dy \\
& - \lambda\int_0^u G(y)dy,
\end{aligned} \tag{69}
$$

where, $G(y) = \int_{-p_1/\beta}^{0} \Phi_3(x;b)(G(y-x)-G(-x))dx + \int_0^y B(x)dx$, since

$$
\int_0^u \left[\int_0^y \Phi_1(x;b)(\lambda+\alpha-\lambda G(y-x))dx\right]dy = \int_0^u \left[\int_x^u \Phi_1(x;b)(\lambda+\alpha-\lambda G(y-x))dy\right]dx. \tag{70}
$$

Substituting (70) into (69) yields (65).

Similarly to the proof of (65), we can obtain (66) and (67). □

Remark 1. *We point out that $\psi_1(u)$, $\psi_2(u)$, and $\psi_3(u)$ are absolutely integrable, and $k_1(u;b)$, $k_2(u;b)$, and $k_3(u;b)$ are all continuous. In accordance with Cai and Dickson [29], $\Phi_1(u;b)$, $\Phi_2(u;b)$ and $\Phi_3(u;b)$ can be approximated recursively by Picards sequence, i.e.,*

$$
\Phi_1(u;b) = \psi_1(u) + \sum_{n=1}^{\infty}\int_0^u k_{1n}(u,x)\psi_1(x)dx, \quad 0 \le u \le b,
$$

where, $k_{11}(u,x) = k_1(u,x)$, $k_{1n}(u,x) = \int_x^u k_1(u,y)k_{1,n-1}(y,x)dy$, $n = 2,3,\ldots$.

$$
\Phi_2(u;b) = \psi_2(u) + \sum_{n=1}^{\infty}\int_b^u k_{2n}(u,x)\psi_2(x)dx, \quad u > b,
$$

where, $k_{21}(u,x) = k_2(u,x)$, $k_{2n}(u,x) = \int_x^u k_2(u,y)k_{2,n-1}(y,x)dy, \ \ n = 2,3,....$

$$\Phi_3(u;b) = \psi_3(u) + \sum_{n=1}^{\infty} \int_{-p_1/\beta}^{u} k_{3n}(u,x)\psi_3(x)dx, \quad 0 \le u \le b,$$

where, $k_{31}(u,x) = k_3(u,x)$, $k_{3n}(u,x) = \int_x^u k_3(u,y)k_{3,n-1}(y,x)dy, \ \ n = 2,3,....$

Hence, at least in theory, if we can provide these values of $\Phi_1(0;b)$, $\Phi_1'(0;b)$, $\Phi_2(b;b)$, $\Phi_2'(b;b)$, $\Phi_3(-p_1/\beta;b)$, and $\Phi_3'(-p_1/\beta;b)$, we can obtain the exact expression of the solutions for $\Phi_1(u;b)$, $\Phi_2(u;b)$, and $\Phi_3(u;b)$, recursively.

5. The Laplace Transform of Absolute Ruin Time

In this section, we set

$$\varphi(u;b) = \begin{cases} \varphi_2(u;b), & u > b, \\ \varphi_1(u;b), & 0 \le u \le b, \\ \varphi_3(u;b), & -p_1/\beta < u < 0. \end{cases} \tag{71}$$

Theorem 6. *When* $0 \le u \le b$*,*

$$\begin{aligned} p_1\varphi_1'(u;b) =& (\lambda+\rho)\varphi_1(u;b) - \lambda\left[\int_0^u \varphi_1(u-y;b)dG(y)\right. \\ &\left. + \int_u^{u+p_1/\beta} \varphi_3(u-y;b)dG(y) + \bar{G}(u+\frac{p_1}{\beta})\right], \end{aligned} \tag{72}$$

and when $u > b$*,*

$$\begin{aligned} (\gamma u + p_2)\varphi_2'(u;b) = (\lambda+\rho)\varphi_2(u;b) - \lambda&\left[\int_0^{u-b} \varphi_2(u-y;b)dG(y)\right. \\ & + \int_{u-b}^{u} \varphi_1(u-y;b)dG(y) + \int_u^{u+p_1/\beta} \varphi_3(u-y;b)dG(y) \\ &\left. + \bar{G}(u+\frac{p_1}{\beta})\right], \end{aligned} \tag{73}$$

and when $-p_1/\beta < u < 0$*,*

$$\begin{aligned} (\beta u + p_1)\varphi_3'(u;b) =& (\lambda+\rho)\varphi_3(u;b) - \lambda\left[\int_0^{u+p_1/\beta} \varphi_3(u-y;b)dG(y)\right. \\ &\left. + \bar{G}(u+\frac{p_1}{\beta})\right], \end{aligned} \tag{74}$$

with the conditions

$$\varphi_1(0+;b) = \varphi_3(0-;b), \tag{75}$$

$$\varphi_1(b-;b) = \varphi_2(b+;b), \tag{76}$$

$$\varphi_1'(0+;b) = \varphi_3'(0-;b), \tag{77}$$

$$p_1\varphi_1'(b-;b) = (\gamma b + p_2)\varphi_2'(b+;b), \tag{78}$$

$$\lim_{u \to -p_1/\beta} \varphi_3(-\frac{p_1}{\beta};b) = \frac{\lambda}{\lambda+\rho}, \tag{79}$$

$$\lim_{u \to \infty} \varphi_2(-\frac{p_1}{\beta};b) = 0, \tag{80}$$

where, Equation (80) is acquired from the fact that $T_b = \infty$ *and* $E[e^{-\rho T_b} I(T_b < \infty) | R_0^{u,b} = u] = 0$ *when* $u \to \infty$.

In the following, we solve the closed form expression for $\varphi(u;b)$ according to the exponential distribution of claims with mean μ. By applying the operator $\left(\frac{d}{du} + \frac{1}{\mu}\right)$ on (72)–(74), respectively, and then rearranging them, one deduces

$$\varphi_1''(u;b) + \left(\frac{1}{\mu} - \frac{\lambda+\rho}{p_1}\right)\varphi_1'(u;b) - \frac{\rho}{\mu p_1}\varphi_1(u;b) = 0, \ \ 0 \le u \le b, \tag{81}$$

$$(\gamma u + p_2)\varphi_2''(u;b) + \left(\frac{\gamma u + p_2}{\mu} + \gamma - \lambda - \rho\right)\varphi_2'(u;b) - \frac{\rho}{\mu}\varphi_1(u;b) = 0, \ \ u > b, \tag{82}$$

$$(\beta u + p_1)\varphi_3''(u;b) + \left(\frac{\beta u + p_1}{\mu} + \beta - \lambda - \rho\right)\varphi_3'(u;b) - \frac{\rho}{\mu}\varphi_3(u;b) = 0, \ \ -p_1/\beta < u \le 0. \tag{83}$$

By comparing (81)–(83) with (36)–(38) respectively, we have

$$\varphi_1(u;b) = m_1 e^{\sigma_1 u} + m_2 e^{\sigma_2 u}, \quad -p_1/\beta < u \le 0, \tag{84}$$

where, m_1 and m_2 are arbitrary constants, $q_1 = \frac{1}{\mu} - \frac{\lambda+\rho}{p_1}$ and $q_2 = -\frac{\rho}{\mu p_1}$ satisfying $q_1^2 - 4q_2 > 0$, i.e.,

$$\sigma_1 = \frac{-q_1 + \sqrt{q_1^2 - 4q_2}}{2}, \quad \sigma_2 = \frac{-q_1 - \sqrt{q_1^2 - 4q_2}}{2}, \tag{85}$$

and

$$\varphi_2(u;b) = m_3\tau_3(u) + m_4\tau_4(u), \quad u > b, \tag{86}$$

$$\varphi_3(u;b) = m_5\tau_5(u) + m_6\tau_6(u), \quad -p_1/\beta < u \le 0, \tag{87}$$

with

$$\tau_3(u) = \exp\{-\frac{\gamma u + p_2}{\gamma\mu}\} \cdot U\left(1 - \frac{\lambda}{\gamma}, 1 - \frac{\lambda+\rho}{\gamma}; \frac{\gamma u + p_2}{\gamma\mu}\right),$$

$$\tau_4(u) = \left(\frac{\gamma u + p_2}{\gamma\mu}\right)^{(\lambda+\rho)/\gamma} \cdot \exp\{-\frac{\gamma u + p_2}{\gamma\mu}\} \cdot M\left(1 + \frac{\rho}{\gamma}, 1 + \frac{\lambda+\rho}{\gamma}; \frac{\gamma u + p_2}{\gamma\mu}\right),$$

$$\tau_5(u) = \exp\{-\frac{\beta u + p_1}{\beta\mu}\} \cdot U\left(1 - \frac{\lambda}{\beta}, 1 - \frac{\lambda+\rho}{\beta}; \frac{\beta u + p_1}{\beta\mu}\right),$$

$$\tau_6(u) = \left(\frac{\beta u + p_1}{\beta\mu}\right)^{(\lambda+\rho)/\beta} \cdot \exp\{-\frac{\beta u + p_1}{\beta\mu}\} \cdot M\left(1 + \frac{\rho}{\beta}, 1 + \frac{\lambda+\rho}{\beta}; \frac{\beta u + p_1}{\beta\mu}\right).$$

If $\beta \neq \lambda + \rho$, we have

$$\lim_{u \downarrow -p_1/\beta} \tau_5(u) = \frac{\Gamma\left(\frac{\lambda+\rho}{\beta}\right)}{\Gamma\left(\frac{\beta+\rho}{\beta}\right)} = \tau_5(-p_1/\beta), \qquad \lim_{u \downarrow -p_1/\beta} \tau_6(u) = 0. \tag{88}$$

From (75)–(80), (84), (86)–(88), it follows that

$$\begin{cases} m_1 + m_2 - m_5\tau_5(0) - m_6\tau_6(0) = 0, \\ m_1 e^{\sigma_1 b} + m_2 e^{\sigma_2 b} - m_3\tau_3(b) - m_4\tau_4(b) = 0, \\ m_1\sigma_1 + m_2\sigma_2 - m_5\tau_5'(0) - m_6\tau_6'(0) = 0, \\ m_1 p_1\sigma_1 e^{\sigma_1 b} + m_2 p_1 \sigma_2 e^{\sigma_2 b} - m_3(\gamma b + p_2)\tau_3'(b) - m_4(\gamma b + p_2)\tau_4'(b) = 0, \\ m_5\tau_5(-p_1/\beta) = \dfrac{\lambda}{\lambda+\rho}, \\ m_3\tau_3(\infty) + m_4\tau_4(\infty) = 0. \end{cases} \tag{89}$$

We let $\mathbf{\Pi}$ be the the matrix, defined as

$$\mathbf{\Pi} = \begin{pmatrix} 1 & 1 & 0 & 0 & -\tau_5(0) & -\tau_6(0) \\ e^{\sigma_1 b} & e^{\sigma_2 b} & -\tau_3(0) & -\tau_4(0) & 0 & 0 \\ \sigma_1 & \sigma_1 & 0 & 0 & -\tau_5'(0) & -\tau_6'(0) \\ p_1\sigma_1 e^{\sigma_1 b} & p_1\sigma_2 e^{\sigma_2 b} & -(\gamma b + p_2)\tau_3'(b) & -(\gamma b + p_2)\tau_4'(b) & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_5(-p_1/\beta) & 0 \\ 0 & 0 & \tau_3(\infty) & \tau_4(\infty) & 0 & 0 \end{pmatrix},$$

and the column vector $\vec{B}$ is defined as

$$\vec{B} = (0,0,0,0,\frac{\lambda}{\lambda+\rho},0)^T,$$

where, T denotes the transpose. Let $\mathbf{\Pi_i}$ denote the matrix except that the ith column of $\mathbf{\Pi}$ is replaced by $\vec{B}$. Then, we have

$$m_i = \frac{det(\mathbf{\Pi_i})}{det(\mathbf{\Pi})}, \quad i = 1,2,3,4,5,6,$$

where, $det(\cdot)$ denotes the determinant of a matrix. Hence, we have provided the closed form expressions for $\varphi_i(u;b), i = 1,2,3$.

6. The Time to Reach the Dividend Barrier

Let us explore how long it takes for the risk reserve process to attain the barrier b from the initial reserve u without absolute ruin. Let χ_b denote the first time that the risk reserve arrives at b, define

$$\Psi(u;b) = E[e^{-\rho\chi_b} I(\chi_b < T)|R_0^{u,b} = u], \ \ \rho > 0. \tag{90}$$

For notational convenience, we set

$$\Psi(u;b) = \begin{cases} \Psi_1(u;b), & 0 \le u \le b, \\ \Psi_2(u;b), & -p_1/\beta < u < 0. \end{cases} \tag{91}$$

Using a method similar to Theorem 1, we have
For $0 \le u \le b$,

$$p_1\Psi_1'(u;b) = (\lambda+\rho)\Psi_1(u;b) - \lambda\left[\int_0^u \Psi_1(u-y;b)dG(y) + \int_u^{u+p_1/\beta} \Psi_2(u-y;b)dG(y)\right], \tag{92}$$

and, for $-p_1/\beta < u < 0$,

$$(\beta u + p_1)\Psi_2'(u;b) = (\lambda+\rho)\Psi_2(u;b) - \lambda\Big[\int_0^{u+p_1/\beta} \Psi_2(u-y;b)dG(y)\Big], \tag{93}$$

with boundary conditions

$$\begin{aligned}
&\Psi_1(0+;b) = \Psi_2(0-;b),\\
&\Psi_1'(0+;b) = \Psi_2'(0-;b)\\
&\lim_{u\to -p_1/\beta} \Psi_2(u;b) = 0,\\
&\Psi_1(b;b) = 1.
\end{aligned}$$

Using the same methods as in Section 5, we can get the explicit expressions for $\Psi_1(u;b)$ and $\Psi_2(u;b)$ when the claim size is exponentially distributed with mean μ. We omit it.

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