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A Three-Dimensional Constrained Ordered Weighted Averaging Aggregation Problem with Lower Bounded Variables

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Abstract: We consider the constrained ordered weighted averaging (OWA) aggregation problem with a single constraint and lower bounded variables. For the three-dimensional constrained OWA aggregation problem with lower bounded variables, we present four types of solution depending on the number of zero elements. According to the computerized experiment we perform, the lower bounds can affect the solution types, thereby affecting the optimal solution of the three-dimensional constrained OWA aggregation problem with lower bounded variables.

Keywords: ordered weighted averaging (OWA) operators; constrained OWA aggregation problem; lower bounded variables

1. Introduction

An ordered weighted averaging (OWA) operator, proposed by Yager [1], is a general class of parametric aggregation operators that appears in many applications such as control, decision making, expert systems, fuzzy system, neural networks, regression analysis and risk analysis [2–6]. A citation-based survey of the literature in all types of optimization problems associated to OWA operators can be found in [7]. In 1996, Yager [8] investigated the constrained OWA aggregation problem [8–15] which is concerned with an optimization problem with an OWA operator. In particular, for the constrained OWA aggregation problem with a single constraint on the sum of all variables, Yager [8] presented the optimal solutions for the three-dimensional case. Furthermore, Carlsson, Fullér and Majlender [9] proposed a simple algorithm for obtaining the optimal solutions for any dimensions. Recently, Coroianu and Fullér [10] presented the optimal solution for the constrained OWA aggregation problem with a single constraint and any coefficients. However, in most practical problems the variables are usually bounded. This paper considers the three-dimensional constrained OWA aggregation problem with lower bounded variables.

The organization of this paper is as follows. Section 2 briefly reviews the constrained OWA aggregation problem. Section 3 discusses the constrained OWA aggregation problem with the same lower bounds. Section 4 presents the solution behaviors of three-dimensional constrained OWA aggregation problems with lower bounded variables. Section 5 outlines the design of the experiment and evaluates the optimal solution behaviors of the three-dimensional constrained OWA aggregation problems with the lower bounded variables. Finally, some concluding remarks are presented.

2. Constrained Ordered Weighted Averaging (OWA) Aggregation Problem

An OWA operator of dimension n is a mapping $F:\mathcal{R}^n \to \mathcal{R}$ that associates a weighting vector $\mathrm{W} = (w_1, w_2, \dots, w_n)$ satisfying:

$$w_1 + w_2 + \cdots + w_n = 1,\ 0 \le w_i \le 1,\ i = 1,2,\dots,n$$

and such that:

$$F(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} w_i y_i, \tag{1}$$

with y_i being the ith largest of $\{x_1, x_2, \dots, x_n\}$.

Consider the following constrained OWA aggregation problem:

$$\begin{aligned} &\text{Max } W^T Y \\ &\text{s.t. } \boldsymbol{AX} \le \boldsymbol{b} \\ &X \ge \boldsymbol{0} \end{aligned} \tag{2}$$

where the column vectors $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{W}$ and $\boldsymbol{b}$, and the $m \times n$ matrix $\boldsymbol{A}$ are:

$$\boldsymbol{X}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ \boldsymbol{Y}=\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},\ \boldsymbol{W}=\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},\ \boldsymbol{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

By introducing the $(n-1) \times n$ matrix:

$$\boldsymbol{G} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

and the column binary vectors $Z_i \in \{0,1\}^n$, $i = 1,2,\dots,n$, Yager [8] transformed the above non-linear programing problem to the following mixed integer linear programming (MIP) problem:

$$\begin{aligned} &\text{Max } W^T Y \\ &\text{s.t. } \boldsymbol{AX} \le \boldsymbol{b} \\ &\quad \mathbf{GY} \le 0 \\ &\quad y_i \mathcal{J} - X - M Z_i \le 0,\ i = 1,2,\dots,n-1 \\ &\quad y_n \mathcal{J} - X \le 0 \\ &\quad \mathcal{J}^T Z_i \le n - i,\ i = 1,2,\dots,n-1 \\ &\quad Z_i \in \{0,1\}^n,\ i = 1,2,\dots,n-1 \\ &\quad X \ge \boldsymbol{0} \end{aligned} \tag{3}$$

where M is a huge positive number and $\boldsymbol{\mathcal{J}}$ is the column vector with all elements equal 1.

For the MIP (3), the number of constraints is:

$$m + n - 1 + n^2 + n - 1 = m + n^2 + 2n - 2,$$

and the number of variables is:

$$n + n + (n-1)n = n^2 + n.$$

In the literature, the constrained OWA aggregation problem with a single constraint on the sum of all variables is as follows:

$$\begin{aligned}
&\text{Max } W^T Y \\
&\text{s.t. } \mathcal{J}^T \boldsymbol{X} \le 1 \\
&\quad \mathbf{GY} \le 0 \\
&\quad y_i \mathcal{J} - X - MZ_i \le 0,\ i = 1,2,\dots,n-1 \\
&\quad y_n \mathcal{J} - X \le 0 \\
&\quad \mathcal{J}^T Z_i \le n - i,\ i = 1,2,\dots,n-1 \\
&\quad Z_i \in \{0,1\}^n,\ i = 1,2,\dots,n-1 \\
&\quad X \ge \mathbf{0}.
\end{aligned} \tag{4}$$

If:

$$\boldsymbol{X}^* = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix}$$

is an optimal solution of (4), then,

$$\begin{bmatrix} x_{\sigma 1}^* \\ x_{\sigma 2}^* \\ \vdots \\ x_{\sigma n}^* \end{bmatrix}$$

is also the optimal solution, for some $\sigma \in S_n$, where S_n is the set of all permutations of the set $\{1,2,\dots,n\}$. To reduce the multiple solutions of the MIP (4), we introduce the following constraints:

$$Z_{i+1} \le Z_i,\ i = 1, 2, \dots, n-2.$$

by inspecting the *j*th element of the constraint $y_i \mathcal{J} - X - MZ_i \le 0$,

$$y_i - x_j - MZ_{ij} \le 0,$$

if $Z_{ij} = 0$, then:

$$y_i \le x_j.$$

From the optimal solution:

$$\mathcal{J}^T Z_i = n - i \text{ and } \mathcal{J}^T Z_{i+1} = n - i - 1,$$

it follows that:

$$Z_{i+1,j} = 0$$

so,

$$y_{i+1} \le x_j.$$

If $Z_{ij} = 1$, then no restriction is imposed on y_i, it implies that:

$$y_{i+1} \le x_j \text{ or } y_{i+1} > x_j$$

so $Z_{i+1,j} = 0$ or 1.

Therefore, the more efficient MIP is as follows:

$$
\begin{aligned}
&\text{Max } W^T Y \\
&\text{s.t. } \mathcal{I}^T \boldsymbol{X} \le 1 \\
&\quad \mathbf{GY} \le 0 \\
&\quad y_i \mathcal{I} - X - M Z_i \le 0,\ i = 1,2,\ldots,n-1 \\
&\quad y_n \mathcal{I} - X \le 0 \\
&\quad \mathcal{I}^T Z_i \le n - i,\ i = 1,2,\ldots,n-1 \\
&\quad Z_{i+1} \le Z_i,\ i = 1,2,\ldots,n-2 \\
&\quad Z_i \in \{0,1\}^n,\ i = 1,2,\ldots,n-1 \\
&\quad X \ge \mathbf{0}.
\end{aligned}
\tag{5}
$$

3. Constrained OWA Aggregation Problem with the Same Lower Bounds

In most practical problems the variables are usually bounded. A typical variable x_i is bounded from below by l_i and from above by u_i, where $l_i < u_i$ and $i = 1, 2, \ldots, n$. If we let $u_i = \infty$, we get the following constrained OWA aggregation problem with lower bounded variables:

$$
\begin{aligned}
&\text{Max } W^T Y \\
&\text{s.t. } \mathcal{I}^T \boldsymbol{X} \le 1 \\
&\quad \mathbf{GY} \le 0 \\
&\quad y_i \mathcal{I} - X - M Z_i \le 0,\ i = 1,2,\ldots,n-1 \\
&\quad y_n \mathcal{I} - X \le 0 \\
&\quad \mathcal{I}^T Z_i \le n - i,\ i = 1,2,\ldots,n-1 \\
&\quad Z_{i+1} \le Z_i,\ i = 1,2,\ldots,n-2 \\
&\quad Z_i \in \{0,1\}^n,\ i = 1,2,\ldots,n-1 \\
&\quad X \ge \boldsymbol{L}
\end{aligned}
\tag{6}
$$

where the column vector:

$$
\boldsymbol{L} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{bmatrix}.
$$

By using the change of variable:

$$
X' = X - \boldsymbol{L}
$$

the lower bound vector can be transformed into the zero vector. The constrained OWA aggregation problem with lower bounded variables is:

$$
\begin{aligned}
&\text{Max } W^T Y \\
&\text{s.t. } \mathcal{I}^T \boldsymbol{X}' \le 1 - \mathcal{I}^T \boldsymbol{L} \\
&\quad \mathbf{GY} \le 0 \\
&\quad y_i \mathcal{I} - X' - M Z_i \le \boldsymbol{L},\ i = 1,2,\ldots,n-1 \\
&\quad y_n \mathcal{I} - X \le 0 \\
&\quad \mathcal{I}^T Z_i \le n - i,\ i = 1,2,\ldots,n-1 \\
&\quad Z_{i+1} \le Z_i,\ i = 1,2,\ldots,n-2 \\
&\quad Z_i \in \{0,1\}^n,\ i = 1,2,\ldots,n-1 \\
&\quad X' \ge \mathbf{0}.
\end{aligned}
\tag{7}
$$

If $1-\mathcal{J}^T\boldsymbol{L}<0$, the constrained OWA aggregation problem has no feasible solution. If $1-\mathcal{J}^T\boldsymbol{L}=0$, the unique optimal solution is $X'^*=\boldsymbol{0}$, so:

$$X^*=\boldsymbol{L}.$$

It remains to discuss the case that $1-\mathcal{J}^T\boldsymbol{L}>0$. More precisely, the three dimensional constrained OWA aggregation problem with lower bounded variables is as follows:

$$\begin{aligned}
&\text{Max } F=w_1y_1+w_2y_2+w_3y_3\\
&\text{s.t. } x_1'+x_2'+x_3'\le 1-l_1-l_2-l_3\\
&\quad y_2-y_1\le 0\\
&\quad y_3-y_2\le 0\\
&\quad y_3-x_1'\le l_1\\
&\quad y_3-x_2'\le l_2\\
&\quad y_3-x_3'\le l_3\\
&\quad y_2-x_1'-\mathrm{M}Z_{21}\le l_1\\
&\quad y_2-x_2'-\mathrm{M}Z_{22}\le l_2\\
&\quad y_2-x_3'-\mathrm{M}Z_{23}\le l_3\\
&\quad Z_{21}+Z_{22}+Z_{23}\le 1\\
&\quad y_1-x_1'-\mathrm{M}Z_{11}\le l_1\\
&\quad y_1-x_2'-\mathrm{M}Z_{12}\le l_2\\
&\quad y_1-x_3'-\mathrm{M}Z_{13}\le l_3\\
&\quad Z_{11}+Z_{12}+Z_{13}\le 2\\
&\quad Z_{21}\le Z_{11}\\
&\quad Z_{22}\le Z_{12}\\
&\quad Z_{23}\le Z_{13}\\
&\quad x_1',x_2',x_3'\ge 0,\ Z_{21},Z_{22},Z_{23},Z_{11},Z_{12},Z_{13}\in\{0,1\}.
\end{aligned}\tag{8}$$

For the special case that the same lower bounds $l_i=l,\ i=1,2,\ldots,n$, by the observing that the ith largest ($x_{\sigma i}$) of $\{x_1,x_2,\ldots,x_n\}$ is the same variable of the ith largest ($x'_{\sigma i}$) of $\{x'_1,x'_2,\ldots,x'_n\}$, let:

$$x_i''=\frac{x_i'}{1-nl}$$

it follows that the optimal solution is the same as that of the constrained OWA aggregation problem [8]. We establish the main results described as follows:

Theorem 1. *Consider the three-dimensional constrained OWA aggregation problem (8).*

(a) If $w_1=\max\limits_{i=1,2,3} w_i$, then the optimal solutions are $X''^*=\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\0\\1\end{bmatrix}$, $X^*=\begin{bmatrix}1-2l\\l\\l\end{bmatrix},\begin{bmatrix}l\\1-2l\\l\end{bmatrix}$ or $\begin{bmatrix}l\\l\\1-2l\end{bmatrix}$, $Y^*=\begin{bmatrix}1-2l\\l\\l\end{bmatrix}$ and $F=w_1+l-3w_1l$.

(b) If $w_2 = \max\limits_{i=1,2,3} w_i$, then the optimal solutions are $X''^* = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$, $X^* = \begin{bmatrix} (1-l)/2 \\ (1-l)/2 \\ l \end{bmatrix}, \begin{bmatrix} (1-l)/2 \\ l \\ (1-l)/2 \end{bmatrix}$ or $\begin{bmatrix} l \\ (1-l)/2 \\ (1-l)/2 \end{bmatrix}$, $Y^* = \begin{bmatrix} (1-l)/2 \\ (1-l)/2 \\ l \end{bmatrix}$ and $F = (1 - w_3 - l + 3w_3 l)/2$ for $w_1 + w_2 \geq 2w_3$, and $X''^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, $X^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, $Y^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, and $F = 1/3$ for $w_1 + w_2 \leq 2w_3$.

(c) If $w_3 = \max\limits_{i=1,2,3} w_i$, then the optimal solutions are $X''^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $X^* = \begin{bmatrix} 1-2l \\ l \\ l \end{bmatrix}, \begin{bmatrix} l \\ 1-2l \\ l \end{bmatrix}$ or $\begin{bmatrix} l \\ l \\ 1-2l \end{bmatrix}$, $Y^* = \begin{bmatrix} 1-2l \\ l \\ l \end{bmatrix}$ and $F = w_1 + l - 3w_1 l$ for $w_2 + w_3 \leq 2w_1$, and $X''^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, $X^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, $Y^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ and $F = 1/3$ for $w_2 + w_3 \geq 2w_1$.

Proof. For the three-dimensional constrained OWA aggregation problem, three cases are considered. Firstly, if:

$$w_1 = \max_{i=1,2,3} w_i$$

the optimal solutions are:

$$X''^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So

$$X^* = \begin{bmatrix} 1-2l \\ l \\ l \end{bmatrix}, \begin{bmatrix} l \\ 1-2l \\ l \end{bmatrix} \text{ or } \begin{bmatrix} l \\ l \\ 1-2l \end{bmatrix}, Y^* = \begin{bmatrix} 1-2l \\ l \\ l \end{bmatrix},$$

and the most favorable value is:

$$F = w_1 + l - 3w_1 l.$$

Secondly, if:

$$w_2 = \max_{i=1,2,3} w_i$$

two subcases are considered. If:

$$w_1 + w_2 \geq 2w_3,$$

then the optimal solutions are:

$$X''^* = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

so:

$$X^* = \begin{bmatrix} (1-l)/2 \\ (1-l)/2 \\ l \end{bmatrix}, \begin{bmatrix} (1-l)/2 \\ l \\ (1-l)/2 \end{bmatrix} \text{ or } \begin{bmatrix} l \\ (1-l)/2 \\ (1-l)/2 \end{bmatrix}, \; Y^* = \begin{bmatrix} (1-l)/2 \\ (1-l)/2 \\ l \end{bmatrix},$$

and the largest objective function value is:

$$F = (1 - w_3 - l + 3w_3 l)/2.$$

If:

$$w_1 + w_2 \leq 2w_3,$$

then the optimal solutions are:

$$X''^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

so:

$$X^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \; Y^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \text{ and } F = 1/3.$$

Finally, if:

$$w_3 = \max_{i=1,2,3} w_i$$

two subcases are considered. If:

$$w_2 + w_3 \leq 2w_1,$$

then the optimal solutions are:

$$X''^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so:

$$X^* = \begin{bmatrix} 1-2l \\ l \\ l \end{bmatrix}, \begin{bmatrix} l \\ 1-2l \\ l \end{bmatrix} \text{ or } \begin{bmatrix} l \\ l \\ 1-2l \end{bmatrix}, \; Y^* = \begin{bmatrix} 1-2l \\ l \\ l \end{bmatrix},$$

and:

$$F = w_1 + l - 3w_1 l.$$

If:

$$w_2 + w_3 \geq 2w_1,$$

then the optimal solutions are:

$$X''^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

so:

$$X^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix},\ Y^* = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \text{ and } F = 1/3. \square$$

4. Constrained OWA Aggregation Problem with Lower Bounded Variables

For simplicity, we consider the three-dimensional constrained OWA aggregation problem with lower bounded variables. From the optimal solution of the first constraint of the model (8):

$$x'_1 + x'_2 + x'_3 = 1 - l_1 - l_2 - l_3 \quad (9)$$

there are four types (I, II, III and IV) of (x'_1, x'_2, x'_3) depending on the number of zero elements. The number of zero elements is two for type I, one for types II and III, and zero for type III. The solutions of $\boldsymbol{X'} = (x'_1, x'_2, x'_3)$ and $\boldsymbol{Y} = (y_1, y_2, y_3)$ for the three-dimensional constrained OWA aggregation problem with lower bounded variables (8) are described as follows:

Theorem 2. *Consider the three-dimensional constrained OWA aggregation problem with lower bounded variables (8). For type I solution, there are three forms* $(1 - l_1 - l_2 - l_3, 0,0)$, $(0,1 - l_1 - l_2 - l_3, 0)$, $(0,0,1 - l_1 - l_2 - l_3)$ *for* $\boldsymbol{X'}$ *and six forms* $(1 - l_2 - l_3, l_2, l_3)$, $(1 - l_2 - l_3, l_3, l_2)$, $(1 - l_1 - l_3, l_1, l_3)$, $(1 - l_1 - l_3, l_3, l_1)$, $(1 - l_1 - l_2, l_1, l_2)$, $(1 - l_1 - l_2, l_2, l_1)$ *for* $\boldsymbol{Y}$. *For type II, there are three forms* $\left(\frac{1-2l_1-l_3}{2}, \frac{1-2l_2-l_3}{2}, 0\right)$, $\left(\frac{1-2l_1-l_2}{2}, 0, \frac{1-l_2-2l_3}{2}\right)$, $(0, \frac{1-l_1-2l_2}{2}, \frac{1-l_1-2l_3}{2})$ *for* $\boldsymbol{X'}$ *and six forms* $(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$, $\left(l_3, \frac{1-l_3}{2}, \frac{1-l_3}{2}\right)$, $\left(\frac{1-l_2}{2}, \frac{1-l_2}{2}, l_2\right)$, $\left(l_2, \frac{1-l_2}{2}, \frac{1-l_2}{2}\right)$, $\left(\frac{1-l_1}{2}, \frac{1-l_1}{2}, l_1\right)$, $(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$ *for* $\boldsymbol{Y}$. *For type III, there are six forms* $(l_3 - l_1, 1 - l_2 - 2l_3, 0)$, $(1 - l_1 - 2l_3, l_3 - l_2, 0)$, $(l_2 - l_1, 0,1 - 2l_2 - l_3)$, $(1 - l_1 - 2l_2, 0, l_2 - l_3)$, $(0, l_1 - l_2, 1 - 2l_1 - l_3)$, $(0,1 - 2l_1 - l_2, l_1 - l_3)$ *for* $\boldsymbol{X'}$ *and six forms* $(l_3, l_3, 1 - 2l_3)$, $(1 - 2l_3, l_3, l_3)$, $(l_2, l_2, 1 - 2l_2)$, $(1 - 2l_2, l_2, l_2)$, $(l_1, l_1, 1 - 2l_1)$, $(1 - 2l_1, l_1, l_1)$ *for* $\boldsymbol{Y}$. *For type IV, there are only one form* $(1/3 - l_1, 1/3 - l_2, 1/3 - l_3)$ *for* $\boldsymbol{X'}$ *and one form* $(1/3,1/3,1/3)$ *for* $\boldsymbol{Y}$.

Proof. For type I, the possible values of (x'_1, x'_2, x'_3) are:

$$(1 - l_1 - l_2 - l_3, 0,0),\ (0,1 - l_1 - l_2 - l_3, 0) \text{ and } (0,0,1 - l_1 - l_2 - l_3).$$

For the case of $(x'_1, x'_2, x'_3) = (1 - l_1 - l_2 - l_3, 0,0)$, we have:

$$(x'_1, x'_2, x'_3) = (1 - l_2 - l_3, l_2, l_3)$$

$$(y_1, y_2, y_3) = (1 - l_2 - l_3, l_2, l_3) \text{ or } (1 - l_2 - l_3, l_3, l_2).$$

For the case of $(y_1, y_2, y_3) = (1 - l_2 - l_3, l_2, l_3)$, if:

$$l_1 + l_2 + l_3 \leq 1,\ l_2 \geq l_3 \text{ and } 2l_2 + l_3 \leq 1,$$

then:

$$(x_1, x_2, x_3) = (1 - l_2 - l_3, l_2, l_3) \text{ and } (y_1, y_2, y_3) = (1 - l_2 - l_3, l_2, l_3)$$

is solution of MIP (8) and the objective value is:

$$F = w_1 + l_2(-w_1 + w_2) + l_3(-w_1 + w_3).$$

Since $w_1 + w_2 + w_3 = 1$, we can express the objective value F in only two weights. Then the other three formats of F are:

$$F = w_1 + l_2(-w_1 + w_2) + l_3(1 - 2w_1 - w_2),$$

$$F = w_1 + l_2(1 - 2w_1 - w_3) + l_3(-w_1 + w_3)$$

and:

$$F = 1 - w_2 - w_3 + l_2(-1 + 2w_1 + w_3) + l_3(-1 + w_2 + 2w_3).$$

Among these four formats, the explicit format adopted is $F = w_1 + l_2(-w_1 + w_2) + l_3(-w_1 + w_3)$ which is the most compact one.

If:

$$l_1 + l_2 + l_3 \le 1,\ l_2 \le l_3 \text{ and } l_2 + 2l_3 \le 1,$$

then:

$$(x_1, x_2, x_3) = (1 - l_2 - l_3, l_2, l_3) \text{ and } (y_1, y_2, y_3) = (1 - l_2 - l_3, l_3, l_2)$$

is the solution of MIP (8) and the objective value is:

$$F = w_1 + l_2(-w_1 + w_3) + l_3(-w_1 + w_2).$$

In Table 1, we display the possible solutions (x_1', x_2', x_3'), (x_1, x_2, x_3), (y_1, y_2, y_3), F and the conditions for the different choices of the type I.

We now consider that the number of zero elements is one. The possible values of (x_1', x_2', x_3') are:

$$(x_1', x_2', 0),\ (x_1', 0, x_3') \text{ and } (0, x_2', x_3').$$

For the case of $(x_1', x_2', 0)$, we have:

$$(x_1, x_2, x_3) = (x_1' + l_1, x_2' + l_2, l_3).$$

At optimal, the possible choices of (x_1', x_2', x_3') are:

$$x_1' + l_1 = x_2' + l_2,\ x_1' + l_1 = l_3 \text{ or } x_2' + l_2 = l_3.$$

We choose $x_1' + l_1 = x_2' + l_2$ for type II, and $x_1' + l_1 = l_3$ or $x_2' + l_2 = l_3$ for type III. For $x_1' + l_1 = x_2' + l_2$, from (9), it follows that:

$$(x_1', x_2', x_3') = (\tfrac{1-2l_1-l_3}{2}, \tfrac{1-2l_2-l_3}{2}, 0).$$

so:

$$(x_1, x_2, x_3) = (\tfrac{1-l_3}{2}, \tfrac{1-l_3}{2}, l_3) \text{ and } (y_1, y_2, y_3) = (\tfrac{1-l_3}{2}, \tfrac{1-l_3}{2}, l_3) \text{ or } (l_3, \tfrac{1-l_3}{2}, \tfrac{1-l_3}{2}).$$

More precisely, if:

$$l_3 \le 1/3,\ 2l_2 + l_3 \le 1 \text{ and } 2l_1 + l_3 \le 1,$$

then:

$$(x_1, x_2, x_3) = (\tfrac{1-l_3}{2}, \tfrac{1-l_3}{2}, l_3) \text{ and } (y_1, y_2, y_3) = (\tfrac{1-l_3}{2}, \tfrac{1-l_3}{2}, l_3)$$

is the solution of MIP (8) and the objective value is:

$$F = \tfrac{1-w_3-l_3+3l_3w_3}{2}.$$

If:

$$l_3 \ge 1/3,\ 2l_2 + l_3 \le 1 \text{ and } 2l_1 + l_3 \le 1,$$

then:

$$(x_1, x_2, x_3) = (\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3) \text{ and } (y_1, y_2, y_3) = (l_3, \frac{1-l_3}{2}, \frac{1-l_3}{2})$$

is the solution of MIP (8) and the objective value is:

$$F = \frac{1-w_1-l_3+3l_3w_1}{2}.$$

For other cases of $(x_1', 0, x_3')$, $(0, x_2', x_3')$, the solutions and conditions are displayed in Table 2.

For type III, we have two possible $x_1' + l_1 = l_3$ or $x_2' + l_2 = l_3$. For $x_1' + l_1 = l_3$, from (9), it follows that if:

$$l_3 \geq l_1,\ l_3 \geq 1/3 \text{ and } l_2 + 2l_3 \leq 1,$$

then:

$(x_1', x_2', x_3') = (l_3 - l_1, 1 - l_2 - 2l_3, 0)$, $(x_1, x_2, x_3) = (l_3, 1 - 2l_3, l_3)$, $(y_1, y_2, y_3) = (l_3, l_3, 1 - 2l_3)$ and

$$F = w_3 + l_3 - 3l_3w_3.$$

If:

$$l_3 \geq l_1,\ l_3 \leq 1/3 \text{ and } l_2 + 2l_3 \leq 1$$

then:

$(x_1', x_2', x_3') = (l_3 - l_1, 1 - l_2 - 2l_3, 0)$, $(x_1, x_2, x_3) = (l_3, 1 - 2l_3, l_3)$, $(y_1, y_2, y_3) = (1 - 2l_3, l_3, l_3)$ and

$$F = w_1 + l_3 - 3l_3w_1.$$

For different choices of type III, detailed results are presented in Table 3.

For type IV, from (9), it follows that:

$$(x_1', x_2', x_3') = (1/3 - l_1, 1/3 - l_2, 1/3 - l_3).$$

So, if:

$$l_1 \leq 1/3,\ l_2 \leq 1/3 \text{ and } l_3 \leq 1/3$$

then the solution of MIP (8) is:

$$(x_1, x_2, x_3) = (1/3, 1/3, 1/3) \text{ and } (y_1, y_2, y_3) = (1/3, 1/3, 1/3) \text{ and } F = 1/3. \square$$

Table 1. The values of (x_1', x_2', x_3'), (x_1, x_2, x_3), (y_1, y_2, y_3), F and the conditions for type I.

Type	(x_1', x_2', x_3')	(x_1, x_2, x_3)	(y_1, y_2, y_3)	F	Conditions
I1	$(1 - l_1 - l_2 - l_3, 0, 0)$	$(1 - l_2 - l_3, l_2, l_3)$	$(1 - l_2 - l_3, l_2, l_3)$	$w_1 + l_2(-w_1 + w_2) + l_3(-w_1 + w_3)$	$l_1 + l_2 + l_3 \leq 1$, $l_2 \geq l_3$, $2l_2 + l_3 \leq 1$
I2	$(1 - l_1 - l_2 - l_3, 0, 0)$	$(1 - l_2 - l_3, l_2, l_3)$	$(1 - l_2 - l_3, l_3, l_2)$	$w_1 + l_2(-w_1 + w_3) + l_3(-w_1 + w_2)$	$l_1 + l_2 + l_3 \leq 1$, $l_2 \leq l_3$, $l_2 + 2l_3 \leq 1$
I3	$(0, 1 - l_1 - l_2 - l_3, 0)$	$(l_1, 1 - l_1 - l_3, l_3)$	$(1 - l_1 - l_3, l_1, l_3)$	$w_1 + l_1(-w_1 + w_2) + l_3(-w_1 + w_3)$	$l_1 + l_2 + l_3 \leq 1$, $l_1 \geq l_3$, $2l_1 + l_3 \leq 1$
I4	$(0, 1 - l_1 - l_2 - l_3, 0)$	$(l_1, 1 - l_1 - l_3, l_3)$	$(1 - l_1 - l_3, l_3, l_1)$	$w_1 + l_1(-w_1 + w_3) + l_3(-w_1 + w_2)$	$l_1 + l_2 + l_3 \leq 1$, $l_1 \leq l_3$, $l_1 + 2l_3 \leq 1$
I5	$(0, 0, 1 - l_1 - l_2 - l_3)$	$(l_1, l_2, 1 - l_1 - l_2)$	$(1 - l_1 - l_2, l_1, l_2)$	$w_1 + l_1(-w_1 + w_2) + l_2(-w_1 + w_3)$	$l_1 + l_2 + l_3 \leq 1$, $l_1 \geq l_2$, $2l_1 + l_2 \leq 1$
I6	$(0, 0, 1 - l_1 - l_2 - l_3)$	$(l_1, l_2, 1 - l_1 - l_2)$	$(1 - l_1 - l_2, l_2, l_1)$	$w_1 + l_1(-w_1 + w_3) + l_2(-w_1 + w_2)$	$l_1 + l_2 + l_3 \leq 1$, $l_1 \leq l_2$, $l_1 + 2l_2 \leq 1$

Table 2. The values of (x_1', x_2', x_3'), (x_1, x_2, x_3), (y_1, y_2, y_3), F and the conditions for type II.

Type	(x_1', x_2', x_3')	(x_1, x_2, x_3)	(y_1, y_2, y_3)	F	Conditions
II1	$(\frac{1-2l_1-l_3}{2}, \frac{1-2l_2-l_3}{2}, 0)$	$(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$	$(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$	$\frac{1-w_3-l_3+3l_3w_3}{2}$	$l_3 \le 1/3,\ 2l_2 + l_3 \le 1,\ 2l_1 + l_3 \le 1$
II2	$(\frac{1-2l_1-l_3}{2}, \frac{1-2l_2-l_3}{2}, 0)$	$(\frac{1-l_3}{2}, \frac{1-l_3}{2}, l_3)$	$(l_3, \frac{1-l_3}{2}, \frac{1-l_3}{2})$	$\frac{1-w_1-l_3+3l_3w_1}{2}$	$l_3 \ge 1/3,\ 2l_2 + l_3 \le 1,\ 2l_1 + l_3 \le 1$
II3	$\left(\frac{1-2l_1-l_2}{2}, 0, \frac{1-l_2-2l_3}{2}\right)$	$(\frac{1-l_2}{2}, l_2, \frac{1-l_2}{2})$	$(\frac{1-l_2}{2}, \frac{1-l_2}{2}, l_2)$	$\frac{1-w_3-l_2+3l_2w_3}{2}$	$l_2 \le 1/3,\ 2l_1 + l_2 \le 1,\ l_2 + 2l_3 \le 1$
II4	$(\frac{1-2l_1-l_2}{2}, 0, \frac{1-l_2-2l_3}{2})$	$(\frac{1-l_2}{2}, l_2, \frac{1-l_2}{2})$	$(l_2, \frac{1-l_2}{2}, \frac{1-l_2}{2})$	$\frac{1-w_1-l_2+3l_2w_1}{2}$	$l_2 \ge 1/3,\ 2l_1 + l_2 \le 1,\ l_2 + 2l_3 \le 1$
II5	$\left(0, \frac{1-l_1-2l_2}{2}, \frac{1-l_1-2l_3}{2}\right)$	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	$(\frac{1-l_1}{2}, \frac{1-l_1}{2}, l_1)$	$\frac{1-w_3-l_1+3l_1w_3}{2}$	$l_1 \le 1/3,\ l_1 + 2l_2 \le 1,\ l_1 + 2l_3 \le 1$
II6	$(0, \frac{1-l_1-2l_2}{2}, \frac{1-l_1-2l_3}{2})$	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	$(l_1, \frac{1-l_1}{2}, \frac{1-l_1}{2})$	$\frac{1-w_1-l_1+3l_1w_1}{2}$	$l_1 \ge 1/3,\ l_1 + 2l_2 \le 1,\ l_1 + 2l_3 \le 1$

Table 3. The values of (x_1', x_2', x_3'), (x_1, x_2, x_3), (y_1, y_2, y_3), F and the conditions for type III.

Type	(x_1', x_2', x_3')	(x_1, x_2, x_3)	(y_1, y_2, y_3)	F	Conditions
III1	$(l_3-l_1, 1-l_2-2l_3, 0)$	$(l_3, 1-2l_3, l_3)$	$(l_3, l_3, 1-2l_3)$	$w_3+l_3-3l_3w_3$	$l_3 \ge l_1,\ l_3 \ge 1/3,\ l_2+2l_3 \le 1$
III2	$(l_3-l_1, 1-l_2-2l_3, 0)$	$(l_3, 1-2l_3, l_3)$	$(1-2l_3, l_3, l_3)$	$w_1+l_3-3l_3w_1$	$l_3 \ge l_1,\ l_3 \le 1/3,\ l_2+2l_3 \le 1$
III3	$(1-l_1-2l_3, l_3-l_2, 0)$	$(1-2l_3, l_3, l_3)$	$(l_3, l_3, 1-2l_3)$	$w_3+l_3-3l_3w_3$	$l_3 \ge l_2,\ l_3 \ge 1/3,\ l_1+2l_3 \le 1$
III4	$(1-l_1-2l_3, l_3-l_2, 0)$	$(1-2l_3, l_3, l_3)$	$(1-2l_3, l_3, l_3)$	$w_1+l_3-3l_3w_1$	$l_3 \ge l_2,\ l_3 \le 1/3,\ l_1+2l_3 \le 1$
III5	$(l_2-l_1, 0, 1-2l_2-l_3)$	$(l_2, l_2, 1-2l_2)$	$(l_2, l_2, 1-2l_2)$	$w_3+l_2-3l_2w_3$	$l_2 \ge l_1,\ l_2 \ge 1/3,\ 2l_2+l_3 \le 1$
III6	$(l_2-l_1, 0, 1-2l_2-l_3)$	$(l_2, l_2, 1-2l_2)$	$(1-2l_2, l_2, l_2)$	$w_1+l_2-3l_2w_1$	$l_2 \ge l_1,\ l_2 \le 1/3,\ 2l_2+l_3 \le 1$
III7	$(1-l_1-2l_2, 0, l_2-l_3)$	$(1-2l_2, l_2, l_2)$	$(l_2, l_2, 1-2l_2)$	$w_3+l_2-3l_2w_3$	$l_2 \ge l_3,\ l_2 \ge 1/3,\ l_1+2l_2 \le 1$
III8	$(1-l_1-2l_2, 0, l_2-l_3)$	$(1-2l_2, l_2, l_2)$	$(1-2l_2, l_2, l_2)$	$w_1+l_2-3l_2w_1$	$l_2 \ge l_3,\ l_2 \le 1/3,\ l_1+2l_2 \le 1$
III9	$(0, l_1-l_2, 1-2l_1-l_3)$	$(l_1, l_1, 1-2l_1)$	$(l_1, l_1, 1-2l_1)$	$w_3+l_1-3l_1w_3$	$l_2 \le l_1,\ l_1 \ge 1/3,\ 2l_1+l_3 \le 1$
III10	$(0, l_1-l_2, 1-2l_1-l_3)$	$(l_1, l_1, 1-2l_1)$	$(1-2l_1, l_1, l_1)$	$w_1+l_1-3l_1w_1$	$l_2 \le l_1,\ l_1 \le 1/3,\ 2l_1+l_3 \le 1$
III11	$(0, 1-2l_1-l_2, l_1-l_3)$	$(l_1, 1-2l_1, l_1)$	$(l_1, l_1, 1-2l_1)$	$w_3+l_1-3l_1w_3$	$l_3 \le l_1,\ l_1 \ge 1/3,\ 2l_1+l_2 \le 1$
III12	$(0, 1-2l_1-l_2, l_1-l_3)$	$(l_1, 1-2l_1, l_1)$	$(1-2l_1, l_1, l_1)$	$w_1+l_1-3l_1w_1$	$l_3 \le l_1,\ l_1 \le 1/3,\ 2l_1+l_2 \le 1$

For the three-dimensional constrained OWA aggregation problem with lower bounded variables (8), there are three forms for $\boldsymbol{X'}$ and six forms for $\boldsymbol{Y}$ for Type I solution. For type II, there are three forms for $\boldsymbol{X'}$ and six forms for $\boldsymbol{Y}$. For type III, there are six forms for $\boldsymbol{X'}$ and six forms for $\boldsymbol{Y}$. Type IV is that the number of zero elements of solution is zero, there are only one form for $\boldsymbol{X'}$ and one form for $\boldsymbol{Y}$.

We illustrate some concrete examples with various (l_1, l_2, l_3) and (w_1, w_2, w_3).

Example 1. *For the case of* $w_1 > \max_{i=2,3} w_i$, *we perform an exhaustive search for* $l_i \in \{-1, -0.9, -0.8, \ldots, 1\}$ *and* $w_i \in \{0, 0.1, 0.2, \ldots, 1\}$, $i = 1,2,3$. *The first type I is* $(l_1, l_2, l_3) = (-1, -1, -1)$ *and* $(w_1, w_2, w_3) = (0.9, 0, 0.1)$. *The optimal solution is* $(y_1, y_2, y_3) = (3, -1, -1)$, $(x_1', x_2', x_3') = (4, 0, 0)$, $(x_1, x_2, x_3) = (3, -1, -1)$ *and* $F = 2.6$.

Example 2. Consider the case of $w_2 > \max_{i=1,3} w_i$. Applying an exhaustive search for $l_i \in \{-1, -0.9, -0.8, \ldots, 1\}$ and $w_i \in \{0, 0.1, 0.2, \ldots, 1\}$, $i = 1,2,3$, the value of $(l_1, l_2, l_3) = (-1, -1, 1)$ and $(w_1, w_2, w_3) = (0, 0.9, 0.1)$ is the first one satisfies type I. The optimal solution is $(y_1, y_2, y_3) = (1, 1, -1)$, $(x_1', x_2', x_3') = (0, 2, 0)$, $(x_1, x_2, x_3) = (-1, 1, 1)$ and $F = 0.8$. For $(l_1, l_2, l_3) = (-1, -1, -1)$ and $(w_1, w_2, w_3) = (0, 0.9, 0.1)$, the type II solution is $(y_1, y_2, y_3) = (1, 1, -1)$, $(x_1', x_2', x_3') = (0, 2, 2)$, $(x_1, x_2, x_3) = (-1, 1, 1)$ and $F = 0.8$. For $(l_1, l_2, l_3) = (-1, -1, 0.4)$ and $(w_1, w_2, w_3) = (0, 0.6, 0.4)$, the type III solution is $(y_1, y_2, y_3) = (0.4, 0.4, 0.2)$, $(x_1', x_2', x_3') = (1.2, 1.4, 0)$, $(x_1, x_2, x_3) = (0.2, 0.4, 0.4)$ and

$\mathrm{F}=0.32$. For $(l_1,l_2,l_3)=(-1,-1,-1)$ and $(w_1,w_2,w_3)=(0,0.6,0.4)$, the type IV solution is $(y_1,y_2,y_3)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $(x_1',x_2',x_3')=(4/3,4/3,4/3)$, $(x_1,x_2,x_3)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ and $\mathrm{F}=1/3$.

Example 3. *Consider the case of* $w_3>\max\limits_{i=2,3} w_i$. *For* $l_i\in\{-1,-0.9,-0.8,\dots,1\}$ *and* $w_i\in\{0,0.1,0.2,\dots,1\}$, $i=1,2,3$, *the value of* $(l_1,l_2,l_3)=(-1,-1,-1)$ *and* $(w_1,w_2,w_3)=(0.4,0,0.6)$ *is the first one satisfies type I. The optimal solution is* $(y_1,y_2,y_3)=(3,-1,-1)$, $(x_1',x_2',x_3')=(4,0,0)$, $(x_1,x_2,x_3)=(3,-1,-1)$ *and* $F=0.6$. *For* $(l_1,l_2,l_3)=(-1,-1,0.4)$ *and* $(w_1,w_2,w_3)=(0,0.1,0.9)$, *the type II solution is* $(y_1,y_2,y_3)=(0.4,0.3,0.3)$, $(x_1',x_2',x_3')=(1.3,1.3,0)$, $(x_1,x_2,x_3)=(0.3,0.3,0.4)$ *and* $F=0.3$. *For* $(l_1,l_2,l_3)=(-1,-0.9,-0.8)$ *and* $(w_1,w_2,w_3)=(0.4,0,0.6)$, *the type III solution is* $(y_1,y_2,y_3)=(2.8,-0.9,-0.9)$, $(x_1',x_2',x_3')=(0.1,0,3.6)$, $(x_1,x_2,x_3)=(-0.9,-0.9,-2.8)$ *and* $F=0.58$. *For* $(l_1,l_2,l_3)=(-1,-1,-1)$ *and* $(w_1,w_2,w_3)=(0,0.1,0.9)$, *the type IV solution is* $(y_1,y_2,y_3)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $(x_1',x_2',x_3')=(4/3,4/3,4/3)$, $(x_1,x_2,x_3)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ *and* $F=1/3$.

Minimizing the objective function of the constrained OWA aggregation problem with bounded variables is also important. One interesting model is the constrained OWA aggregation problem with upper bounded variables described as follows:

$$\begin{aligned}
&\text{Min } W^TY\\
&\text{s.t. } \mathcal{J}^T\boldsymbol{X}\leq 1\\
&\quad \mathbf{GY}\leq 0\\
&\quad y_i\mathcal{J}-X-MZ_i\leq 0,\ i=1,2,\dots,n-1\\
&\quad y_n\mathcal{J}-X\leq 0\\
&\quad \mathcal{J}^TZ_i\leq n-i,\ i=1,2,\dots,n-1\\
&\quad Z_{i+1}\leq Z_i,\ i=1,2,\dots,n-2\\
&\quad Z_i\in\{0,1\}^n,\ i=1,2,\dots,n-1\\
&\quad X\leq \boldsymbol{U}
\end{aligned}\tag{10}$$

where the column vector:

$$\boldsymbol{U}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}.$$

By using the change of variable:

$$X'=U-\boldsymbol{X}\ \ y_i=-y'_{n+1-i}\ \text{and}\ \boldsymbol{Y}'=\begin{bmatrix}y'_n\\y'_{n-1}\\\vdots\\y'_1\end{bmatrix},$$

minimizing the objective function of the constrained OWA aggregation problem with upper bounded variables is:

$$\begin{aligned}
&\text{Max } W^TY'\\
&\text{s.t. } \mathcal{J}^T\geq \mathcal{J}^T\boldsymbol{U}-1\\
&\quad \mathbf{GY}'\leq 0\\
&\quad y_1'\mathcal{J}-X'\geq -\boldsymbol{U}\\
&\quad y_i'\mathcal{J}-X'+MZ_i\geq -\boldsymbol{U},\ i=2,3,\dots,n\\
&\quad \mathcal{J}^TZ_i\leq n-i,\ i=1,2,\dots,n-1\\
&\quad Z_{i+1}\leq Z_i,\ i=1,2,\dots,n-2
\end{aligned}\tag{11}$$

$Z_i \in \{0,1\}^n,\ i = 1,2,\dots,n-1$

$X' \geq \mathbf{0}$

If $\mathcal{J}^T\boldsymbol{U} - 1 < 0$, the constrained OWA aggregation problem has unbounded solution. If $\mathcal{J}^T\boldsymbol{U} - 1 = 0$, the unique optimal solution is $X'^* = \mathbf{0}$, so:

$$X^* = \boldsymbol{U}.$$

For the case of $1 - \mathcal{J}^T\boldsymbol{L} > 0$, the similar results as Theorem 2 can be derived.

5. Numerical Results

To evaluate the optimal solution behaviors of the three-dimensional constrained OWA aggregation problem with lower bounded variables, we present some numerical experiments.

In Table 4, we display the number of solution type I, II, III and IV for different choices of the weights and the lower bounds. To this end, we consider four types of solution forms I, II, III and IV and six types of weights:

$$w_1 = \max_{i=1,2,3} w_i,\ w_2 = \max_{i=1,2,3} w_i,\ w_3 = \max_{i=1,2,3} w_i,\ w_1 > \max_{i=2,3} w_i,\ w_2 > \max_{i=1,3} w_i,\ w_3 > \max_{i=2,3} w_i.$$

Each cell is associated to a pair (W, S) and gives the number of different instances of $(l_1, l_2, l_3, w_1, w_2, w_3)$ satisfying weight (W) and solution (S) conditions. We restrict our attention to:

$$W \in \left\{w_1 = \max_{i=1,2,3} w_i, w_2 = \max_{i=1,2,3} w_i,\ w_3 = \max_{i=1,2,3} w_i, w_1 > \max_{i=2,3} w_i,\ w_2 > \max_{i=1,3} w_i, w_3 > \max_{i=2,3} w_i\right\},\ S \in$$

$$\{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}\},\ l_i \in \{-1, -0.9, -0.8, \dots, 1\} \text{ and } w_i \in \{0, 0.1, 0.2, \dots, 1\},\ i = 1, 2, 3.$$

For each cell, the instances $(l_1, l_2, l_3, w_1, w_2, w_3)$ of the test problem are 179,760 for $w_1 = \max_{i=1,2,3} w_i,\ w_2 = \max_{i=1,2,3} w_i,\ w_3 = \max_{i=1,2,3} w_i$ and 119,840 for $w_1 > \max_{i=2,3} w_i,\ w_2 > \max_{i=1,3} w_i,\ w_3 > \max_{i=2,3} w_i$. The total instances of the test problem are 898,800. An examination of the table reveals that the type IV is not optimal solution for $w_1 = \max_{i=1,2,3} w_i$. In particular, for $w_1 > \max_{i=2,3} w_i$, the optimal solution type is always type I solution. If the lower bounds $(l_1, l_2, l_3) = (0,0,0)$, then the optimal solution is types II, III and IV for $w_2 = \max_{i=1,2,3} w_i$ and $w_2 > \max_{i=1,3} w_i$, and types I and IV for $w_3 = \max_{i=1,2,3} w_i$ and $w_3 > \max_{i=2,3} w_i$. However, from Table 4, the possible optimal solutions are all the types I, II, III and IV for $w_2 = \max_{i=1,2,3} w_i,\ w_3 = \max_{i=1,2,3} w_i,\ w_2 > \max_{i=1,3} w_i$ and $w_3 > \max_{i=2,3} w_i$. Among a set of four optimal solution types, the largest number of instances of the test problem is the solution type II. Therefore, the optimal solution type is I for $w_1 = \max_{i=1,2,3} w_i$ and $w_1 > \max_{i=2,3} w_i$, and types I, II, III and IV for $w_2 = \max_{i=1,2,3} w_i$, $w_3 = \max_{i=1,2,3} w_i,\ w_2 > \max_{i=1,3} w_i$ and $w_3 > \max_{i=2,3} w_i$.

Table 4. The number of different instances satisfying weight (W) and solution type (S).

W	I	II	III	IV
$w_1 \geq w_2, w_1 \geq w_3$	168,101	6826	4833	0
$w_2 \geq w_1, w_2 \geq w_3$	47,133	114,240	7411	10,976
$w_3 \geq w_1, w_3 \geq w_2$	51,302	56,618	16,960	54,880
$w_1 > w_2, w_1 > w_3, w_2 \neq w_3$	119,840	0	0	0
$w_2 > w_1, w_2 > w_3,\ w_1 \neq w_3$	28,856	80,164	5332	5488
$w_3 > w_1, w_3 > w_2,\ w_1 \neq w_2$	30,720	40,656	10,048	38,416

For the three-dimensional constrained OWA aggregation problem with lower bounded variables, from the numerical experiments the solution type I is the same as that of the constrained OWA aggregation problem without lower bounded variables for $w_1 > \max_{i=2,3} w_i$. However, for $w_2 >$

$\max_{i=1,3} w_i$ and $w_3 > \max_{i=2,3} w_i$, there are all solution types. For the constrained OWA aggregation problem without lower bounded variables, the solution are types II, III, IV and types I, IV, for $w_2 > \max_{i=1,3} w_i$ and $w_3 > \max_{i=2,3} w_i$, respectively. The four solution types may be too simple for the three-dimensional constrained OWA aggregation problem with lower bounded variables. From this result, we anticipate more complication in the higher dimensions of the constrained OWA aggregation problem with lower bounded variables.

6. Conclusions

For the constrained OWA aggregation problem with one constraint on the sum of all variables, this paper introduces some constraints to reduce the multiple solution problem. For the three-dimensional constrained OWA aggregation problem with the same lower bounds, by using the change of variables, the optimal solution is the same as that of the constrained OWA aggregation problem without lower bounded variables. For the three-dimensional constrained OWA aggregation problem with lower bounded variables, this paper presents four types (I, II, III and IV) of solutions depending on the number of zero elements. When the number of zero elements of solution is two (type I), there are three closed-form expressions of $\boldsymbol{X'}$ and six closed-form expressions of $\boldsymbol{Y}$. When the number of zero elements of the solution is one (types II and III), there are three closed-form expressions of $\boldsymbol{X'}$ and six closed-form expressions of $\boldsymbol{Y}$ for type II, and six closed-form expressions of $\boldsymbol{X'}$ and six closed-form expressions of $\boldsymbol{Y}$ for type III. When the number of zero elements of the solution is zero (type IV), there is only one closed-form expression of $\boldsymbol{X'}$ and one closed-form expression of $\boldsymbol{Y}$. According to the computerized experiment we perform for the three-dimensional constrained OWA aggregation problem with lower bounded variables, the optimal solution type is I for $w_1 = \max_{i=1,2,3} w_i$ and $w_1 > \max_{i=2,3} w_i$, and types I, II, III and IV for $w_2 = \max_{i=1,2,3} w_i$, $w_3 = \max_{i=1,2,3} w_i$, $w_2 > \max_{i=1,3} w_i$ and $w_3 > \max_{i=2,3} w_i$.

Worthy of future research is that the analysis is extended to the lower and upper bounded variables for the constrained OWA aggregation problem, especially for the three-dimensional constrained OWA aggregation problem with upper bounded variables. Thus, the analysis of the constrained OWA aggregation problem with bounded variables is a subject of considerable ongoing research.

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