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Optimal Dividend and Capital Injection Problem with Transaction Cost and Salvage Value: The Case of Excess-of-Loss Reinsurance Based on the Symmetry of Risk Information

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Abstract: This paper considers the optimal dividend and capital injection problem for an insurance company, which controls the risk exposure by both the excess-of-loss reinsurance and capital injection based on the symmetry of risk information. Besides the proportional transaction cost, we also incorporate the fixed transaction cost incurred by capital injection and the salvage value of a company at the ruin time in order to make the surplus process more realistic. The main goal is to maximize the expected sum of the discounted salvage value and the discounted cumulative dividends except for the discounted cost of capital injection until the ruin time. By considering whether there is capital injection in the surplus process, we construct two instances of suboptimal models and then solve for the corresponding solution in each model. Lastly, we consider the optimal control strategy for the general model without any restriction on the capital injection or the surplus process.

Keywords: optimal dividend; capital injection; salvage value; transaction cost; excess-of-loss reinsurance

1. Introduction

The expansion of the economic activities in the sense of time and space triggered the need for managing the exposure to risk for different types of businesses (Aniunas et al. [1], Lakstutiene et al. [2], and Kurach [3]). However, the increasing scale and scope of the activities of the insurance companies require identification of effective strategies for risk management in the insurance business itself. Therefore, there have been attempts to model the optimal strategies allowing for stable operation of the insurance companies.

Since the stochastic control theory is a primal approach towards handling the risk issues in recent years, more and more scholars pay attention to the aspect of the optimal dividend for an insurance company. A plethora of mathematical models for managing these issues have been developed. For instance, Assmussen and Taksar [4] applied a controlled diffusion approach in order to address the issue of the optimal dividend in a more advanced framework. They showed that a singular type of control indicating pay out when the sum to be paid out (the accumulated surplus) exceeds a certain level (and no payment in case the threshold is not reached) that can be used as the optimal strategy. Gerber and Shiu [5] pointed out that the barrier strategies solve the mathematical problems. However, the dividend stream corresponding to this solution is not acceptable in reality. Belhaj [6] considered a Brownian risk and a Poisson risk as the two kinds of liquidity risk within the unified framework.

The resulting model implied that the barrier strategy is still the optimal one. Taking an insurance company into consideration, Azue and Muler [7] examined the optimal dividend exercise in case the uncontrolled reserve process follows a classical Cramér-Lundberg model, which involves claim-size distribution of an unknown type. The closed-form solutions for different cases were obtained by Meng et al. [8] who studied an optimal dividend problem taking nonlinear insurance risk processes into consideration. The nonlinearity was related to internal competition factors.

Although these papers argued that the optimal dividend strategy is a barrier strategy where the expected cumulative discounted value of the dividend flow is maximized within the time horizon until the ruin event. By assuming different conditions, ruin happens for the insurance company following the risk process with probability 1. Apparently, it is actually unrealistic. Taking some pharmaceutical or petroleum companies, for example, the shareholders focus on the economic returns and the social benefits as well. Therefore, once their company is on the edge of bankruptcy, they will prevent that from happening by raising sufficient funds. Therefore, in the real financial market, a company always raises funds and, subsequently, reduces exposure to risk by the virtue of capital injection. Therefore, when capital injection is taken into account, the company is assumed to survive forever. Afterward, the expected cumulative discounted dividends are less than the expected discounted cost of capital injection in the infinite time horizon, which is regarded as a critical value and should be maximized when deciding on the strategies for dividend and capital injection management for a certain company. There have also been a number of papers studying this aspect. The conventional risk model was taken into consideration by Kulenko and Schmidli [9] to streamline the dividend payments and capital injections. The issue of the dividend payments and capital injections was also tackled by Yao et al. [10] in terms of the dual risk model involving fixed transaction costs. The latter study identified the bond strategy with an upper and lower barriers as the optimal one. In the context of the random time horizon and a ruin penalty, Zhao and Yao [11] investigated the optimal dividend and capital injection strategy. Yin and Yuen [12] studied the issue of optimal control at the company level when there is a surplus process characterized by an upward jump diffusion and random return on investment.

Besides capital injection, reinsurance is also considered an effective method for a company to control its risk exposure. This is because an appropriate reinsurance strategy can protect a company against the potentially large loss and, therefore, reduce the earning volatility. In practice, there are many different types of insurance policies adopted by companies. Due to its great value both in theory and practice, the issue of the combined dividend and reinsurance has attracted substantial attention and now there is plenty of research on this issue, which includes Høgaard and Taksar [13], Peng et al. [14], Yao et al. [15,16], Yao and Fan [17], and other references. Among the possible strategies, options such as the proportional reinsurance and the excess-of-loss reinsurance have also been investigated extensively (see, e.g., Candenillas et al. [18], Meng and Siu [19], Xu and Zhou [20], Yao et al. [21], A et al. [22], Yao et al. [23]). In these papers, the excess-of-loss reinsurance and dividend strategies are explored and the corresponding solutions to the value function are obtained as well. However, the fixed transaction cost incurred by capital injection has not been discussed in-depth, which is crucial.

Accordingly, we focus our research on the optimal dividend and capital injection policies for an insurance company that manages its risk by the virtue of both the excess-of-loss reinsurance and capital injection based on the symmetric of risk information. In this paper, the symmetry of risk information requires the reinsurance and insurance companies to have complete information on each other. In other words, the possible loss is the common information of both sides in order for there to be no moral hazard caused by asymmetric information of risk. We also add the fixed and proportional transaction cost incurred by capital injection and the salvage value of the company at the ruin time into the surplus process. In reality, transaction cost is unavoidable when the managers run the business especially since the fixed transaction cost is always generated by advisories and consultants when capital injection happens, which makes the impulse control problems more difficult (see, e.g., Paulsen [24], Bai et al. [25], Peng et al. [14], Liu and Hu [26]). In addition, the salvage value of the company can be interpreted as a company's liquidation value at the time of bankruptcy such as the company's brand name or agency network. There has also been more research that examines the optimal dividend policy for an insurance company in the presence of the salvage value for bankruptcy (see, e.g., Loeffen and Renaud [27], Liang and Young [28], Yao et al. [15]). Therefore, introducing the fixed transaction cost and salvage value make our model closer to reality. In order to find out a strategy that maximizes the expected sum of the discounted salvage value and the discounted cumulative dividends minus the expected discounted cost of capital injection until the ruin time, we construct two auxiliary suboptimal models in which one never goes bankrupt by capital injection and the other is a classical model without capital injection. After identifying the corresponding solutions to these two auxiliary models and the corresponding optimal strategy, we solve the general control problem without any restrictions on capital injection or the surplus process.

The outline of the paper is as follows. Section 2 presents the optimal control problem and then gives the definition of the value function by using a diffusion approximation to the compound Poisson model with excess-of-loss reinsurance. Sections 3 and 4 consider two auxiliary suboptimal models, respectively. Section 5 explores the solution to the general control problem. The last Section concludes the study.

2. Model Formulation and the Control Problem

Let $(\Omega, F, \mathbb{P})$ be a probability space with the filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions. In the classical risk theory, without reinsurance and dividend payments, an insurance company's surplus following the compound Poisson risk process on this filtered probability space is given by the equation below.

$$X_t = x + pt - \sum_{i=1}^{N_t} Y_i, \tag{1}$$

where x is the initial surplus, $p > 0$ is the premium rate, $\{N_t\}$ is a Poisson process with intensity λ, and the individual claim size $Y_1, Y_2, \dots$, independent of $\{N_t\}$ are independent identically distributed positive random variables with a common continuous distribution $F(y) = 1 - \overline{F}(y) = P(Y_i < y)$ where the corresponding finite first and second moments are $\mu^{(1)} = E[Y_1] > 0$ and $\mu^{(2)} = E\left[Y^2\right] > 0$. Define $M = \sup\{y : F(y) < 1\}$, in this paper we only consider the case where the claim distribution has an upper bound, which means $M < \infty$.

If the excess-of-loss reinsurance is taken by an insurance company to cede the potential risk (denote by $m \in [0, M]$ the excess-of-loss retention level), then, for each claim Y_i, the retained risk level is $Y_i^{(m)} = Y_i \wedge m$, and its first and second moments are shown below:

$$\mu^{(1)}(m) = E\left[Y_i^{(m)}\right] = \int_0^m \overline{F}(y)dy, \tag{2}$$

$$\mu^{(2)}(m) = E\left[(Y_i^{(m)})^2\right] = \int_0^m 2y\overline{F}(y)dy. \tag{3}$$

Assume that the excess-of-loss reinsurance premium rate is calculated based on the expected value principle with the safe loading $\theta > 0$. Then the company's surplus process with reinsurance can be rewritten as the equation below:

$$X_t^m = x + (p - p^{(m)})t - \sum_{i=1}^{N_t} Y_i^{(m)}, \tag{4}$$

where $p^{(m)} = (1+\theta)E\left[\sum_{i=1}^{N_t}\left(Y_i - Y_i^{(m)}\right)\right] = (1+\theta)\lambda\left(\mu^{(1)} - \mu^{(1)}(m)\right)$. According to many previous studies, the diffusion approximation can be described by the formula below:

$$X_t^m = x + \left[\theta\lambda\mu^{(1)}(m) + p - (1+\theta)\lambda\mu^{(1)}\right]t + \sqrt{\lambda\mu^{(2)}(m)}B_t, \tag{5}$$

where $\{B_t\}$ is a standard Brownian motion, which is adapted to the filtration $\mathcal{F}_t^B = \sigma\{B_s : 0 \leq s \leq t\}$. In this paper, we consider the cheap reinsurance, which is shown as $p = (1+\theta)\lambda\mu^{(1)}$ and the insurer can dynamically control the retention level m to expose the risk, which means the surplus process becomes the equation below:

$$X_t^m = x + \theta\lambda\mu^{(1)}(m_t)t + \sqrt{\lambda\mu^{(2)}(m_t)}B_t.$$

Now, we incorporate dividend payments and capital injection into the model. Let $\{L_t\}$ denote the cumulative amount of dividend pay until time t and $\{G\}$ denote the capital injection described by a sequence of increasing stopping times $\{\tau_n | n = 1, 2, 3, \ldots\}$ and the corresponding amount $\{\eta_n | n = 1, 2, 3, \ldots\}$. With a control strategy $\pi = \{m_t^\pi; L_t^\pi; G^\pi\} = \{m_t^\pi; L_t^\pi; \tau_1^\pi, \ldots, \tau_n^\pi; \eta_1^\pi, \ldots, \eta_n^\pi\}$, at time t, the surplus process becomes the equation below:

$$X_t^\pi = x + \theta\lambda\mu^{(1)}(m_t^\pi)t + \sqrt{\lambda\mu^{(2)}(m_t^\pi)}B_t - L_t^\pi + \sum_{n=1}^{\infty} I_{\{\tau_n^\pi \leq t\}}\eta_n^\pi. \tag{6}$$

Definition 1. *A control strategy π is admissible if it meets the following conditions.*

(i) $\{m_t^\pi\}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process with $m_t^\pi \in [0, M]$ for all $t \geq 0$.
(ii) $\{L_t^\pi\}$ is an increasing, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted cádlág process and $\Delta L_t^\pi \leq X_{t-}^\pi$.
(iii) $\{\tau_n^\pi\}$ is a sequence stopping times with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ and $0 \leq \tau_1^\pi \leq \cdots \leq \tau_n^\pi \leq \cdots$, a.s.
(iv) $\eta_n^\pi (n = 1,\ 2,\ 3, \ldots)$ is measurable and non-negative with respect to $\{\mathcal{F}_{\tau_n^\pi}\}$.
(v) $\forall T > 0$, it has $P(\lim_{n\to\infty} \tau_n^\pi < T) = 0$.

For each admissible strategy π, we establish respective ruin time as $\tau^\pi := \inf\{t \geq 0; X_t^\pi < 0\}$, which is a $\{\mathcal{F}_t\}_{t\geq 0}$ stopping time. If capital injection occurs, this stopping time could be infinite.

Therefore, we estimate the value of an insurance company by exploiting the performance index function. The performance index function is defined as the expected sum of the discounted salvage value and the discounted cumulative dividends except for the expected discounted costs of capital injection until the ruin time:

$$V(x, \pi) = E_x\left[\int_0^{\tau^\pi} \beta_1 e^{-\delta s} dL_s^\pi - \sum_{n=1}^{\infty} e^{-\delta\tau_n^\pi}(\beta_2\eta_n^\pi + K)I_{\{\tau_n^\pi \leq \tau^\pi\}} + Pe^{-\delta\tau^\pi}\right], \tag{7}$$

where $\delta > 0$ is the interest force, $P \geq 0$ is the salvage value of an insurance company at the ruin time, $\beta_1 < 1$ means the proportional transaction cost in the dividend payout process, and $\beta_2 > 1$ and $K > 0$ are the proportional and fixed transaction costs associated with the capital injection, respectively.

With the initial surplus x, the objective is to obtain the value function

$$V(x) = \sup_{\pi \in \Pi_x} V(x, \pi), \tag{8}$$

and the corresponding optimal control strategy $\pi^* = \left\{m^{\pi^*}; L^{\pi^*}; G^{\pi^*}\right\}$ such that $V(x) = V(x, \pi^*)$.

Remark 1. *The compound Poisson risk model is applied to descript the surplus process of an insurance company in this research. In fact, the compound Poisson risk process also known as the Cramér–Lundberg process is a commonly used jump process. Lots of work related to the jump process has been done in various contexts and the literature includes as Nguyen et al. [29], Nguyen and Vuong [30], and Hoang and Vuong [31].*

Remark 2. *In this paper, we only focus on the instance in which $P \geq 0$. As for the case of $P < 0$, it is not included in our study because the surplus will be always non-negative if the insurer can cede all the risk to* *the reinsurer. Following the optimality of the value function, we have $V(0) \geq 0$. Therefore, the case where $V(0) = P < 0$ is impossible.*

Aiming to proceed with the results, we first present some useful operators. For a function $v \in C^2$, the operator of maximum capital injection $\mathcal{M}$ is defined below:

$$\mathcal{M}v(x) := \sup_{y \geq 0}\{v(x+y) - \beta_2 y - K\},$$

and the operator $\mathcal{L}^{m}$ is represented by the equation below:

$$\mathcal{L}^{m}v(x) = \frac{1}{2}\lambda\mu^{(2)}(m)v''(x) + \theta\lambda\mu^{(1)}(m)v'(x) - \delta v(x).$$

3. The Solution to the Problem Where Bankruptcy is Not Allowed

In this section, we consider one suboptimal control model that the insurance company will not go bankrupt in finite time horizon due to capital injection. Then, the objective is to maximize the expected present value of the discounted cumulative dividend payout except for the discounted cost of capital injection in the infinite time horizon.

Defined by $\pi_c = \{m^{\pi_c}; L^{\pi_c}; G^{\pi_c}\} \in \Pi_x$, the control strategy of the insurance company won't ruin. Therefore, for each admissible strategy, the performance index function becomes the equation below

$$V(x, \pi_c) = E_x\left[\int_0^{\infty} \beta_1 e^{-\delta s} dL_s^{\pi_c} - \sum_{n=1}^{\infty} e^{-\delta \tau_n^{\pi_c}}(\beta_2 \eta_n^{\pi_c} + K) I_{\{\tau_n^{\pi_c} \leq \infty\}}\right]. \tag{9}$$

The objective is to obtain the value function shown below

$$V_c(x) = \sup_{\pi_c \in \Pi_x} V(x, \pi_c), \tag{10}$$

and the associated optimal strategy π_c^* with $V_c(x) = V(x, \pi_c^*)$.

Assume that the value function defined by Equation (10) is sufficiently smooth. Then, based on the stochastic control theorem, one can derive that the HJB Equation along with the boundary condition of this suboptimal control problem are given below:

$$\max\left\{\max_{0 \leq m \leq M} \mathcal{L}^{m}V_c(x),\ \beta_1 - V_c'(x),\ \mathcal{M}V_c(x) - V_c(x)\right\} = 0, \tag{11}$$

$$\max\{\mathcal{M}V_c(0) - V_c(0),\ -V_c(0)\} = 0. \tag{12}$$

The operator $\mathcal{M}V_c(x)$ denotes the value of a strategy to choose the optimal immediate capital injection. If x is the starting point for the surplus process and the process is governed in line with the optimal strategy, then the performance index function associated with this strategy is $V_c(x)$. However, suppose that the surplus process still starts at x. If we choose an appropriate time to inject the capital and, after that, the surplus process is also governed in line with the optimal strategy, then the performance index function becomes $\mathcal{M}V_c(x)$. It is easy to show that the performance index function with the first strategy is greater than the second one. Moreover, the two functions can be equal if and only if the time of capital injection is optimal. Therefore, it follows that $\mathcal{M}V_c(x) \leq V_c(x)$. Furthermore, the time value of money implies that the optimal time to inject the capital only comes at the moment when the surplus becomes zero. Mathematically, it has stated that $\mathcal{M}V_c(0) = V_c(0)$ and $\mathcal{M}V_c(x) < V_c(x)$ for all $x > 0$.

Furthermore, when an insurance company is on the edge of bankruptcy, it usually has two ways to tackle the risk. The first one is to inject new capital and its surplus immediately jumps to some level $\eta^* > 0$. If the time for this capital injection is optimal, by the definition of the operator M, the optimal amount of capital injection should be $\eta^* := \inf\{x : V'_c(x) = \beta_2\}$ and the boundary condition satisfies $\mathcal{M}V_c(0) = V_c(0) = V_c(\eta^*) - \beta_2\eta^* - K > 0$. Therefore, the optimal strategy of capital injection $G^{\pi_c^*}$ is constructed in the equations below:

$$\tau_1^{\pi_c^*} = \inf\{t \geq 0 : X_{t-}^{\pi_c^*} = 0\}, \tag{13}$$

$$\tau_n^{\pi_c^*} = \inf\{t \geq \tau_{n-1}^{\pi_c^*} : X_{t-}^{\pi_c^*} = 0\}, n = 2,\ 3,\ \ldots\ , \tag{14}$$

$$\eta_n^{\pi_c^*} \equiv \eta^*, n = 1,\ 2,\ 3,\ \ldots \tag{15}$$

The second one is to cede all the potential risks to a reinsurance company and keep the insurance company's surplus at the barrier 0 forever. Since the insurance company never goes bankrupt, it does not need any capital injection. If this choice is optimal, we can deduce that the boundary condition should satisfy $V_c(0) = 0$ and $\mathcal{M}V_c(0) < V_c(0)$. Correspondingly, the optimal strategy of capital injection $G^{\pi_c^*}$ is shown below

$$G^{\pi_c^*} \equiv 0. \tag{16}$$

In addition, if there is some value $x_{1c}^* := \inf\{x : V'(x) = \beta_1\}$ such that $x_{1c} \geq \eta^*$, the optimal dividend strategy is a linear barrier strategy with the barrier x_{1c}^*. That is, $L^{\pi_c^*}$ satisfies the equation below

$$L^{\pi_c^*} = (x - x_{1c}^*)^+ + \int_0^t I_{(x_s^{\pi_c^*} = x_{1c}^*)} dL_s^{\pi_c^*}, \text{ for all } t \geq 0. \tag{17}$$

Therefore, the optimal excess-of-loss reinsurance retention level $m^{\pi_c^*}(x)$ should satisfy the equation below

$$\mathcal{L}^{m^{\pi_c^*}(x)} V_c(x) = \max_{0 \leq m \leq M} \mathcal{L}^m V_c(x) = 0, \text{ for } 0 \leq x \leq x_{1c}^*. \tag{18}$$

Theorem 1. *Let $g(x)$ be an increasing concave and twice continuously differentiable solution to the Equations (11) and (12). In this case, one arrives at the following outcomes.*

(i) For each admissible strategy π_c, there exists $g(x) \geq V(x, \pi_c)$ and, therefore, $g(x) \geq V_c(x)$ for all $x \geq 0$.
(ii) In case of the strategy, $\pi_c^* = \left\{m^{\pi_c^*}; L^{\pi_c^*}; G^{\pi_c^*}\right\}$ is constructed by Equations (13)–(18) with $g(x) = V(x, \pi_c^*)$. Then $g(x) = V_c(x)$ and π_c^* is the optimal control strategy.

Proof. (i) Fixing a strategy $\pi_c \in \Pi_x$, define the sets $\Lambda = \left\{s : L_{s-}^{\pi_c} \neq L_s^{\pi_c}\right\}$ and $\Lambda' = \left\{s : G_{s-}^{\pi_c} \neq G_s^{\pi_c}\right\} = \left\{\tau_1^{\pi}, \ldots, \tau_n^{\pi}, \ldots\right\}$, then let $\hat{L}_t^{\pi_c} = \sum\limits_{s \in \Lambda, s \leq t} (L_s^{\pi_c} - L_{s-}^{\pi_c})$ and $\widetilde{L}_t^{\pi_c} = L_t^{\pi_c} - \hat{L}_t^{\pi_c}$ be the discontinuous and continuous parts of $L_t^{\pi_c}$, respectively. By the virtue of Itô formula, it can be shown by the equation below.

$$\begin{aligned} e^{-\delta(t \wedge \tau^{\pi_c})} g(X_t^{\pi_c}) \quad = \quad & g(x) + \int_0^{t \wedge \tau^{\pi_c}} e^{-\delta s} \mathcal{L}^{\, m^{\pi_c}} g(X_s^{\pi_c})\, ds + \int_0^{t \wedge \tau^{\pi_c}} e^{-\delta s} \sqrt{\lambda \mu^{(2)}(m_s^{\pi_c})} g'(X_s^{\pi_c})\, dB_s \\ & - \int_0^{t \wedge \tau^{\pi_c}} e^{-\delta s} g'(X_s^{\pi_c})\, d\widetilde{L}_s^{\pi_c} + \sum_{s \,\in\, \Lambda \cup \Lambda', s \,\leq\, t \,\wedge\, \tau^{\pi_c}} e^{-\delta s} \left[g(X_s^{\pi_c}) - g(X_{s-}^{\pi_c})\right]. \end{aligned} \tag{19}$$

The sum of discontinuous parts of $e^{-\delta t} g(X_t^{\pi_c})$ can be rearranged in the following manner:

$$\sum_{s \in \Lambda \cup \Lambda', s \leq t \wedge \tau^{\pi_c}} e^{-\delta s}\left[g(X_s^{\pi_c}) - g(X_{s-}^{\pi_c})\right]$$
$$= \sum_{s \in \Lambda, s \leq t \wedge \tau^{\pi_c}} e^{-\delta s}\left[g(X_s^{\pi_c}) - g(X_{s-}^{\pi_c})\right] + \sum_{\tau_n^{\pi_c} \leq t \wedge \tau^{\pi_c}} e^{-\delta s}\left[g(\eta_n^{\pi_c}) - g(0)\right] \tag{20}$$
$$\leq - \sum_{s \in \Lambda, s \leq t \wedge \tau^{\pi_c}} e^{-\delta s}\beta_1(L_s^{\pi_c} - L_{s-}^{\pi_c}) + \sum_{i=1}^{\infty} e^{-\delta\tau_n^{\pi_c}}(\beta_2\eta_n^{\pi_c} + K)I_{\{\tau_n^{\pi_c} \leq t \wedge \tau^{\pi_c}\}}.$$

Since $g(x)$ satisfies the HJB Equation with $g'(x) \geq \beta_1$ and $\mathcal{M}g(0) \leq g(0)$, we can see that the above inequality holds. Moreover, the second term on the right side of Equation (19) is non-positive. Then inserting Equation (20) into Equation (19) yields the equation below

$$\begin{aligned} e^{-\delta(t\wedge\tau^{\pi_c})}g(X_t^{\pi_c}) \quad &\leq g(x) + \int_0^{t\wedge\tau^{\pi_c}} e^{-\delta s}\sqrt{\lambda\mu^{(2)}(m_s^{\pi_c})}g'(X_s^{\pi_c})\, dB_s \\ &-\int_0^{t\wedge\tau^{\pi_c}} e^{-\delta s}\beta_1 dL_s^{\pi_c} + \sum_{i=1}^{\infty} e^{-\delta\tau_n^{\pi_c}}(\beta_2\eta_n^{\pi_c} + K)I_{\{\tau_n^{\pi_c} \leq t\wedge\tau^{\pi_c}\}}. \end{aligned} \tag{21}$$

Owing to capital injection or ceding all the risk to the reinsurer, the bankruptcy never happens, which means $\tau^{\pi_c} = \infty$. $X_t^{\pi_c}$ has a "continuous" path and $g(x)$ is increasing, which is defined by the equation below

$$\liminf_{t\to\infty} e^{-\delta t}g(X_t^{\pi_c}) \geq \lim_{t\to\infty} e^{-\delta t}g(0) = 0.$$

Note that the stochastic integral with respect to Brownian motion is a uniformly integral martingale. Therefore, applying the expectations on both sides of Equation (21) and setting $t \to \infty$, one arrives at the formula below

$$g(x) \geq V(x, \pi_c).$$

From Equation (10), it follows that $g(x) \geq V_c(x)$.

(ii) If the strategy π_c^* is constructed according to Equations (13)–(18), by replacing π_c with π_c^* in Equation (19) and taking some simple calculations, we have outlined the equation below for $x \leq x_{1c}^*$.

$$\begin{aligned} &\int_0^{t\wedge\tau^{\pi_c}} e^{-\delta s}\beta_1 dL_s^{\pi_c^*} - \sum_{n=1}^{\infty} e^{-\delta\tau_n^{\pi_c^*}}(\beta_2\eta_n^{\pi_c^*} + K)\, I_{\{\tau_n^{\pi_c^*} \leq t\wedge\tau^{\pi_c^*}\}} \\ &= -e^{-\delta(t\wedge\tau^{\pi_c^*})}g(X_t^{\pi_c^*}) + g(x) + \int_0^{t\wedge\tau^{\pi_c^*}} e^{-\delta s}\sqrt{\lambda\mu^{(2)}(m_s^{\pi_c^*})}g'(X_s^{\pi_c^*})\, dB_s. \end{aligned} \tag{22}$$

Applying the expectations and the limits on both sides of Equation (22) and noting that the controlled process $X_t^{\pi_c^*}$ is a double barrier reflecting process, the theorem can be proven. □

Next, before obtaining the closed-form solution of the value function and the retention level $m^{\pi_c^*}(x)$, we present the following lemma, which plays a key role in the solution procedure.

Lemma 1. *Let $\hat{m}_0 \in (0, M)$ be the unique solution to the equation given below*

$$C\exp\left[\int_0^{G(M)-G(\hat{m}_0)} \frac{\theta}{G^{-1}[x + G(\hat{m}_0)]}ds\right] = \beta_2, \tag{23}$$

where $C > 0$ is a constant and $G(x) = \int_0^x \frac{\mu^{(2)}(y)}{\frac{2\delta}{\theta\lambda}y^2 + 2\theta y\mu^{(1)}(y) - \theta\mu^{(2)}(y)}dy$. For each $m_0 \in [0, \hat{m}_0]$, define a function

$$F(x, m_0) = C\exp\left[\int_x^{x_0} \frac{\theta}{G^{-1}[x + G(m_0)]}ds\right],\ 0 \leq x \leq x_0, \tag{24}$$

where $x_0 = G(M) - G(m_0)$. As such, there is a unique $\eta \in (0, x_0)$ such that $F(\eta, m_0) = \beta_2$. Furthermore, the function below:

$$\Phi(m_0) = \int_0^{\eta(m_0)} [F(x, m_0) - \beta_2] dx \tag{25}$$

is decreasing in m_0 in the range $[0,\ \Phi(0)]$.

Proof. From the analytic form of $F(x, m_0)$, differentiating the function with respect to x yields the formula below

$$\frac{\partial F}{\partial x} = -\frac{C}{G^{-1}[x + G(m_0)]} \exp\left[\int_x^{x_0} \frac{\theta}{G^{-1}[x + G(m_0)]} ds\right] < 0,\ 0 < x < x_0.$$

Therefore, $F(x, m_0)$ is a decreasing function. Then, we let $m(x) = G^{-1}[x + G(m_0)]$ and we have the equation below:

$$m(x_0) = M,$$
$$m'(x) = \frac{1}{\mu^{(2)}(m)}\left[\frac{2\delta}{\theta\lambda}m^2 + 2\theta m\mu^{(1)}(m) - \theta\mu^{(2)}(m)\right].$$

Doing a variable change of $y = m(s)$ for $F(x, m_0) = C\exp\left[\int_x^{x_0} \frac{\theta}{m(s)} ds\right]$ and combining with the above two Equations, we have the equation below

$$F(x, m_0) = C\exp\left[\int_{m(x)}^{M} \frac{\mu^{(2)}(y)}{2y^2\left(\frac{\delta}{\theta^2\lambda}y + \mu^{(1)}(y) - \frac{\mu^{(2)}(y)}{2y}\right)} ds\right],\ 0 \le x \le x_0. \tag{26}$$

Clearly, $F(x, m_0)$ can be also viewed as a decreasing function of m_0. Then, Equation (23) can be rewritten as $F(0, \hat{m}_0) = \beta_2$. Therefore, we can deduce that if $m_0 \in [0, \hat{m}_0]$, there is a unique solution $\eta(m_0) \in [0, x_0]$ to the Equation $F(\eta, m_0) = \beta_2$. Furthermore, $\eta(m_0)$ is a decreasing function of m_0. Therefore, the minimum $\eta_{\min} = \eta(\hat{m}_0) = 0$ and the maximum $\eta_{\max} = \eta(0) < x_0$, which is uniquely determined by $F(\eta(0), 0) = C\exp\left[\int_{\eta(0)}^{G(M)} \frac{\theta}{G^{-1}(s)} ds\right] = \beta_2$. It's easy to see that $\Phi(m_0)$ is non-negative and decreasing in $[0, \hat{m}_0]$, which satisfies $\Phi(\hat{m}_0) = 0$ and $\Phi(0) = \int_0^{\eta(0)} [F(\eta(0), 0) - \beta_2]\, dx$. Therefore, it holds $\Phi(m_0) \in [0, \Phi(0)]$ for $m_0 \in [0, \hat{m}_0]$. □

In the following part, we will solve the explicit solution to the HJB Equation with the boundary condition. From Theorem 1, we know that Equation (11) can be rewritten using the formula below.

$$\max_{0 \le m \le M}\left[\frac{1}{2}\lambda\mu^{(2)}(m)g''(x) + \theta\lambda\mu^{(1)}(m)g'(x) - \delta g(x)\right] = 0, \text{ for } 0 \le x \le x_{1c}^* \tag{27}$$

and

$$g'(x) = \beta_1, \text{ for } x \ge x_{1c}^*.$$

For $0 \le x \le x_{1c}^*$, by differentiating on both sides of Equation (27) with respect to m and setting the derivative equal to zero, we have the equation below:

$$m(x) = -\theta\frac{g'(x)}{g''(x)}. \tag{28}$$

Substituting Equation (28) back into Equation (27) leads to the formula below:

$$\theta\lambda\left[\mu^{(1)}(m)-\frac{\mu^{(2)}(m)}{2m}\right]g'(x)-\delta g(x)=0. \tag{29}$$

Differentiating the above Equation with respect to x and using Equation (28) again, we obtain the formula below:

$$m'(x)=\frac{1}{\mu^{(2)}(m)}\left[\frac{2\delta}{\theta\lambda}m^2+2\theta m\mu^{(1)}(m)-\theta\mu^{(2)}(m)\right]. \tag{30}$$

Let:

$$G(x)=\int_0^x\frac{\mu^{(2)}(y)}{\frac{2\delta}{\theta\lambda}y^2+2\theta y\mu^{(1)}(y)-\theta\mu^{(2)}(y)}dy$$

It is not hard to verify that $G'(x)>0$ and then the inverse function of $G(x)$ exists. Therefore, the equation is shown below:

$$m(x)=G^{-1}[x+G(m_0)]. \tag{31}$$

From Equation (28), we can see that $m(x)$ is strictly increasing, which implies that there exists $x^*_{0c}<x^*_{1c}$ such that the insurance company will keep all the claims and not cede any risk to the insurer if the surplus exceeds x^*_{0c}. In the view of (28), we have the equation below:

$$g(x)=k_1\int_0^x\exp\left[\int_z^{x_0}\frac{\theta}{m(s)}ds\right]dz+k_2,\ 0\le x\le x^*_{0c}. \tag{32}$$

In addition, for $x^*_{0c}<x\le x^*_{1c}$, taking $m(x)\equiv M$, we have $g(x)$ satisfying the following ODE

$$\frac{1}{2}\lambda\mu^{(2)}g''(x)+\theta\lambda\mu^{(1)}g'(x)-\delta g(x)=0.$$

It has the solution shown below:

$$g(x)=k_3e^{r_+(x-x^*_{1c})}+k_4e^{r_-(x-x^*_{1c})},\ x^*_{0c}<x\le x^*_{1c}.$$

where $r_\pm$ are the two roots of the equation $\frac{1}{2}\lambda\mu^{(2)}r^2+\theta\lambda\mu^{(1)}r-\delta=0$, and:

$$r_\pm=\frac{-\theta\mu^{(1)}\pm\sqrt{\left(\theta\mu^{(1)}\right)^2+2\mu^{(2)}\delta}}{\mu^{(2)}}. \tag{33}$$

Lastly, for $x>x^*_{1c}$, $g'(x)\equiv\beta_1$ and $g(x^*_{1c})=k_3+k_4$ yields the following formula:

$$g(x)=\beta_1(x-x^*_{1c})+k_3+k_4.$$

The constants k_1, k_2, k_3, k_4, and the critical values x^*_{0c}, x^*_{1c} are determined by the principle of smooth fit. From the first and second derivatives of $g(x)$ at the points x^*_{0c} and x^*_{1c}, we can have the following equalities:

$$k_3r_++k_4r_-=\beta_1,$$

$$k_3(r_+)^2+k_4(r_-)^2=0,$$

$$k_3r_+e^{r_+(x^*_{0c}-x^*_{1c})}+k_4r_-e^{r_-(x^*_{0c}-x^*_{1c})}=k_1,$$

$$k_3(r_+)^2e^{r_+(x^*_{0c}-x^*_{1c})}+k_4(r_-)^2e^{r_-(x^*_{0c}-x^*_{1c})}=-\frac{\theta}{M}k_1.$$

Solving the above equations and doing some calculations, we have found the following formula:

$$k_3 = -\frac{r_-\beta_1}{r_+(r_+ - r_-)} > 0, \tag{34}$$

$$k_4 = -\frac{r_+\beta_1}{r_-(r_+ - r_-)} > 0, \tag{35}$$

$$k_1 = \frac{M\beta_1}{M + \theta/r_+}\left[\frac{M + \theta/r_-}{M + \theta/r_+}\right]^{\frac{r_-}{r_+ - r_-}}. \tag{36}$$

It's obvious that $k_3 + k_4 = \frac{\theta\lambda\mu^{(1)}\beta_1}{\delta}$. Therefore, substituting the constants back into $g'(x^*_{0c}) = k_1$ yields the formula below:

$$x^*_{1c} = x^*_{0c} + \frac{1}{r_+ - r_-}\ln\left[\frac{M + \theta/r_+}{M + \theta/r_-}\right], \tag{37}$$

where x^*_{0c} satisfies the following equation:

$$x^*_{0c} = G(M) - G(m^{\pi^*_c}(0)). \tag{38}$$

In the view of Equation (29), Let $x = 0$ and performing the same variable change as in Lemma 1, we obtain the following formula:

$$k_2 = \frac{\theta\lambda}{\delta}k_1\left(\mu^{(1)}(m^{\pi^*_c}(0)) - \frac{\mu^{(2)}(m^{\pi^*_c}(0))}{2m^{\pi^*_c}(0)}\right)\exp\left(\int_{m^{\pi^*_c}(0)}^{M}\frac{\mu^{(2)}(y)}{2y^2\left(\frac{\delta}{\theta^2\lambda}y + \mu^{(1)}(y) - \frac{\mu^{(2)}(y)}{2y}\right)}dy\right). \tag{39}$$

Therefore, the unknown constants x^*_{0c}, x^*_{1c}, and k_2 are clear from Equations (37)–(39) once $m^{\pi^*_c}(0)$ is determined. Considering the analysis before and the boundary condition $\mathcal{M}g(0) = g(0)$, we can obtain the value of $m^{\pi^*_c}(0)$ in the following two cases.

(i) If $0 < K \le \Phi(0)$, it's conjectured that $\mathcal{M}g(0) = g(0)$ holds under the condition that there exist some $m^{\pi^*_c}(0) \in [0, M]$ and $\eta^*(m^{\pi^*_c}(0)) > 0$ in which the following equations are true:

$$g'(\eta^*) = \beta_2, \tag{40}$$

$$g(0) = g(\eta^*) - \beta_2\eta^* - K = \mathcal{M}g(0). \tag{41}$$

which can be rewritten as:

$$K = \int_0^{\eta^*}[g'(x) - \beta_2]dx = \int_0^{\eta^*(m^{\pi^*_c}(0))}[F(x, m^{\pi^*_c}(0)) - \beta_2]\,dx = \Phi(m^{\pi^*_c}(0)). \tag{42}$$

By Lemma 1, it follows that $m^{\pi^*_c}(0) \in [0, \hat{m}_0)$ and η^* exist if and only if $0 < K \le \Phi(0)$.

(ii) If $K > \Phi(0)$, the value $\eta^*(m^{\pi^*_c}(0)) > 0$ satisfying Equations (40) and (41) doesn't exist and $\mathcal{M}g(0) < g(0)$, which implies that $G^{\pi^*_c} \equiv 0$. Therefore, in order to meet the boundary condition in Equation (12), we have $g(0) = 0$ and $m^{\pi^*_c}(0) \equiv 0$. Now, summarizing the above discussions, we have the following result.

Theorem 2. *If the insurance company doesn't allow for bankruptcy, the value function $V_c(x)$ coincides with the formulas below.*

$$g(x) = \begin{cases} k_1\int_0^x \exp\left[\int_z^{x_0}\frac{\theta}{m(s)}ds\right]dz + k_2, & 0 \le x \le x^*_{0c}, \\ k_3e^{r_+(x - x^*_{1c})} + k_4e^{r_-(x - x^*_{1c})}, & x^*_{0c} \le x \le x^*_{1c}, \\ \beta_1(x - x^*_{1c}) + \frac{\theta\lambda\mu^{(1)}\beta_1}{\delta}, & x > x^*_{1c}, \end{cases} \tag{43}$$

where the constants k_1, k_2, k_3, and k_4 are given by Equations (36), (39), (34), and (35), respectively. In addition, the critical values x^*_{0c} and x^*_{1c} are shown in Equations (37) and (38). Correspondingly, the optimal dividends strategy $L^{\pi^*_c}$ satisfies the equation below:

$$L^{\pi^*_c} = (x - x^*_{1c})^+ + \int_0^t I_{\{X^{\pi^*_c}_s = x^*_{1c}\}} dL^{\pi^*_c}_s, \text{ for all } t \geq 0. \tag{44}$$

In addition, the optimal retention level of excess-of-loss reinsurance is shown below:

$$m^{\pi^*_c}(x) = \begin{cases} G^{-1}(x + G(m^{\pi^*_c}(0))), & 0 \leq x \leq x^*_{0c}, \\ M, & x \geq x^*_{0c}. \end{cases} \tag{45}$$

The value $m^{\pi^*_c}(0)$ and the optimal injection strategy $G^{\pi^*_c}_t$ are determined in the following cases.

(i) If $0 < K \leq \Phi(0)$, $m^{\pi^*_c}(0) = m_0 \in [0, \hat{m}_0]$ is the unique solution to the equation $\Phi(m_0) = K$. The optimal injection strategy $G^{\pi^*_c}_t$ is given by Equations (13)–(15) and the optimal amount of capital injection η^* is obtained by Equations (40) and (41). This means that, by injecting the capital, the insurance company's surplus immediately jumps to η^* when it hits the barrier 0. In this case, the boundary condition are $\mathcal{M}g(0) = g(0)$ and $g(0) \geq 0$.

(ii) If $K > \Phi(0)$, then $m^{\pi^*_c}(0) = 0$. The optimal strategy of capital injection satisfies $G^{\pi^*_c}_t \equiv 0$, which means the capital injection never happens. It suggests that, if the insurance company's surplus attains zero, it will cede all the potential risk to the reinsurance company and keep the surplus stay at 0. Therefore, the bankruptcy will never happen.

Remark 3. *We can easily verify that $g(x)$ is concave by checking its second derivative symbol and prove that $g(x)$ given in the three cases is the solution to the HJB Equation by substituting all the forms of $g(x)$ back into Equation (11) and applying the analysis before the Theorem. Therefore, we omit all the details here.*

Remark 4. *From Theorem 2, $\Phi(0)$ could be viewed as the maximum fixed transaction cost that the insurance company is willing to pay when the capital injection happens. With the increase of the fixed cost K, the company should reduce the retention level and raise the dividend barrier to increase the size of capital injection in order to reduce its amount. When K is larger than $\Phi(0)$, the best way to avoid raising new funds is to keep the company away from bankruptcy, which coincides with the real market.*

4. The Solution to the Problem without Capital Injection

In this section, we consider the other suboptimal control model for an insurance company seeking to maximize the expected discounted value of the cumulative dividend payout until the ruin time plus the salvage value at the ruin time.

Defined by $\pi_d = \{m^{\pi_d}; L^{\pi_d}; 0\} \in \Pi_x$ the control strategy without capital injection, the performance index function associated with this strategy becomes the equation below:

$$V(x, \pi_d) = E_x\left[\int_0^{\tau^{\pi_d}} \beta_1 e^{-\delta s} dL^{\pi_d}_s + Pe^{-\delta\tau^{\pi_d}}\right]. \tag{46}$$

The objective is to find the value function

$$V_d(x) = \sup_{\pi_d \in \Pi_x} V(x, \pi_d), \tag{47}$$

and the associated optimal control strategy π^*_d so that $V_d(x) = V(x, \pi^*_d)$.

Let the value function defined by Equation (47) be sufficiently smooth. Based on the stochastic control theorem, we obtain the corresponding HJB Equation below:

$$\max\left\{\max_{0\le m\le M}\mathcal{L}^{\,m}V_d(x),\ \beta_1 - V'_d(x)\right\}=0 \tag{48}$$

with the boundary condition $V_d(0)=P$.

It's easy to see that it is a typical optimal dividend control problem. Inspired by some research on this issue, we conjecture that the optimal dividend strategy $L_t^{\pi_d^*}$ is still a barrier strategy with some critical values $x_{1d}^*=\inf\{x: V_d{}'(x)=\beta_1\}$. Mathematically, we define the formula below:

$$L_t^{\pi_d^*}=(x-x_{1d}^*)^+ + \int_0^t I_{\{X_s^{\pi_d^*}=x_{1d}^*\}}\,dL_s^{\pi_d^*},\ \text{for all } t\ge 0. \tag{49}$$

Furthermore, the optimal excess-of-loss reinsurance retention level $m^{\pi_d^*}(x)$ satisfies the equation below.

$$\mathcal{L}^{\,m^{\pi_d^*}(x)}V_d(x)=\max_{0\le m\le M}\mathcal{L}^{\,m}V_d(x)=0,\ \text{for } 0\le x\le x_{1d}^*. \tag{50}$$

Therefore, based on the above analysis, we have the following result.

Theorem 3. *Let $f(x)$ be an increasing, concave, and twice continuously differential solution to the HJB Equation (48) with the boundary condition. In this case, one can obtain the following results.*

(i) For each $\pi_d\in\Pi_x$, it shows that $f(x)\ge V(x,\pi_d)$. Therefore, $f(x)\ge V_d(x)$ for all $x\ge 0$.

(ii) If the strategy $\pi_d^* = \{m^{\pi_d^*}; L^{\pi_d^*}; 0\}$ is constructed by Equations (49) and (50) so that $f(x) = V(x,\pi_d^*)$ is constructed by Equations (49) and (50) so that , then $f(x)=V_d(x)$ and π_d^* is optimal.

Proof. Fixing a strategy $\pi_d\in\Pi_x$, define the sets $\Lambda = \left\{s: L_{s-}^{\pi_d}\ne L_s^{\pi_d}\right\}$ and then let $\hat{L}_t^{\pi_d} = \sum\limits_{s\in\Lambda,s\le t}(L_s^{\pi_d}-L_{s-}^{\pi_d})$ and $\widetilde{L}_t^{\pi_d} = L_t^{\pi_d}-\hat{L}_t^{\pi_d}$ be the discontinuous and continuous parts of $L_t^{\pi_d}$, respectively. Then, by the general Itô formula, we obtain the equation below.

$$\begin{aligned} e^{-\delta(t\wedge\tau^{\pi_d})}f(X_t^{\pi_d}) = {} & f(x)+\int_0^{t\wedge\tau^{\pi_d}}e^{-\delta s}\mathcal{L}^{\,m^{\pi_d}}f(X_s^{\pi_d})\,ds+\int_0^{t\wedge\tau^{\pi_d}}e^{-\delta s}\sqrt{\lambda\mu^{(2)}(m_s^{\pi_d})}f'(X_s^{\pi_d})\,dB_s \\ & -\int_0^{t\wedge\tau^{\pi_d}}e^{-\delta s}f'(X_s^{\pi_d})\,d\widetilde{L}_s^{\pi_d}+\sum_{s\,\in\,\Lambda,\,s\,\le\,t\,\wedge\,\tau^{\pi_d}}e^{-\delta s}\left[f(X_s^{\pi_d})-f(X_{s-}^{\pi_d})\right] \end{aligned} \tag{51}$$

due to $f(x)$ satisfying the HJB Equation with $f'(x)=\beta_1$, the equation below shows:

$$\sum_{s\,\in\,\Lambda,\,s\,\le\,t\,\wedge\,\tau^{\pi_d}}e^{-\delta s}\left[f(X_s^{\pi_d})-f(X_{s-}^{\pi_d})\right]\le-\sum_{s\,\in\,\Lambda,\,s\,\le\,t\,\wedge\,\tau^{\pi_d}}e^{-\delta s}\beta_1(L_s^{\pi_d}-L_{s-}^{\pi_d}). \tag{52}$$

Moreover, the second term on the right side of Equation (51) is non-positive. Therefore, substituting Equation (52) into Equation (51) leads to the findings below:

$$e^{-\delta(t\wedge\tau^{\pi_d})}f(X_t^{\pi_d})\ \le\ f(x)+\int_0^{t\wedge\tau^{\pi_d}}e^{-\delta s}\sqrt{\lambda\mu^{(2)}(m_s^{\pi_d})}f'(X_s^{\pi_d})\,dB_s-\int_0^{t\wedge\tau^{\pi_d}}e^{-\delta s}\beta_1 dL_s^{\pi_d}. \tag{53}$$

since $f(x)$ is an increasing function and $f(0)=P$, the following equation was found.

$$\liminf_{t\to\infty}e^{-\delta(t\wedge\tau^{\pi_d})}f(X_t^{\pi_d})\ge\lim_{t\to\infty}e^{-\delta(t\wedge\tau^{\pi_d})}f(0)=Pe^{-\delta\tau^{\pi_d}}.$$

The stochastic integral with respect to the Brownian motion is a uniformly integral martingale. Taking expectations on both sides of Equation (53) and letting $t \to \infty$, we have $f(x) \geq V(x, \pi_d)$ and, therefore, $f(x) \geq V_d(x)$.

(ii) If the strategy π_d^* is constructed according to Equations (49) and (50), replacing π_d by π_d^* in Equation (51) and taking some simple calculations, we found the equation below:

$$\int_0^{t\wedge\tau^{\pi_d}} e^{-\delta s}\beta_1 dL_s^{\pi_d} = -e^{-\delta(t\wedge\tau^{\pi_d})} f(X_t^{\pi_d^*}) + f(x) + \int_0^{t\wedge\tau^{\pi_d}} e^{-\delta s}\sqrt{\lambda\mu^{(2)}(m_s^{\pi_d})} f'(X_s^{\pi_d})\, dB_s. \tag{54}$$

Taking the expectation and the limits on both sides of Equation (54), we can obtain the result. □

We can see that both $V_d(x)$ and $V_c(x)$ satisfy the same HJB Equation but meet different boundary conditions. Therefore, we can get the expression of $V_d(x)$ and the retention level $m^{\pi_d^*}(x)$ by the same method where the value $m^{\pi_d^*}(0) \in [0, M]$ is determined by the boundary condition $V_d(0) = P$. In order to save space, we avoid the repeated calculations and give the result as follows.

Theorem 4. *If the insurance company does not allow for capital injection, according to the salvage value $P \geq 0$, the value function $V_d(x)$ coincides with $f(x)$ in the following three cases.*

(i) If $0 \leq P \leq \frac{\theta\lambda}{\delta}k_1\left(\mu^{(1)} - \frac{\mu^{(2)}}{2M}\right)$, then $f(x)$ has the form:

$$f(x) = \begin{cases} k_1\int_0^x \exp\left[\int_z^{x_0} \frac{\theta}{m(s)}ds\right] dz + P, & 0 \leq x < x_{0d}^*, \\ k_3 e^{r_+(x - x_{1d}^*)} + k_4 e^{r_-(x - x_{1d}^*)}, & x_{0d}^* \leq x < x_{1d}^*, \\ \beta_1(x - x_{1d}^*) + \frac{\theta\lambda\mu^{(1)}\beta_1}{\delta}, & x \geq x_{1d}^*, \end{cases} \tag{55}$$

where the constants k_1, k_3, and k_4 are given by Equations (36), (34), and (35), respectively. The critical level x_{0d}^* and x_{1d}^* satisfy the equation below:

$$x_{1d}^* = x_{0d}^* + \frac{1}{r_+ - r_-}\ln\left[\frac{M + \theta/r_+}{M + \theta/r_-}\right], \tag{56}$$

$$x_{0d}^* = G(M) - G(m^{\pi_d^*}(0)). \tag{57}$$

In addition, $m^{\pi_d^*}(0)$ is the solution to the following equation

$$\frac{\theta\lambda}{\delta}k_1\left(\mu^{(1)}(m^{\pi_d^*}(0)) - \frac{\mu^{(2)}(m^{\pi_d^*}(0))}{2m^{\pi_d^*}(0)}\right)\exp\left(\int_{m^{\pi_d^*}(0)}^{M} \frac{\mu^{(2)}(y)}{2y^2\left(\frac{\delta}{\theta^2\lambda}y + \mu^{(1)}(y) - \frac{\mu^{(2)}(y)}{2y}\right)}dy\right) = P. \tag{58}$$

Accordingly, the optimal dividend strategy $L_t^{\pi_d^*}$ should satisfy the equation below:

$$L_t^{\pi_d^*} = (x - x_{1d}^*)^+ + \int_0^t I_{\{X_s^{\pi_d^*} = x_{1d}^*\}} dL_s^{\pi_d^*}, \text{ for all } t \geq 0. \tag{59}$$

In addition, the optimal excess-of-loss reinsurance retention level $m^{\pi_d^*}(x)$ is shown below:

$$m^{\pi_d^*}(x) = \begin{cases} G^{-1}(x + G(m^{\pi_d^*}(0))), & 0 \leq x \leq x_{0d}^*, \\ M, & x \geq x_{0d}^*. \end{cases} \tag{60}$$

(ii) If $\frac{\theta\lambda}{\delta}k_1\left(\mu^{(1)}-\frac{\mu^{(2)}}{2M}\right) < P \leq \frac{\theta\lambda\mu^{(1)}}{\delta}$, then $f(x)$ has the form below:

$$f(x)=\begin{cases} k_3e^{r_+(x-x_{1d}^*)}+k_4e^{r_-(x-x_{1d}^*)}, & 0\leq x<x_{1d}^*, \\ \beta_1(x-x_{1d}^*)+\frac{\theta\lambda\mu^{(1)}\beta_1}{\delta}, & x\geq x_{1d}^*, \end{cases} \tag{61}$$

where the constants k_3 and k_4 are given by Equations (34) and (35). The critical level x_{1d}^* is determined by the equation below:

$$k_3e^{-r_+x_{1d}^*}+k_4e^{-r_-x_{1d}^*}=P. \tag{62}$$

Correspondingly, the optimal dividend strategy $L_t^{\pi_d^*}$ should satisfy the formula below:

$$L_t^{\pi_d^*}=(x-x_{1d}^*)^+ + \int_0^t I_{\{X_s^{\pi_d^*}=x_{1d}^*\}}dL_s^{\pi_d^*}, \text{ for all } t\geq 0. \tag{63}$$

In addition, the optimal excess-of-loss reinsurance retention level is $m^{\pi_d^*}(x)\equiv M$.

(iii) If $P>\frac{\theta\lambda\mu^{(1)}}{\delta}$, then $f(x)=\beta_1x+P$. The optimal dividend strategy is to pay the whole initial surplus x as the dividends and declare bankruptcy at once. Then, the salvage P is realized.

Remark 5. *As shown in Theorem 4, the determination of the value function and the optimal control strategy depends on P and the retained risk level of the insurance company is increasing with P. In the case of* $P=0$*, the optimal retention is zero. This means that the insurer will cede all the risk to the reinsurance company and keep the surplus at zero. Therefore, the ruin will never happen. When the salvage value is great enough, it's optimal to announce the bankruptcy and realize the salvage value at once.*

5. The Solution to the General Control Problem

If there are no restrictions on the capital injection or the surplus process, the general control problem seeks to maximize the expected sum of discounted salvage value and the discounted dividends except for the expected discounted cost of capital injection over all the admissible strategies. Therefore, the corresponding HJB Equation takes the following form:

$$\max\left\{\max_{0\leq m\leq M}\mathcal{L}^mV(x),\ \beta_1-V'(x),\ \mathcal{M}V(x)-V(x)\right\}=0. \tag{64}$$

The boundary condition is shown below:

$$\max\{\mathcal{M}V(0)-V(0),\ P-V(0)\}=0. \tag{65}$$

Theorem 5. *Let* $v(x)$ *be a concave, increasing, and twice continuously differentiable solution to Equations (64) and (65). We have the following result:*

(i) For each $\pi\in\Pi_x$, it shows that $v(x)\geq V(x,\pi)$. So $v(x)\geq V(x)$ for all $x\geq 0$.

(ii) If there is a strategy of $\pi^*=\left\{m^{\pi^*};L^{\pi^*};G^{\pi^*}\right\}$ so that $v(x)=V(x,\pi^*)$, then $v(x)=V(x)$ and π^* is optimal.

Proof. The proof of (i) is similar to the first statement's proof of Theorems 1 and 3 and the result (ii) can be obtained by considering the optimality of $v(x)$. We omit this here. □

Before deriving the optimal strategy by solving the general control problem, we give the two Lemmas to show that the sign of some important properties are determined by the relationships of parameters. This shows the equation below:

$$\hat{P} = \frac{\theta\lambda k_1}{\delta}\left[\mu^{(1)}(\hat{m}_0) - \frac{\mu^{(2)}(\hat{m}_0)}{2\hat{m}_0}\right]\exp\left(\int_{\hat{m}_0}^{M}\frac{\mu^{(2)}(y)}{2y^2\left(\frac{\delta}{\theta^2\lambda}y + \mu^{(1)}(y) - \frac{\mu^{(2)}(y)}{2y}\right)}dy\right).$$

One can note that $\hat{P} \in [0, \frac{\theta\lambda k_1}{\delta}(\mu^{(1)} - \frac{\mu^{(2)}}{2M})]$ with a unique root $\hat{m}_0 \in [0, M]$.

Lemma 2. *The sign of $\mathcal{M}V_c(0) - V_c(0)$ and $P - V_c(0)$ are determined in the following manner.*

(i) If $0 < K \leq \Phi(0)$, $P \leq \hat{P}$ and $m^{\pi_c^*}(0) \leq m^{\pi_d^*}(0)$, it has $\mathcal{M}V_c(0) - V_c(0) = 0$ and $P - V_c(0) \geq 0$.
(ii) If $0 < K \leq \Phi(0)$, $P \leq \hat{P}$ and $m^{\pi_c^*}(0) > m^{\pi_d^*}(0)$, it has $\mathcal{M}V_c(0) - V_c(0) = 0$ and $P - V_c(0) < 0$.
(iii) If $0 < K \leq \Phi(0)$ and $P > \hat{P}$, it has $\mathcal{M}V_c(0) - V_c(0) < 0$.
(iv) If $K > \Phi(0)$, it has $\mathcal{M}V_c(0) - V_c(0) < 0$.

Proof. If the function $h(x) = \mu^{(1)}(x) - \frac{\mu^{(2)}(x)}{2x}$, in Section 3, it follows that:

$$\begin{aligned} V_c(0) &= \tfrac{\theta\lambda k_1}{\delta}\left[\mu^{(1)}(m_0) - \tfrac{\mu^{(2)}(m_0)}{2m_0}\right]\exp\left(\int_{m_0}^{M}\frac{\mu^{(2)}(y)}{2y^2\left(\frac{\delta}{\theta^2\lambda}y + \mu^{(1)}(y) - \frac{\mu^{(2)}(y)}{2y}\right)}dy\right) \\ &= \tfrac{\theta\lambda k_1}{\delta}g(m_0), \end{aligned}$$

where the function $g(x)$ is given below:

$$g(x) = h(x)\exp\left(\int_x^M \frac{h'(y)}{\frac{\delta}{\theta^2\lambda}y + h(y)}dy\right),\ x > 0.$$

By checking $g(0+) = 0, g(M) = \mu^{(1)} - \frac{\mu^{(2)}}{2M}$ and the derivative below are discovered:

$$g'(x) = h'(x)\exp\left(\int_x^M \frac{h'(y)}{\frac{\delta}{\theta^2\lambda}y + h(y)}dy\right)\left(1 - \frac{h(x)}{\frac{\delta}{\theta^2\lambda}x + h(x)}\right) > 0.$$

We can deduce that $g(x)$ is increasing in $[0, M]$ and $\hat{P} = \theta\lambda k_1 g(\hat{m}_0)/\delta$ and the case of $P \leq \hat{P}$ leads to $m^{\pi_d^*}(0) \leq \hat{m}_0$. From Theorem 2, it follows that $\mathcal{M}V_c(0) - V_c(0) = 0$ holds with some $m^{\pi_c^*}(0) \in [0, \hat{m}_0)$. When $0 < K \leq \Phi(0)$, it's clear that $V_c(0) = \theta\lambda k_1 g(m^{\pi_c^*}(0))/\delta < \hat{P}$ since $m^{\pi_c^*}(0) < \hat{m}_0$. Therefore, if $m^{\pi_c^*}(0) \leq m^{\pi_d^*}(0)$, then $P - V_c(0) \geq 0$. On the other hand, if $m^{\pi_c^*}(0) > m^{\pi_d^*}(0)$, then $P - V_c(0) < 0$. In this paper, we have the statements (i) and (ii). As for $P > \hat{P}$, it shows that $V_c(0) < \hat{P} < P$ and $\mathcal{M}V_c(0) - V_c(0) = 0$, which holds with $m^{\pi_c^*}(0) \in [0, \hat{m}_0)$. The statement (iv) is a direct result of Theorem 2.

The proof is completed. □

Lemma 3. *If the equality $V_d(0) - P = 0$ holds, the sign of $\mathcal{M}V_d(0) - V_d(0)$ is determined by the different cases, which is shown below.*

(i) If $0 < K \leq \Phi(0)$, $P \leq \hat{P}$, and $m^{\pi_c^*}(0) < m^{\pi_d^*}(0)$, we find that $\mathcal{M}V_d(0) - V_d(0) < 0$.
(ii) If $0 < K \leq \Phi(0)$, $P \leq \hat{P}$, and $m^{\pi_c^*}(0) \geq m^{\pi_d^*}(0)$, we find that $\mathcal{M}V_d(0) - V_d(0) \geq 0$.
(iii) If $0 < K \leq \Phi(0)$ and $P > \hat{P}$, we find that $\mathcal{M}V_d(0) - V_d(0) < 0$.
(iv) If $K > \Phi(0)$, we find that $\mathcal{M}V_d(0) - V_d(0) < 0$.

Proof. In the case of $P \leq \hat{P}$, the equality $V_d(0) - P = 0$ has a unique solution of $m^{\pi_d^*}(0) \leq \hat{m}_0$. By Lemma 1, when $0 < K \leq \Phi(0)$, it suggests that there exists some $\eta(m^{\pi_d^*}(0)) \in [0, x_0]$ such that $V'_d(\eta(m^{\pi_d^*}(0))) = F(\eta, m^{\pi_d^*}(0)) = \beta_2$. Since $\Phi(x)$ is a decreasing function, the following equation was found:

$$\begin{aligned}\mathcal{M}V_d(0) - V_d(0) &= \max_{y\geq 0}\{V_d(y) - \beta_2 y - K - V_d(0)\} = \max_{y\geq 0}\{\textstyle\int_0^y (V'_d(x) - \beta_2)dx\} - K \\ &= \textstyle\int_0^{\eta(m^{\pi_d^*}(0))} (V'_d(x) - \beta_2)dx - K \leq \int_0^{\eta(m^{\pi_c^*}(0))} (V'_d(x) - \beta_2)dx - K = 0,\end{aligned}$$

where the inequality follows from $m^{\pi_d^*}(0) \geq m^{\pi_c^*}(0)$. Clearly, the following equation holds:

$$\mathcal{M}V_d(0) - V_d(0) = \Phi(\eta(m^{\pi_d^*}(0))) - K > 0,$$

then the condition satisfies $m^{\pi_c^*}(0) > m^{\pi_d^*}(0)$. As for the case of $P > \hat{P}$, the solution $m^{\pi_d^*}(0) \in [0, \hat{m}_0]$ doesn't exist. Therefore, there isn't some value $\eta\left(m^{\pi_d^*}(0)\right) > 0$ such that $V'_d(\eta(m^{\pi_d^*}(0))) = F(\eta, m^{\pi_d^*}(0)) = \beta_2$. It implies that $V'_d(x) < \beta_2$ holds for all $x \geq 0$. Since $V_d(x)$ is concave, then the following equation is found:

$$\mathcal{M}V_d(0) - V_d(0) = \max_{y\geq 0}\left\{\int_0^y (V'_d(x) - \beta_2)dx\right\} - K < 0.$$

From Lemma 1, we know that the maximum of $\mathcal{M}V_d(0) - V_d(0)$ is $\Phi(0) - K$. Therefore, it follows that $\mathcal{M}V_d(0) - V_d(0) < 0$ when $\Phi(0) < K$.

The proof is completed. □

Comparing the two different suboptimal models in Sections 3 and 4 and using the above two Lemmas, we obtain the following Theorem.

Theorem 6. *For any given initial surplus $x > 0$, if the general control problem seeks to maximize the performance index function over all admissible strategies, $g(x)$ and $f(x)$ are the solution to the HJB Equation in Theorems 2 and 4, respectively. Then the solution is given in the following two cases:*

Case 1. If $\mathcal{M}g(0) - g(0) = 0$ and $P - g(0) \leq 0$, the following equivalent condition is valid

$$0 < K \leq \Phi(0), P \leq \hat{P} \text{ and } m^{\pi_c^*}(0) > m^{\pi_d^*}(0),$$

then $V(x) = V_c(x) = g(x)$ and the optimal strategy π^* is the same as the corresponding strategy $\pi_c^* = \left\{m^{\pi_c^*}; L^{\pi_c^*}; G^{\pi_c^*}\right\}$ in Theorem 2.

Case 2. If $\mathcal{M}f(0) - f(0) < 0$ and $P - f(0) = 0$, one of the following equivalent conditions holds.

(i) $0 < K \leq \Phi(0), P \leq \hat{P}$ and $m^{\pi_c^*}(0) < m^{\pi_d^*}(0)$;
(ii) $0 < K \leq \Phi(0)$ and $P > \hat{P}$;
(iii) $K > \Phi(0)$.

then $V(x) = V_d(x) = f(x)$ and the optimal strategy π^* is the same as the corresponding strategy $\pi_d^* = \left\{m^{\pi_d^*}; L^{\pi_d^*}; 0\right\}$ in Theorem 4.

Proof. The proofs of (i) and (ii) resemble the second statement in Theorems 1 and 3, which we omit here. □

Remark 6. *Xu and Zhou [20] explored the optimal dividend policies with the terminal value and excess-of-loss reinsurance. It is mainly the general control problem with Case 2 in this paper. Liu and Hu [26] studied the* *optimal financing and dividend policies with excess-of-loss reinsurance in the case where $P = K = 0$. Those results can be perceived as the limiting form of our results when $P \to 0$, $K \to 0$. Since there exists the fixed transaction cost and the salvage value, whether the insurance company decides to inject new capital or declare bankruptcy relies on the underlying cost of injections and also the potential profits in the future.*

6. Conclusions

This paper investigated the optimal control problem for an insurance company with transaction costs and salvage value where the company controls the risk exposure by the excess-of-loss reinsurance and capital injection based on the symmetry of risk information. Besides the proportional cost, the fixed cost incurred by capital injection is also incorporated. The insurance company's objective is to maximize the expected discounted sum of the salvage value and the cumulative dividends minus the expected discounted cost of capital injection until the ruin time. By considering whether there is capital injection in the surplus process, we construct two categories of suboptimal models and then solve for the corresponding solution in each model. Lastly, we consider the optimal control strategy for the general model without any restriction on the capital injection or the surplus process.

The result shows that, with the excess-of-loss reinsurance, if the insurance company does not intend to inject the capital and allows for the possibility of bankruptcy, the determinations of both the value function and the optimal dividend strategy depend on the salvage value of the company at the ruin time. Furthermore, the retained risk level of excess-of-loss reinsurance also increases with this salvage value. In particular, if the salvage value is zero, the optimal retention is zero and, therefore, the insurance company should cede all the risk to the reinsurance company. However, if the salvage value is great enough, it's optimal to announce the bankruptcy and realize the salvage value at once. If the insurance company is willing to prevent itself from going bankrupt by injecting the capital, it should give more attention to the maximum fixed cost that can be paid when the capital injection occurs. In addition, with the increase of the fixed cost, the company should reduce the retention risk level and raise the dividend barrier at the same time. By doing this, the insurance company can increase the size of capital injection and, therefore, reduce the amount of the fixed cost. However, if the cost is large enough, the best way forward for the insurance company is not to collect new money in order to keep itself from going bankrupt. This coincides with the real-world market situation.

As it is widely known, dividends and capital injection are two important economic activities in an insurance company's operations. Therefore, how to decide the corresponding control strategies is always an imperative problem that remains to be solved. From the main result, we can see that, when an insurance company takes the excess-of-loss reinsurance as the main insurance strategy to manage and control the exposure to risk, the choice of the optimal strategy for the general control problem is determined by some key parameters in the surplus process. Furthermore, due to the existence of the fixed transaction cost and the salvage value, the insurance company should consider the cost of injections and the potential profits in the future when deciding to inject new capital or declare bankruptcy.

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