Review

Group-Theoretic Exploitations of Symmetry in Novel Prestressed Structures

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Abstract: In recent years, group theory has been gradually adopted for computational problems of solid and structural mechanics. This paper reviews the advances made in the application of group theory in areas such as stability, form-finding, natural vibration and bifurcation of novel prestressed structures. As initial prestress plays an important role in prestressed structures, its contribution to structural stiffness has been considered. General group-theoretic approaches for several problems are presented, where certain stiffness matrices and equilibrium matrices are expressed in symmetry-adapted coordinate system and block-diagonalized neatly. Illustrative examples on structural stability analysis, force-finding analysis, and generalized eigenvalue analysis on cable domes and cable-strut structures are drawn from recent studies by the authors. It shows how group theory, through symmetry spaces for irreducible representations and matrix decompositions, enables remarkable simplifications and reductions in the computational effort to be achieved. More importantly, before any numerical computations are performed, group theory allows valuable and effective insights on the behavior or intrinsic properties of a prestressed structure to be gained.

Keywords: symmetry; cable-strut structures; structural stability; form-finding; bifurcation analysis

1. Introduction

Symmetry is one of the most common and important features in nature. Different forms of symmetry can be easily observed from a microscopic view to macroscopic view, from atoms and crystals to large-scale space structures. At the same time, innovative methods that consider the inherent symmetry of a system will be systematic and effective [1–3]. As an important branch of mathematics and vector algebra, group theory is a powerful tool for systematic analysis on symmetric systems. It simplifies the complex computation process. Moreover, it has a direct and qualitative understanding on the intrinsic properties of the system. Thus, group theory has been extensively applied in many fields such as computational physics, electromagnetics, crystal chemistry [4,5] and molecular vibration [6,7].

Notably, applying group theory into structural analysis can significantly improve computational efficiency [8,9]. In recent years, group theory has been gradually adopted by some researchers from structural engineering to solve the involved computational mechanics of symmetric structures [10–14]. Based on group theory, Zlokoviâc [15] and Zingoni [16] showed how to evaluate irreducible representations and symmetry subspaces for symmetric structures. They also summarized the main advantage of group theory through its applications in structural stability, vibration and control. Kangwai et al. [3] briefly described how to utilize group theory to establish a symmetry-adapted coordinate system and perform static analysis on symmetric structures. They concluded that the Fourier method was a special case of the symmetry method using group theory. During nonlinear buckling analysis or frequency analysis of a symmetric structure, if symmetry subspaces associated with certain irreducible representations are established, the tangential

stiffness matrix and mass matrix of the structure would be neatly block-diagonalized, and also different types of buckling modes and vibration shapes can be predicted without further computations [10,16]. Pandia Raj and Guest [17] successfully introduced group theory to the form-finding analysis of tensegrity structures. According to the distribution patterns of the symmetry-adapted equilibrium matrices, analytic solutions for the internal forces of different types of members were effectively obtained. Then, Zhang et al. [18,19] combined group theory and the force-density method to investigate prestress stability and super stability of symmetric tensegrity structures. Subsequently, considerable progress has been made, and important applications have arisen in areas such as vibration, stability, bifurcation and finite element analysis [20–24].

Unfortunately, group theory has not been widely adopted in structural analysis because it is abstract (i.e., mathematically represented) and not commonly used by most structural engineers. In addition, conventional symmetry mentioned in structural mechanics generally refers to mirror symmetry or cyclic symmetry [25]. However, many prestressed space structures are neither cyclically symmetric nor mirror-symmetric, such as the dihedral star tensegrity structures [19] and the two-orbit switch-pitch deployable structures [26]. It should be explained that, in this study, novel prestressed structures refer to a type of engineering structures whose stiffness would be dominated/improved by initial prestresses or similar measures. For example, cable-strut structures, tension structures, tensegrity and cable domes. In fact, many seemingly asymmetric structures still retain a number of symmetry operations [14,27], including proper rotations, reflections, improper rotations, and the inverse [28]. It should be pointed out that using group theory to investigate novel prestressed structures can make full use of all the inherent symmetry. Importantly, group-theoretic approaches can simplify the involved computations [23], and also get fruitful and effective conclusions that might be difficult to obtain from conventional methods [16,29]. Here, through different areas of structural mechanics and illustrative examples drawn from recent work of the authors, this study will describe important developments and applications of group theory for novel prestressed structures. As initial prestress plays an important role in prestressed structures, its contribution to structural stiffness is considered. General group-theoretic approaches for the involved problems are presented, where certain stiffness matrices and equilibrium matrices are expressed in symmetry-adapted coordinate system and block-diagonalized.

2. Group Theory and Its Matrix Representations

A symmetry group G describes all the inherent symmetry of a structure by a set of elements $G = \{g_i, i \in [1, \tau]\}$, where each element g_i denotes an independent symmetry operation, and τ denotes the total number of symmetry operations. This group must obey the following four group properties:

i.   Identity: $\exists E \in G$, for any element $g_i \in G$, it satisfies $g_i \cdot E = E \cdot g_i = g_i$.
ii.   Inverses: $g_i \in G$, $\exists g_i^{-1} \in G \Leftrightarrow g_i \cdot g_i^{-1} = E$.
iii.   Closure: $\forall g_i, g_j \in G \Rightarrow g_i \cdot g_j \in G$.
iv.   Associativity: $\forall g_i, g_j, g_k \in G \Rightarrow (g_i \cdot g_j) \cdot g_k = g_i \cdot (g_j \cdot g_k)$.

where $\cdot$ denotes the multiplication between different elements. In fact, each element g_i in the symmetry group describes an independent symmetry operation for the structure [2]. Under every symmetry operation, the structure can be transformed into a new configuration that is equivalent or indistinguishable from the original configuration. For structures with finite nodes and members, all symmetry operations can be divided into the following five types [14,30]: (a) the identity E; (b) rotation C_n about an n-fold symmetry axis; (c) reflection σ_n along a symmetry plane; (d) inverse S_2; and (e) improper rotation S_n.

Mathematically speaking, each symmetry operation can be taken as a linear transformation in 2D/3D space, and thus the configuration can be expressed as a matrix form:

$$X_S = R_S \cdot X \tag{1}$$

where the matrices X and X_S are the generalized nodal coordinates for the initial configuration and the transformed configuration by the symmetry operation S, and R_S is the corresponding transformation matrix. As the base vectors are variable for different coordinate systems, the transformation matrix R_S is not unique. Fortunately, group theory can be utilized to neatly construct a symmetry-adapted coordinate system, and to decompose the reducible transformation matrix R_S into a series of irreducible representations [23,29]:

$$R_S = \alpha_1 \Gamma^{(1)} \oplus \alpha_2 \Gamma^{(2)} + \cdots \tag{2}$$

where the parameters α_1 and α_2 are the coefficients for the first and second irreducible representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$. Generally, a symmetry group G has $\mu \geq 1$ independent irreducible representations, $\Gamma^{(i)}$ ($i \in [1, \mu]$). The character $\chi\left(\Gamma^{(i)}\right)$ for each irreducible representation keeps constant in different coordinate systems. Specific values associated with different kinds of symmetry operations can be directly read from group theory books [15,28].

3. Stability Analysis on Prestressable Pin-Jointed Structures

Note that novel prestressed structures described in this study refer to various types of prestressable pin-jointed structures, including cable domes, cable nets, tensegrity structures and cable-strut tension structures [31–33]. These structures have innovative and attractive geometries, and often contain internal mechanism modes. Thus, the stability of a prestressable pin-jointed structure, that is, whether it can keep a stable state by the initial prestresses, is the basic and key problem that structural engineers need to solve [29,34,35].

3.1. Positive Definiteness of the Tangent Stiffness Matrix for a Prestressed Pin-Jointed Structure

It is known that the potential energy Π_R of a stable structure states at a minimum [36]. In other words, for an arbitrary virtual nodal displacement vector δd, the second-order variation of the potential energy $\delta^2 \Pi_R$ should satisfy

$$\delta^2 \Pi_R = \delta d^{\mathrm{T}} K_T \delta d > 0 \qquad \text{for all } \delta d \tag{3}$$

which is expressed by the quadratic form of the tangent stiffness matrix K_T of the structure. Therefore, stability analysis of the structure is transformed into the positive definiteness problem of the tangent stiffness matrix [34,35,37]. Guest [38,39] has established a unifying approach for the tangent stiffness matrix of a general prestressed pin-jointed structure:

$$K_T = K_E + K_G \tag{4}$$

In Equation (4), K_E is the elastic stiffness matrix, and K_G is the geometric stiffness matrix contributed by the initial prestresses. Unlike conventional prestressed structures with positive definite matrix K_E, novel prestressed structures have internal mechanisms M, which come from the null space of the positive semi-definite matrix K_E and satisfy

$$K_E M = 0 \tag{5}$$

For a symmetric structure, the tangent stiffness matrix K_T can be block-diagonalized in the symmetry-adapted coordinate system. According to the great orthogonal theorem [3], the transformation matrix U for expressing generalized nodal displacements and external loads is established:

$$U = \sum_{i=1}^{\mu} \sum_{h=1}^{l_i} \oplus U^{(i-h)} \tag{6}$$

where $\boldsymbol{U}^{(i-h)}$ denotes the symmetry subspace associated with the irreducible representation $\Gamma^{(i)}$, μ is the number of irreducible representations of a symmetry group, and l_i is the dimension of the irreducible representation $\Gamma^{(i)}$. Then, the symmetry-adapted tangent stiffness matrix $\overline{\boldsymbol{K}}_T$ of the structure is

$$
\begin{aligned}
\overline{\boldsymbol{K}}_T = \boldsymbol{U}^{\mathrm{T}} \boldsymbol{K}_T \boldsymbol{U} \quad &= \mathrm{diag}\left[\overline{\boldsymbol{K}}_T^{(1-1)}, \cdots, \overline{\boldsymbol{K}}_T^{(i-h)}, \cdots, \overline{\boldsymbol{K}}_T^{(\mu-l_\mu)}\right] \\
&= \begin{bmatrix}
\overline{\boldsymbol{K}}_T^{(1-1)} & 0 & 0 & 0 & 0 \\
0 & \ddots & 0 & 0 & 0 \\
0 & 0 & \overline{\boldsymbol{K}}_T^{(i-h)} & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & \overline{\boldsymbol{K}}_T^{(\mu-l_\mu)}
\end{bmatrix}
\end{aligned}
\tag{7}
$$

where diag [ ] denotes a diagonal matrix, and $\overline{\boldsymbol{K}}_T^{(i-h)}$ represents a typical small-sized block matrix along the diagonal of the matrix $\overline{\boldsymbol{K}}_T$. Equation (7) shows that the original tangential stiffness matrix is decomposed into a series of block matrices, and the total number of the block matrices is computed by

$$
q = \sum_{i=1}^{\mu} l_i
\tag{8}
$$

On condition that the structural configuration and initial prestresses are fully symmetric, the stiffness matrices $\boldsymbol{K}_E$ and $\boldsymbol{K}_G$ can be similarly decomposed into the block-diagonalized ones:

$$
\overline{\boldsymbol{K}}_E = \boldsymbol{U}^{\mathrm{T}} \boldsymbol{K}_E \boldsymbol{U} = \mathrm{diag}\left[\overline{\boldsymbol{K}}_E^{(1-1)}, \cdots, \overline{\boldsymbol{K}}_E^{(i-h)}, \cdots, \overline{\boldsymbol{K}}_E^{(\mu-l_\mu)}\right]
\tag{9}
$$

$$
\overline{\boldsymbol{K}}_G = \boldsymbol{U}^{\mathrm{T}} \boldsymbol{K}_G \boldsymbol{U} = \mathrm{diag}\left[\overline{\boldsymbol{K}}_G^{(1-1)}, \cdots, \overline{\boldsymbol{K}}_G^{(i-h)}, \cdots, \overline{\boldsymbol{K}}_G^{(\mu-l_\mu)}\right]
\tag{10}
$$

where $\overline{\boldsymbol{K}}_E$ and $\overline{\boldsymbol{K}}_G$ are symmetry-adapted elastic stiffness matrix and geometric stiffness matrix. For a positive integer $h \in [1, l_i]$, the block stiffness matrices $\overline{\boldsymbol{K}}_T^{(i-h)}$, $\overline{\boldsymbol{K}}_E^{(i-h)}$, and $\overline{\boldsymbol{K}}_G^{(i-h)}$ are associated with the irreducible representation $\Gamma^{(i)}$.

Therefore, to evaluate the positive definiteness of the original tangent stiffness matrix $\boldsymbol{K}_T$, each block matrix $\overline{\boldsymbol{K}}_T^{(i-h)}$ can be independently evaluated by numerical approaches, e.g., the eigenvalue decomposition

$$
\overline{\boldsymbol{K}}_T^{(i-h)} \overline{\boldsymbol{\Psi}}^{(i-h)} = \overline{\boldsymbol{\Psi}}^{(i-h)} \overline{\boldsymbol{\lambda}}^{(i-h)}, \quad \forall\, i \in [1, \mu],\ j \in [1, l_i]
\tag{11}
$$

where $\overline{\boldsymbol{\lambda}}^{(i-h)}$ and $\overline{\boldsymbol{\Psi}}^{(i-h)}$ are the eigenvalue and eigenvector extracted from the symmetry subspace associated with the irreducible representation $\Gamma^{(i)}$. Because any similarity transformations do not alter the eigenvalues of a matrix [16,40], the eigenvalues obtained from Equation (11) belong to the eigenvalues λ of the original tangent stiffness matrix

$$
\overline{\boldsymbol{\lambda}}^{(i-h)} \subset \boldsymbol{\lambda}, \quad \forall\, i \in [1, \mu],\ h \in [1, l_i]
\tag{12}
$$

and the corresponding eigenvector $\boldsymbol{\Psi}^{(i-h)}$ can be expressed as

$$
\boldsymbol{\Psi}^{(i-h)} = \boldsymbol{U}^{(i-h)} \overline{\boldsymbol{\Psi}}^{(i-h)}, \quad \forall\, i \in [1, \mu],\ h \in [1, l_i]
\tag{13}
$$

where the symmetry subspace (see Equation (6)) reveals that the eigenvector $\mathbf{\Psi}^{(i-h)}$ holds the same symmetry associated with the irreducible representation $\Gamma^{(i)}$. Subsequently, the full eigenvector matrix $\mathbf{\Psi}$ can be obtained by

$$\mathbf{\Psi} = \sum_{i=1}^{\mu} \sum_{h=1}^{l_i} \oplus \mathbf{\Psi}^{(i-h)} \tag{14}$$

Hence, by using group theory, the positive definiteness problem of the original tangent stiffness matrix is transformed into many independent sub-problems. Consequently, the computational complexity required for eigenvalue decomposition will be significantly reduced.

3.2. Necessary Conditions for Structural Stability of Prestressed Pin-Jointed Structures

In the symmetry-adapted coordinate system, Equation (3) can be rewritten as

$$\delta\overline{\boldsymbol{d}}^{\mathrm{T}}\overline{\boldsymbol{K}}_T\delta\overline{\boldsymbol{d}} = \boldsymbol{\alpha}^{\mathrm{T}}\overline{\boldsymbol{M}}^{\mathrm{T}}\left(\overline{\boldsymbol{K}}_E + \overline{\boldsymbol{K}}_G\right)\overline{\boldsymbol{M}}\boldsymbol{\alpha} = \boldsymbol{\alpha}^{\mathrm{T}}\overline{\boldsymbol{M}}^{\mathrm{T}}\overline{\boldsymbol{K}}_G\overline{\boldsymbol{M}}\boldsymbol{\alpha} > 0 \tag{15}$$

where $\delta\boldsymbol{d} = \overline{\boldsymbol{M}}\boldsymbol{\alpha}$ is the symmetry-adapted nodal displacement vector, $\overline{\boldsymbol{M}}$ is the symmetry-adapted mechanism mode, and $\overline{\boldsymbol{K}}_E\overline{\boldsymbol{M}} = \mathbf{0}$ (refer to Equation (5)). To satisfy the structural stability, the matrix $\overline{\boldsymbol{K}}_G$ should be positive definite or positive semi-definite, i.e., $\sum \lambda(\overline{\boldsymbol{K}}_G) > 0$. Then, a necessary condition for stability of the symmetric prestressed structure can be obtained [36]. The structure must have at least a single state of self-stress with full symmetry, given by

$$\Gamma^{(1)} \subset \Gamma_s \tag{16}$$

where Γ_s is the symmetry representation of the self-stress states. On the other hand, when the structure has only self-stress states without full symmetry ($\Gamma^{(1)}\Gamma_s$), the symmetry of the self-stress states is associated with the *i*-th type irreducible representation $\Gamma^{(i)}$, where $1 < i \leq \mu$. Thereafter, the symmetry-adapted geometric stiffness matrix $\overline{\boldsymbol{K}}_G$ cannot be decomposed into block-diagonalized matrices. However, it has certain regularity, and all the diagonal blocks are zero matrices. At this point, the trace of the matrix $\overline{\boldsymbol{K}}_G$ is 0, $\sum \lambda(\overline{\boldsymbol{K}}_G) = 0$. Therefore, this matrix is neither a positive definite nor a negative definite matrix, and does not satisfy the necessary condition for the stability (see Equations (3) and (15)). In this case, the structure must be an unstable (finite) structure.

3.3. Example: C_{2v} Symmetric Cable-Strut Structures

Stability analysis on two simple two-dimensional symmetric cable-strut structures is presented, to explain the basic group-theoretic process. As shown in Figure 1, both of the structures consist of two vertical struts and six cables. The length of struts is L_c = 2000 mm, the distance between the struts is 2000 mm, and the minimum distance from the boundary nodes (nodes 5 and 6) to the struts is 1000 mm. The elastic modulus of the cables and that of the struts is $E_t = 1.9 \times 10^5$ MPa and $E_c = 2 \times 10^5$ MPa; the cross-sectional area of the cables and that of the struts is $A_t = 500$ mm^2 and $A_c = 3000$ mm^2. Both structures have a single mechanism mode ($m = 1$) and a self-stress state ($s = 1$). Note that the initial prestress of the vertical struts is -100 kN, and prestresses of other members can be uniquely determined by the self-stress state.
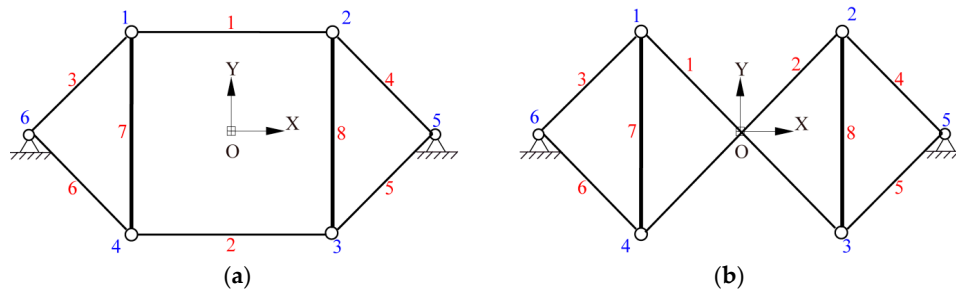
Figure 1. Two-dimensional C_{2v} symmetric cable-structures (**a**) with horizontal cables; (**b**) with cross cables.

The structures are C_{2v} symmetric as they keep equivalent by the following four symmetry operations: Identity, rotation by π, and the reflections along X and Y axes. These four symmetry operations constitute a $C_{2v} = \{E, C_2, \sigma_X, \sigma_Y\}$ symmetry group. The C_{2v} group has $\mu = 4$ one-dimensional irreducible representations, $\Gamma^{(1)} = A_1$, $\Gamma^{(2)} = A_2$, $\Gamma^{(3)} = B_1$, $\Gamma^{(4)} = B_2$, and $l_1 = l_2 = l_3 = l_4 = 1$. Through Equations (6) and (7), the tangential stiffness matrices can be decomposed into four 2×2 block matrices, as listed in Table 1.

Table 1. Different blocks $\overline{K}_T^{(i)}$ of symmetry-adapted tangent stiffness matrix ($\times E_c A_c / L_c$).

Irreducible Representation $\Gamma^{(i)}$	A_1	A_2	B_1	B_2
Figure 1a	$\begin{bmatrix} 2.1121 & 0.1118 \\ 0.1118 & 0.4288 \end{bmatrix}$	$\begin{bmatrix} 0.1125 & 0.1118 \\ 0.1118 & 0.1121 \end{bmatrix}$	$\begin{bmatrix} 2.1125 & 0.1118 \\ 0.1118 & 0.1121 \end{bmatrix}$	$\begin{bmatrix} 0.1121 & 0.1118 \\ 0.1118 & 0.4285 \end{bmatrix}$
Figure 1b	$\begin{bmatrix} 2.2241 & 0 \\ 0 & 0.2241 \end{bmatrix}$	$\begin{bmatrix} 0.2241 & 0 \\ 0 & 0.2238 \end{bmatrix}$	$\begin{bmatrix} 2.1120 & 0.1119 \\ 0.1119 & 0.1120 \end{bmatrix}$	$\begin{bmatrix} 0.1120 & 0.1119 \\ 0.1119 & 0.1117 \end{bmatrix}$

Therefore, the eigenvalues and eigenvectors can be independently extracted from each of the block matrices. Figure 2 plots the eigenvalue decomposition results of the tangent stiffness matrix of the structure shown in Figure 1a. The endpoints of the polyline and the discrete dots are the solutions obtained by the symmetry method and the conventional method, respectively.
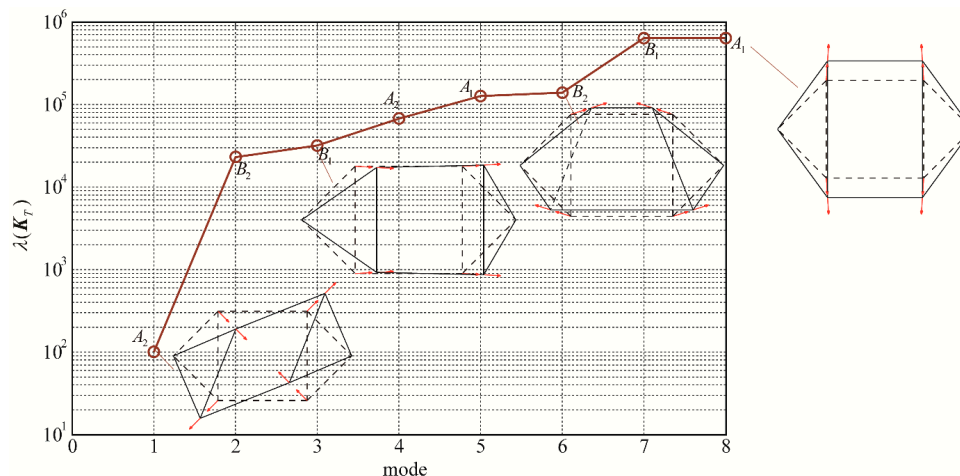

Figure 2. Eigenvalues extracted from different symmetry subspaces for of the cable-strut structure shown in Figure 1a.

As the minimum eigenvalue is positive, $\lambda_{\min}(K_T) > 0$, this structure is stable. However, the minimum eigenvalue is much smaller than the others, as shown in Figure 2. This is because the structure has an internal and infinitesimal mechanism mode, and the first-order eigenvalue is

mainly determined by the initial prestress [40,41]. Besides, the results obtained by the symmetry method are exactly the same as those of the conventional method. Because the tangent stiffness matrix has been diagonalized into four small-sized blocks, computation cost of the symmetry method is only 0.004 s. However, the conventional method concerns the original matrix and takes 0.0156 s.

In Figure 2, the first, third, sixth, and eighth modes of the eigenvectors obtained from different symmetry spaces are also plotted, where A_1 indicates full symmetry and the others indicate lower-order symmetry. The short arrows indicate the motion trend of the free nodes. It turns out the eigenvectors keep consistent with the inherent symmetry properties of specific symmetry subspaces. For instance, the fifth and eighth eigenvectors come from the full symmetry subspace and thus they have full symmetry of C_{2v}. Similarly, since the first and fourth eigenvectors are from the symmetry subspace A_2, they exhibit rotational symmetry of C_{2v}. The third and seventh eigenvectors keeps C_v symmetry along the X axis, while the second and the sixth eigenvectors keeps C_v symmetry along the Y axis.

On the other hand, stability of the structure shown in Figure 1b is also studied, where the upper and lower horizontal cables are replaced by two cross cables. As shown in Figure 3, this structure has a single state of self-stress with full symmetry (i.e., C_{2v} symmetry) and an internal mechanism mode with lower-order symmetry (i.e., C_v symmetry). However, it has been verified that the internal mechanism mode cannot be rigidified by prestressing (see Figure 3), as the minimum eigenvalue $\lambda_{\min}(\boldsymbol{K}_T) = -0.037 < 0$. Consequently, this structure is unstable, although it satisfies the necessary conditions given by Equation (16). It should be pointed out that the results for this structure are consistent with the reported ones [42].
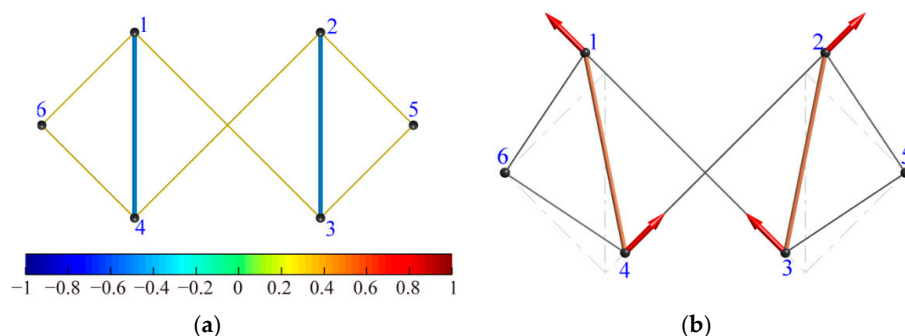

Figure 3. Static and kinematic indeterminacy of the cable-strut shown in Figure 1b: (**a**) C_{2v} symmetric self-stress state; (**b**) C_v symmetric mechanism mode.

4. Form-Finding Analysis on Tensegrity Structures

A significance of novel prestressed structures mentioned in this study is that these structures rely on initial prestresses to obtain or enhance the structural stiffness. Therefore, form-finding (or force-finding) analysis on these structures is important [43,44]. For a structure with multiple self-stress states and complex geometry, Yuan et al. [45] considered the symmetry of the structure and proposed the concepts of integral self-stress state and feasible prestress to seek an effective and proper distribution for initial prestresses. They proposed the double singular value decomposition(DSVD) method for form-finding, which manually classified the members into different types and imposed symmetry constraints on the internal forces in the members. Zhang et al. [46] have utilized the energy method to obtain the integral self-stress state. Recently, some researchers [17,18,32] have combined the force density method with the structural symmetry to simplify the form-finding analysis for novel prestressed structures.

Admittedly, these methods rely on the correct classification for the members. When the configuration becomes complicated or the number of members rises [47], the involved form-finding becomes difficult. Notably, in a symmetry-adapted coordinate system, each block matrix is associated with a certain symmetry subspace, and thus the null space of the block with full symmetry

necessarily contains independent self-stress states that satisfy the requirements of full symmetry [36,48]. In this section, the authors will describe how to adopt group theory for computing the integral self-stress states.

4.1. Integral Self-Stress State Obtained from the Block with Full Symmetry

According to the force equilibrium condition for each node of a structure, the equilibrium equation of the structure can be established:

$$\boldsymbol{Ht} = \boldsymbol{P} \tag{17}$$

where $\boldsymbol{H}$ is the equilibrium matrix, $\boldsymbol{t}$ is the internal force vector, and $\boldsymbol{P}$ is the external load vector. Because most novel prestressed structures are statically indeterminate, the self-stress states $\boldsymbol{S}$ come from the null space of the singular matrix $\boldsymbol{H}$ (i.e., $\boldsymbol{HS} = \boldsymbol{0}$). In the symmetry-adapted coordinate system, the symmetry-adapted equilibrium matrix $\overline{\boldsymbol{H}}$ can be transformed from the original matrix $\boldsymbol{H}$:

$$\overline{\boldsymbol{H}} = \boldsymbol{V}_P^{\mathrm{T}}\boldsymbol{H}\boldsymbol{V}_t = \mathrm{diag}\left[\overline{\boldsymbol{H}}^{(1-1)}, \cdots, \overline{\boldsymbol{H}}^{(i-h)}, \cdots, \overline{\boldsymbol{H}}^{(\mu-l_\mu)}\right] \tag{18}$$

where $\boldsymbol{V}_P$ is the transformation matrix associated with the load vector $\boldsymbol{P}$, and $\boldsymbol{V}_t$ is the transformation matrix associated with the internal force vector $\boldsymbol{t}$; the positive integer $i \in [1, \mu]$, $h \in [1, l_i]$, and l_i and l_μ are the dimensions of the irreducible representations $\Gamma^{(i)}$ and $\Gamma^{(\mu)}$. Equation (18) indicates that each block matrix $\overline{\boldsymbol{H}}^{(i-h)}$ corresponds to a symmetry subspace associated with the irreducible representation $\Gamma^{(i)}$. The first block matrix is associated with full symmetry, while the last block indicates the lowest-order symmetry. Thus, integral self-stress states necessarily come from the first block matrix of the matrix $\overline{\boldsymbol{H}}$. Notably, as each block matrix is independent, the first block matrix $\overline{\boldsymbol{H}}^{(1-1)}$ can be directly computed by

$$\overline{\boldsymbol{H}}^{(1-1)} = \left(\boldsymbol{V}_P^{(1-1)}\right)^{\mathrm{T}}\boldsymbol{H}\boldsymbol{V}_t^{(1-1)} \tag{19}$$

where $\boldsymbol{V}_P^{(1-1)}$ and $\boldsymbol{V}_t^{(1-1)}$ are the full symmetry subspaces for the load vector and the internal force vector, respectively. Both matrices are associated with the first irreducible representation $\Gamma^{(1-1)}$. Then, the null space $\overline{\boldsymbol{S}}^{(1-1)}$ of the block matrix $\overline{\boldsymbol{H}}^{(1-1)}$ can be solved, which satisfies

$$\overline{\boldsymbol{H}}^{(1-1)}\overline{\boldsymbol{S}}^{(1-1)} = \boldsymbol{0} \tag{20}$$

The integral self-stress state of the structure is given by [48]

$$\boldsymbol{S}' = \boldsymbol{V}_t^{(1-1)}\overline{\boldsymbol{S}}^{(1-1)} \tag{21}$$

Note that the integral self-stress state in Equation (21) has considered the whole symmetry of the structure, which allows the members of the same type to retain equal prestress. Thus, the solution space for the form-finding problem of prestressed structures can be effectively reduced, especially for the structures with multiple self-stress states [48,49]. Moreover, this symmetry method does not need to manually classify the members into different groups in advance, and it avoids repeating calculations of conventional methods. Thus, the involved computation process is simple and convenient, and particularly suitable for structures with complex geometry [50].

4.2. Example: A D_3 Symmetric Tensegrity Structure

Figure 4 shows the geometric configuration and symmetry rotations for a simple prismatic tensegrity structure, which consists of six pin-joints, six tension cables and three compression struts. The twist angle between the bottom and top triangles is $\pi/6$; the radius of the circumcircle of the triangles is unit length, and the vertical height of the structure is unit length.
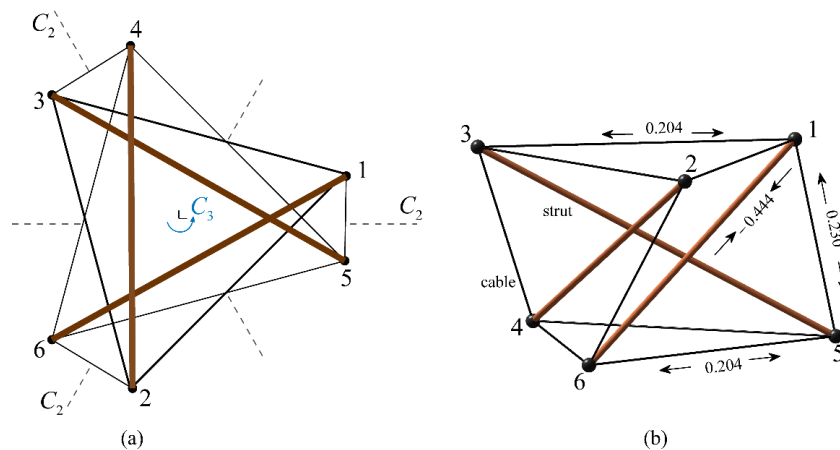
Figure 4. A simple tensegrity structure with D_3 symmetry [48]: (**a**) three two-fold rotations; (**b**) integral prestress mode with full symmetry.

This symmetric structure has $\tau = 6$ independent symmetry operations: the identity, two rotations around the vertical C_3 axis, and three two-fold rotations C_2 around the axes connecting the center of the structure and the mid-points of the diagonal cables (indicated by the dotted lines in Figure 4a). Thus, this structure belongs to the D_3 group [34,51]. Thereafter, the full symmetry subspaces corresponding to the internal force vector and the load vector are

$$
V_t^{(1-1)} = \begin{bmatrix} 0.41 & 0.41 & 0.41 & 0.41 & 0.41 & 0.41 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.58 & 0.58 & 0.58 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.58 & 0.58 & 0.58 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{22}
$$

$$
V_P^{(1-1)} = \begin{bmatrix} -0.41 & 0 & 0 & 0.2 & 0.35 & 0 & 0.2 & -0.35 & 0 & -0.41 & 0 & 0 & 0.2 & 0.35 & 0 & 0.2 & -0.35 & 0 \\ 0 & -0.41 & 0 & -0.35 & 0.2 & 0 & 0.35 & 0.2 & 0 & 0 & 0.41 & 0 & 0.35 & -0.2 & 0 & -0.35 & -0.2 & 0 \\ 0 & 0 & -0.41 & 0 & 0 & -0.41 & 0 & 0 & -0.41 & 0 & 0 & 0.41 & 0 & 0 & 0.41 & 0 & 0 & 0.41 \end{bmatrix}^{\mathrm{T}} \tag{23}
$$

Then, the first block matrix $\overline{H}^{(1-1)}$ of the symmetry-adapted equilibrium matrix is obtained by Equation (19)

$$
\overline{H}^{(1-1)} = \left(V_P^{(1-1)} \right)^{\mathrm{T}} H V_t^{(1-1)} = \begin{bmatrix} 0.447 & -0.562 & 0 \\ -1.669 & -0.324 & -1.253 \\ 0 & 1.267 & 0.656 \end{bmatrix} \tag{24}
$$

Because this 3×3 matrix $\overline{H}^{(1-1)}$ is a rank-deficient (its rank $\bar{r}^{(1-1)} = 2$), it has a $\bar{s}^{(1-1)} = 1$-dimensional null space

$$
\overline{S}^{(1-1)} = \begin{bmatrix} 0.500 & 0.398 & -0.769 \end{bmatrix}^{\mathrm{T}} \tag{25}
$$

Therefore, the integral self-stress state of the structure is:

$$
\begin{aligned}
S' = V_t^{(1-1)} \overline{s}^{(1-1)} &= [t_1 \cdots t_6 \mid t_7 \cdots t_9 \mid t_{10} \cdots t_{12}]^{\mathrm{T}} \\
&= [0.204 \cdots 0.204 \mid 0.230 \cdots 0.230 \mid -0.444 \cdots -0.444]^{\mathrm{T}}
\end{aligned} \tag{26}
$$

As shown in Figure 4b, the obtained integral self-stress state satisfies the feasibility condition for the members, where six horizontal cables, three vertical cables, or three struts respectively belong to the same type. All the cables are in tension while the struts are in compression. It should be pointed out that this symmetry method just computes the null space of the first block matrix and extract integral self-stress states from all the independent self-stress. Because the dimension of the first block matrix is significantly smaller than that of the original matrix, the computational complexity is effectively reduced. In comparison with the conventional DSVD method, this method avoids manual

classification on the members and repeated computations on the matrices. As far as a highly symmetric structure with many self-stress states and members is concerned, the computational efficiency of this symmetry method is considerably improved [48].

5. Generalized Eigenvalue Problems of Symmetric Prestressed Structures

5.1. Symmetry-Adapted Frequency Analysis

Generalized eigenvalue problems can be frequently found among structural analysis [16,40], which generally is a challenge from a computational perspective. For example, frequency analysis can be taken as a well-known generalized eigenvalue problem in terms of the tangent stiffness matrix $\boldsymbol{K}_T$ and the mass matrix $\boldsymbol{M}$

$$\boldsymbol{K}_T \boldsymbol{\Psi} - \omega^2 \boldsymbol{M} \boldsymbol{\Psi} = 0 \tag{27}$$

which describes free vibrations of a general structure. In Equation (27), ω denotes a circular frequency and $\boldsymbol{\Psi}$ is the corresponding mode shape. On condition that this structure is symmetric, group-theoretic process considers smaller block matrices rather than the full matrices. Using the transformation matrix $\boldsymbol{U}$ in Equation (7), the mass matrix $\boldsymbol{M}$ can be decomposed into similar block-diagonalized forms

$$\overline{\boldsymbol{M}} = \boldsymbol{U}^\mathrm{T} \boldsymbol{M} \boldsymbol{U} = \mathrm{diag}\left[\overline{\boldsymbol{M}}^{(1-1)}, \cdots, \overline{\boldsymbol{M}}^{(i-h)}, \cdots, \overline{\boldsymbol{M}}^{(\mu-l_\mu)} \right] \tag{28}$$

where $\overline{\boldsymbol{M}}$ is a symmetry-adapted mass matrix. For any positive integer $h \in [1, l_i]$, the block matrices $\overline{\boldsymbol{K}}_T^{(i-h)}$ and $\overline{\boldsymbol{M}}^{(i-h)}$ come from the specific symmetry subspace associated with the irreducible representation $\Gamma^{(i)}$. Therefore, to analyze the vibration characteristics of the structure, eigenvalue problems with smaller matrices can be independently solved. For the block matrices $\overline{\boldsymbol{K}}_T^{(i-h)}$ and $\overline{\boldsymbol{M}}^{(i-h)}$, it satisfies

$$\overline{\boldsymbol{K}}_T^{(i-h)} \overline{\boldsymbol{\Psi}}^{(i-h)} - \left[\overline{\omega}^{(i-h)} \right]^2 \overline{\boldsymbol{M}}^{(i-h)} \overline{\boldsymbol{\Psi}}^{(i-h)} = 0, \quad \forall\, i \in [1, \mu],\ h \in [1, l_i] \tag{29}$$

where $\overline{\omega}^{(i-h)}$ and $\overline{\boldsymbol{\Psi}}^{(i-h)}$ are the eigenvalues and eigenvectors extracted from the symmetry subspace associated with the irreducible representation $\Gamma^{(i)}$. Since the similarity transformation does not change the eigenvalues of the matrix, the obtained generalized eigenvalues are included in the solutions to the original eigenvalue problem [40],

$$\overline{\omega}^{(i-h)} \subset \omega, \quad \forall\, i \in [1, \mu],\ h \in [1, l_i] \tag{30}$$

The corresponding mode shape can be expressed as

$$\boldsymbol{\Psi}^{(i-h)} = \boldsymbol{U}^{(i-h)} \overline{\boldsymbol{\Psi}}^{(i-h)}, \quad \forall\, i \in [1, \mu],\ h \in [1, l_i] \tag{31}$$

where the symmetry subspace $\boldsymbol{U}^{(i-h)}$ associated with $\Gamma^{(i)}$ indicates the symmetry of the eigenvector $\boldsymbol{\Psi}^{(i-h)}$. It is helpful for predicting the vibration modes for a symmetric structure. In addition, the complete vibration mode matrix $\boldsymbol{\Psi}$ can be obtained from

$$\boldsymbol{\Psi} = \sum_{i=1}^{\mu} \sum_{h=1}^{l_i} \oplus \boldsymbol{\Psi}^{(i-h)} \tag{32}$$

In summary, using the symmetry method based on group theory, the original frequency analysis has been transformed into a series of independent sub-problems.

5.2. Symmetry-Adapted Buckling Analysis

Buckling analysis of a structure is essentially about the generalized eigenvalue problem of the tangent stiffness matrix and the geometric stiffness matrix:

$$f(\Phi, \lambda) = (K_T + \lambda K_G)\Phi = 0 \tag{33}$$

where the function f is defined to describe the buckling problem of the structure, λ is the eigenvalue and Φ is the corresponding eigenvector. According to the inherent symmetry of the structure, the above equation satisfies [52]:

$$f(R_s\Phi, \lambda) = R_s f(\Phi, \lambda), \forall s \in [1, \tau] \tag{34}$$

where R_s is the transformation matrix for a symmetry operation S. According to the external loads and their symmetries, symmetry subspaces need to be established in a certain subgroup $G_1 \subseteq G$. Then, the stiffness matrices K_T and K_G in Equation (33) can be decomposed into a similar block-diagonal form:

$$\overline{K_T} = V^T K_T V = \text{diag}\left[\overline{K_T}^{(1-1)}, \cdots, \overline{K_T}^{(1-l_1)}, \overline{K_T}^{(2-1)}, \cdots, \overline{K_T}^{(2-l_2)}, \cdots, \overline{K_T}^{(\mu-1)}, \cdots, \overline{K_T}^{(\mu-l_\mu)}\right] \tag{35}$$

$$\overline{K_G} = V^T K_G V = \text{diag}\left[\overline{K_G}^{(1-1)}, \cdots, \overline{K_G}^{(1-l_1)}, \overline{K_G}^{(2-1)}, \cdots, \overline{K_G}^{(2-l_2)}, \cdots, \overline{K_G}^{(\mu-1)}, \cdots, \overline{K_G}^{(\mu-l_\mu)}\right] \tag{36}$$

where the matrix V is an orthogonal transformation matrix for expression in the symmetry-adapted coordinate system. Because each block matrix is independent, the original buckling problem can be solved by solving the subproblems in parallel [52]

$$(\overline{K_T}^{(i-j)} + \overline{\lambda}^{(i-j)}\overline{K_G}^{(i-j)})\overline{\Phi}^{(i-j)} = 0, \quad \forall i \in [1, \mu], j \in [1, l_i] \tag{37}$$

Recall that similarity transformations do not alter the generalized eigenvalues of the original matrix, the eigenvalues $\overline{\lambda}^{(i-j)}$ obtained from Equation (37) are included in the solutions to the original buckling problem,

$$\overline{\lambda}^{(i-j)} \subset \lambda, \quad \forall i \in [1, \mu], \ j \in [1, l_i] \tag{38}$$

and the corresponding buckling shape can be expressed as

$$\Phi^{(i-j)} = V^{(i-j)} \cdot \overline{\Phi}^{(i-j)}, \quad \forall i \in [1, \mu], \ j \in [1, l_i] \tag{39}$$

5.3. Illustrative Example: A C_{12v} Symmetric Cable Dome Structure

The well-known Levy cable dome is a typical prestressed cable-strut structure [40,53]. Figure 5 shows a highly symmetric Levy cable dome with a diameter of 100 m. This structure is composed of 84 pin-joints, 36 compression struts and 168 tension cables, where the boundary nodes 1–12 are constrained in three directions. Note that cross-sectional areas and initial prestresses of different types of members of this cable dome structure have been reported [40].
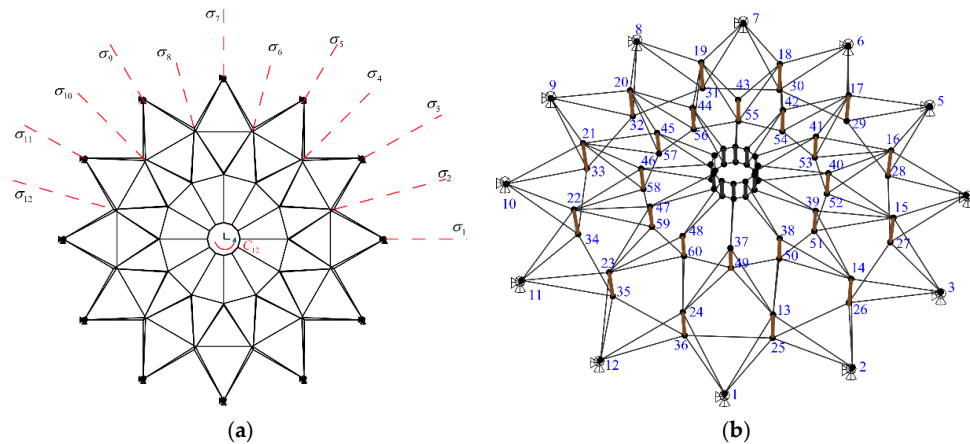
Figure 5. A C_{12v} symmetric cable dome structures [40]: (**a**) twelve rotations and twelve reflections indicated in the plan view; (**b**) 3D geometric configuration.

In the view of symmetry, this cable dome structure keeps unshifted by twelve rotations along the 12-fold rotation axis C_{12}, and twelve reflections $\sigma_1 - \sigma_{12}$, as shown in Figure 5a. Thus, this structure is C_{12v} symmetric. According to the point-group theory table [28], the C_{12v} symmetry group has four one-dimensional ($l_{i \in [1,4]} = 1$) irreducible representations A_1, A_2, B_1 and B_2, and five two-dimensional ($l_{i \in [5,9]} = 2$) irreducible representations $E_1 - E_5$. To establish the symmetry-adapted coordinate system, the 216×216 orthogonal transformation matrix U for the cable dome is obtained from Equation (6). This matrix is composed by independent vectors associated with fourteen symmetry subspaces, where the first vector $U^{(A_1)}$ for the full symmetry subspace is a 216×12 matrix, $U^{(A_2)}$ is a 216×6 matrix, $U^{(B_1)}$ is a 216×10 matrix, $U^{(B_2)}$ is a matrix, and all the vectors $U^{(E_{11})} - U^{(E_{52})}$ for the lower-order symmetry subspaces are 216×18 matrices. These 14 independent symmetry subspaces are utilized to decompose the 216×216 stiffness matrix or the mass matrix into 14 block matrices with smaller dimensions. For example, the distribution patterns of nonzero entries of the original tangent stiffness matrix and the block-diagonalized matrix are shown in Figure 6. As shown in Figure 6b, all the block matrices are independent and distribute along the main diagonal.
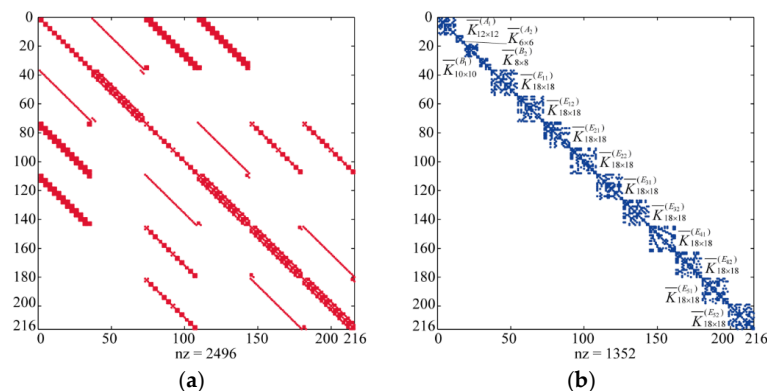

Figure 6. Distribution patterns of nonzero entries of stiffness matrices of the C_{12v} symmetric cable dome: (**a**) original tangent stiffness matrix K_T; (**b**) block-diagonalized stiffness matrix $\overline{K_T}$.

As far as the frequency analysis is concerned, the original problem is neatly simplified into fourteen sub-problems involved with small-sized matrices. Figure 7 shows the first 100 frequencies obtained by the proposed symmetry method, which is compared with the results by the conventional numerical method. In addition, to study the influence of initial prestresses on the natural frequencies, the C_{12v} symmetric cable dome with different prestress levels are analyzed by the proposed method

(the symmetry subspaces keep invariant). The prestress levels are respectively 0, 0.25*t*, 0.5*t*, 2*t* and 4*t*, and the initial prestresses *t* are determined by the feasible prestress modes [40].
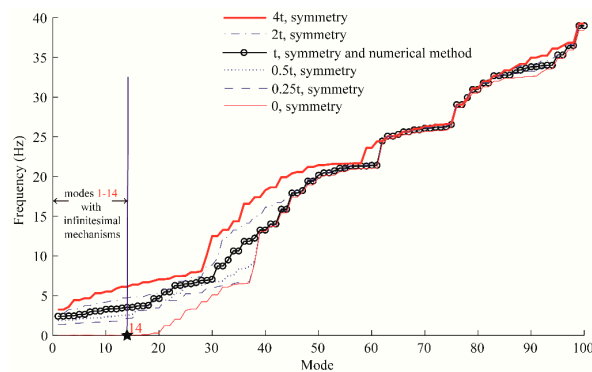

Figure 7. First 100 frequencies of the C_{12v} symmetric cable dome with different prestress levels [40].

It can be noticed from Figure 7 that the obtained results are in good agreement with those of the numerical method, where the maximum error is 10^{-9}. Lower-order natural frequencies are low and intensively distributed. Notably, the first 14 frequencies are dominated by the prestress level, as they are associated with $m = 14$ infinitesimal mechanism modes. In fact, the initial prestresses also improve the structural stiffness of the cable dome. After increasing the initial prestress level, the first 50 frequencies of this structure increase significantly.

On the other hand, Figure 8 depicts the first eight vibration shapes of the C_{12v} symmetric cable dome [40]. It can be noticed that this symmetric dome structure has many repeated eigenvalues and equivalent eigenvectors, such as modes 1–2, modes 3–4, and modes 5–6. In fact, this phenomenon is ubiquitous for most symmetric structures. This is because the roots computed from the symmetry spaces for multi-dimensional irreducible representations are identical, and the generalized eigenvalues for these symmetry subspaces are exactly the same. In addition, each vibration shape obtained from lower-order symmetry subspace $\boldsymbol{U}^{(i-h)}$ does not maintain its full symmetry (i.e., C_{12v}). Then, it may be reduced to a low-order symmetry. Because the seventh vibration shape is obtained from the symmetry subspace $\Gamma^{(2-1)} = A_2$, it has rotational symmetry (i.e., C_{12}). In other words, the symmetry of all the vibration modes can be predicted from symmetry subspaces in advance [16,29], without numerical computing.
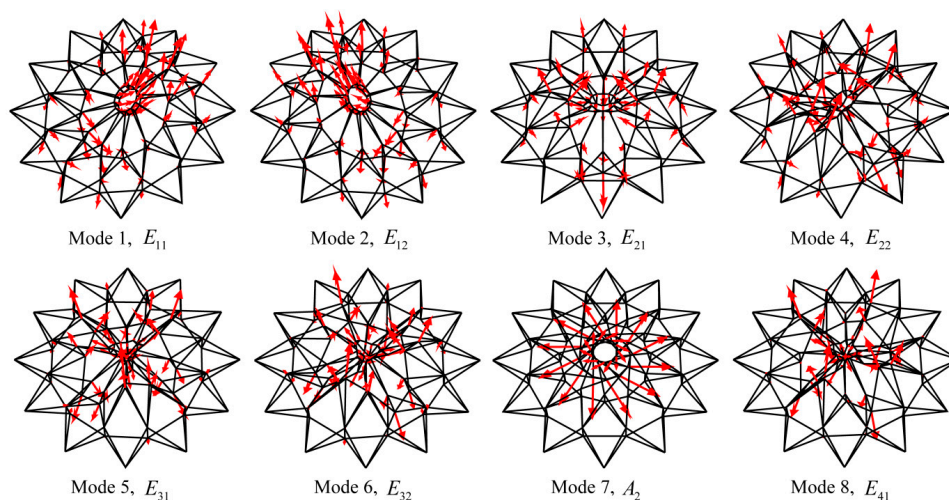

Figure 8. First eight vibration shapes of the C_{12v} symmetric cable dome [40].

6. Conclusions

This paper has described some new developments of symmetry analysis on novel prestressed structures using group theory. Through theoretical group-theoretic approaches and illustrative examples, basic process and key features of group-theoretic approaches in stability analysis, form-finding analysis, frequency analysis and buckling analysis have been identified. A significant advantage of these group-theoretic approaches is that the involved large-sized matrices can be neatly block-diagonalized into many smaller-sized and independent matrices. Thus, the computational complexity for dealing with the original problem will be significantly reduced. More importantly, these methods can qualitatively reveal certain physical meanings and obtain effective insights into the involved problems. It is worth mentioning that group-theoretic approaches are systematic and general. When a structure has only the lowest-order symmetry, the method still works and degenerates into a conventional method. This research is helpful for enriching developments in the field of symmetric structures.

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