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Gauss Map and Its Applications on Ruled Submanifolds in Minkowski Space

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Abstract: We study ruled submanifolds in Minkowski space in regard to the Gauss map satisfying some partial differential equation. As a generalization of usual cylinders, cones and null scrolls in a three-dimensional Minkowski space, a cylinder over a space curve, a product manifold of a right cone and a k-plane, a product manifold of a hyperbolic cone and a k-plane which look like kinds of cylinders over cones in 3-space, and the generalized B-scroll kind in Minkowski space are characterized with the partial differential equation regarding the Gauss map, where k is a positive integer.

Keywords: finite-type immersion; pointwise 1-type Gauss map of the second kind; generalized B-scroll kind

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1. Introduction

According to Nash's imbedding theorem, a Riemannian manifold can be imbedded in a Euclidean space with considerably high codimension. That naturally enables us to study Riemannian manifolds as submanifolds of a Euclidean space. In the late 1970's, the notion of finite-type immersion of Riemannian manifolds into Euclidean space was introduced, which is a generalization of the so-called eigenvalue problem of the immersion [1]: An isometric immersion of x of a Riemannian manifold M into a Euclidean space $\mathbb{E}^m$ is said to be of finite-type if it can be expressed as

$$x = x_0 + x_1 + \cdots + x_k$$

for some positive integer k, where x_0 is a constant vector and $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$. Here, Δ denotes the Laplace operator defined on M. If $\lambda_1, \ldots, \lambda_k$ are mutually different, M is said to be of k-type. We may assume that a finite-type immersion of x of a Riemannian manifold into a Euclidean space is of k-type for some non-negative integer k.

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space of signature $(m-s, s)$. The notion of finite-type immersion was extended to that of submanifolds in pseudo-Euclidean space $\mathbb{E}^m_s$ and to that of smooth maps defined on submanifolds of Euclidean space $\mathbb{E}^m$ or pseudo-Euclidean space $\mathbb{E}^m_s$. In particular, the study of finite-type immersions and finite-type Gauss map of submanifolds in the Minkowski m-space $\mathbb{E}^m_1$ denoted by $\mathbb{L}^m$ has been made extensively ([2–15]).

On the other hand, the Gauss map of some nice surfaces in the three-dimensional Euclidean space $\mathbb{E}^3$ has an interesting property regarding the Laplacian. The helicoid in $\mathbb{E}^3$ parameterized by

$$x(u, v) = (u \cos v, u \sin v, av), \ a \neq 0$$

has the Gauss map G satisfying

$$\Delta G = \frac{2a^2}{(a^2 + u^2)^2} G.$$

The Gauss map of the right (or circular) cone in $\mathbb{E}^3$ with parametrization

$$x(u,v) = (u\cos v, u\sin v, au),\ a \geq 0$$

satisfies

$$\Delta G = \frac{1}{u}(G + (0, 0, \frac{1}{\sqrt{1+a^2}}))$$

(cf. [16,17]). The Gauss map of those surfaces is similar to of 1-type, but obviously not of 1-type in the usual sense. We need to know what other manifolds have such a property. Based on these examples, the following definition was introduced.

Definition 1 ([18])**.** *An oriented n-dimensional submanifold M of the Euclidean space $\mathbb{E}^m$ or the pseudo-Euclidean space $\mathbb{E}^m_s$ is said to have pointwise 1-type Gauss map or the Gauss map is of pointwise 1-type if it satisfies*

$$\Delta G = f(G + \boldsymbol{C}), \tag{1}$$

where f is a non-zero smooth function on M and $\boldsymbol{C}$ a constant vector in the ambient space. In particular, if $\boldsymbol{C}$ is zero, the Gauss map G is said to be of pointwise 1-type of the first kind. Otherwise, it is said to be of the second kind ([19–24]).

The notion of ruled submanifold is a concept of great interest in the Riemannian geometry, which has been investigated by many authors. Several results involving ruled submanifolds in manifolds equipped with remarkable geometric structures were recently obtained in [25–28].

In [19,20], the authors of the present paper et al. studied ruled submanifolds in the Euclidean space $\mathbb{E}^m$ with pointwise 1-type Gauss map and proved that the ruled submanifold M in the Euclidean space $\mathbb{E}^m$ is minimal if and only if the Gauss map G of M is of pointwise 1-type Gauss map of the first kind. Further, we showed that the only non-cylindrical ruled submanifold M in the Euclidean space $\mathbb{E}^m$ with pointwise 1-type Gauss map of the second kind is the generalized right cone.

In [29], the authors of the present paper and et al. investigated the ruled submanifolds in the Lorentz-Minkowski m-space $\mathbb{L}^m$ with pointwise 1-type Gauss map of the first kind and then established the equivalent conditions for the minimality of the ruled submanifold in the Lorentz-Minkowski m-space $\mathbb{L}^m$ by means of the Gauss map.

In this paper, we will study ruled submanifolds in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the second kind and thereby complete the classification of the ruled submanifolds in $\mathbb{L}^m$ with pointwise 1-type Gauss map.

2. Preliminaries

A curve in $\mathbb{E}^m_s$ is said to be *space-like*, *time-like* or *null* if its tangent vector field is space-like, time-like or null, respectively.

Let $x : M \to \mathbb{E}^m_s$ be an isometric immersion of an n-dimensional pseudo-Riemannian manifold M into $\mathbb{E}^m_s$. Throughout the present paper, a submanifold in $\mathbb{E}^m_s$ always means pseudo-Riemannian, in other words, each tangent space of the submanifold in $\mathbb{E}^m_s$ is non-degenerate.

Let $(x_1, x_2, \ldots, x_n)$ be a local coordinate system of M in $\mathbb{E}^m_s$. For the components g_{ij} of the pseudo-Riemannian metric $\langle\cdot,\cdot\rangle$ on M induced from that of $\mathbb{E}^m_s$, we denote by (g^{ij}) (respectively, $\mathcal{G}$) the inverse matrix (respectively, the determinant) of the matrix (g_{ij}) of the components of the induced metric $\langle\cdot,\cdot\rangle$. Then, the Laplacian Δ defined on M is given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}}\sum_{i,j}\frac{\partial}{\partial x_i}(\sqrt{|\mathcal{G}|}g^{ij}\frac{\partial}{\partial x_j}).$$

We now define the Gauss map G on M. Consider the map $G : M \to G(n,m)$ of a point p of M mapped to an oriented tangent space at p, where $G(n,m)$ is the Grassmannian manifold consisting of all oriented n-planes passing through the origin. Roughly speaking it can be achieved by parallel displacement of the oriented tangent space at p to the origin of $\mathbb{L}^m$. By an isomorphism, $G(n,m)$ can be identified with $G(m-n,m)$ in a natural manner. Let us express the Gauss map rigorously. Choose an adapted local orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ in $\mathbb{E}^m_s$ such that $e_1, e_2, \ldots, e_n$ are tangent to M and $e_{n+1}, e_{n+2}, \ldots, e_m$ normal to M. Define the map $G : M \to G(n,m) \subset \mathbb{R}^N$ ($N = {}_mC_n$), $G(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$.

An indefinite scalar product $\ll \cdot, \cdot \gg$ on $G(n,m) \subset \mathbb{R}^N$ is defined by

$$\ll e_{i_1} \wedge \cdots \wedge e_{i_n}, e_{j_1} \wedge \cdots \wedge e_{j_n} \gg = \det(\langle e_{i_l}, e_{j_k} \rangle).$$

Then, $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} | 1 \le i_1 < \cdots < i_n \le m\}$ is an orthonormal basis of $\mathbb{E}^N_k$ for some positive integer k.

Now, let us recall the notion of a ruled submanifold M in $\mathbb{L}^m$ ([7–10]). A non-degenerate $(r+1)$-dimensional submanifold M in $\mathbb{L}^m$ is called a *ruled submanifold* if M is foliated by r-dimensional totally geodesic submanifolds $E(s,r)$ of $\mathbb{L}^m$ along a regular curve $\alpha = \alpha(s)$ on M defined on an open interval I. Thus, a parametrization of a ruled submanifold M in $\mathbb{L}^m$ can be given by

$$x = x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s), \ \ s \in I, \ t_i \in I_i,$$

where I_i's are some open intervals for $i = 1, 2, \ldots, r$. Without loss of generality, we may assume that $0 \in I_i$ for all $i = 1, 2, \ldots, r$. For each s, $E(s,r)$ is open in Span$\{e_1(s), e_2(s), \ldots, e_r(s)\}$, which is the linear span of linearly independent vector fields $e_1(s), e_2(s), \ldots, e_r(s)$ along the curve α. Here, we assume that $E(s,r)$ are either non-degenerate or degenerate for all s along α. We call $E(s,r)$ the *rulings* and α the *base curve* of the ruled submanifold M. In particular, the ruled submanifold M is said to be *cylindrical* if $E(s,r)$ are parallel along α, or *non-cylindrical* otherwise.

Remark 1 ([7,8]). *(1) If the rulings of M are non-degenerate, then the base curve α can be chosen to be orthogonal to the rulings as follows: Let V be a unit vector field on M which is orthogonal to the rulings. Then α can be taken as an integral curve of V.*

(2) If the rulings are degenerate, we can choose a null base curve which is transversal to the rulings: Let V be a null vector field on M which is not tangent to the rulings. An integral curve of V can be the base curve.

By solving a system of ordinary differential equations similarly set up relative to a frame along a curve in $\mathbb{L}^m$ as given in [30], we have

Lemma 1 ([8]). *Let $V(s)$ be a smooth l-dimensional non-degenerate distribution in the Minkowski m-space $\mathbb{L}^m$ along a curve $\alpha = \alpha(s)$, where $l \ge 2$ and $m \ge 3$. Then, we can choose orthonormal vector fields $e_1(s), \ldots, e_{m-l}(s)$ along α which generate the orthogonal complement $V^\perp(s)$ satisfying $e_i'(s) \in V(s)$ for $1 \le i \le m-l$.*

3. Characterization of Cylinders over Spatial Base Curves

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with non-degenerate rulings. Then, by Remark 1, the base curve α can be chosen to be orthogonal to the rulings. Without loss of generality, we may assume that α is a unit speed curve, that is, $\langle \alpha'(s), \alpha'(s) \rangle = \varepsilon (= \pm 1)$. From now on, the prime $'$ denotes d/ds unless otherwise stated. By Lemma 1, we may choose orthonormal vector fields $e_1(s), \ldots, e_r(s)$ along α satisfying

$$\langle \alpha'(s), e_i(s) \rangle = 0, \ \langle e_i'(s), e_j(s) \rangle = 0, \ \ i, j = 1, 2, \ldots, r. \tag{2}$$

A parametrization of M is given by

$$x = x(s, t_1, t_2, \dots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s). \tag{3}$$

In this section, we always assume that the parametrization (3) satisfies condition (2). Then, the Gauss map G of M is given by

$$G = \frac{1}{\|x_s\|} x_s \wedge x_{t_1} \wedge \cdots \wedge x_{t_r},$$

or, equivalently

$$G = \frac{1}{|q|^{1/2}} (\Phi + \sum_{i=1}^{r} t_i \Psi_i), \tag{4}$$

where q, Φ and Ψ_i are the function and the vectors respectively, defined by

$$q = \langle x_s, x_s \rangle, \quad \Phi = \alpha' \wedge e_1 \wedge \cdots \wedge e_r \quad \text{and} \quad \Psi_i = e_i' \wedge e_1 \wedge \cdots \wedge e_r.$$

First, we consider the case of cylindrical ruled submanifolds that are one of two typical types of ruled submanifolds, which are cylindrical or non-cylindrical. Before discussing cylindrical ruled submanifolds, we cite the following lemma.

Lemma 2 ([29]). *Suppose that a unit speed curve $\alpha(s)$ in the m-dimensional Minkowski space $\mathbb{L}^m$ defined on an open interval I satisfies*

$$\alpha'''(s) = g(s)(\alpha'(s) + \mathbf{C}), \tag{5}$$

where g is a function of the parameter s and $\mathbf{C}$ a constant vector in $\mathbb{L}^m$. Then, the curve α lies in a 3-dimensional affine space in $\mathbb{L}^m$. In particular, if the constant vector $\mathbf{C}$ is zero, we see that α is a plane curve.

We now prove that if an $(r+1)$-dimensional cylindrical ruled submanifold M in $\mathbb{L}^m$ has pointwise 1-type Gauss map of the second kind satisfying (1), then it is part of a $(r+1)$-plane or a cylinder over a curve in 3-dimensional affine space.

Let M be a cylindrical $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ generated by non-degenerate rulings which is parameterized by (3). Without loss of generality, we may assume that $e_1, e_2, \dots, e_r$ generating the rulings are constant vectors.

The Laplacian Δ of M is then naturally expressed by

$$\Delta = -\varepsilon \frac{\partial^2}{\partial s^2} - \sum_{i=1}^{r} \varepsilon_i \frac{\partial^2}{\partial t_i^2},$$

where $\varepsilon_i = \langle e_i(s), e_i(s) \rangle = \pm 1$ and the Gauss map G of M is given by

$$G = \varepsilon \alpha' \wedge e_1 \wedge \cdots \wedge e_r = \varepsilon \Phi.$$

We now suppose that the Gauss map G is of pointwise 1-type of the second kind, that is, $\Delta G = f(G + \mathbf{C})$ for some non-zero smooth function f and some non-zero constant vector $\mathbf{C}$. Then, the equation $\Delta G = f(G + \mathbf{C})$ is written as

$$-\varepsilon \Phi'' = f(\varepsilon \Phi + \mathbf{C}). \tag{6}$$

From Equation (6), we see that f is a function of s. We may assume that f is non-zero on the open interval $I = \mathrm{dom}(\alpha)$. Then, differentiation of Equation (6) with respect to s gives

$$\frac{\varepsilon f'}{f^2} \Phi'' - \frac{\varepsilon}{f} \Phi''' - \varepsilon \Phi' = \mathbf{0}, \tag{7}$$

or, equivalently

$$\frac{f'}{f^2}\alpha''' - \frac{1}{f}\alpha^{(4)} - \alpha'' = \mathbf{0}$$

which implies that $-\frac{1}{f}\alpha''' - \alpha' = \mathbf{D}$ for some constant vector $\mathbf{D}$, where $\mathbf{0}$ denotes zero vector. Namely, if we denote by Δ' the Laplacian of α, we have

$$\Delta'\alpha' = -\alpha''' = f(\alpha' + \mathbf{D}). \tag{8}$$

According to Lemma 2, we see that the curve α lies in a 3-dimensional affine space in $\mathbb{L}^m$.

If a cylindrical ruled submanifold M is part of an $(r+1)$-plane or a cylinder over a 3-dimensional affine space satisfying (8), it is obvious that the Gauss map G is of pointwise 1-type of the second kind. Thus, we have

Theorem 1. *Let M be an $(r+1)$-dimensional cylindrical ruled submanifold of $\mathbb{L}^m$. Then, M has pointwise 1-type Gauss map of the second kind if and only if M is part of an $(r+1)$-dimensional plane or a cylinder over a curve in a 3-dimensional affine space in $\mathbb{L}^m$ satisfying (8).*

Next, we consider the case that non-cylindrical ruled submanifolds have pointwise 1-type Gauss map of the second kind. Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by (3) in $\mathbb{L}^m$. Then, we have

$$x_s = \alpha'(s) + \sum_{j=1}^{r} t_j e_j'(s), \quad x_{t_i} = e_i(s)$$

for $i = 1, 2, \ldots, r$. The function q defined in the beginning of this section is given by

$$q = \langle x_s, x_s \rangle = \varepsilon + \sum_{i=1}^{r} 2u_i t_i + \sum_{i,j=1}^{r} w_{ij} t_i t_j, \tag{9}$$

where $u_i(s) = \langle \alpha', e_i' \rangle$ and $w_{ij}(s) = \langle e_i', e_j' \rangle$ for $i, j = 1, \ldots, r$. Note that q is a polynomial in $t = (t_1, \ldots, t_r)$ with functions in s as coefficients.

From now on, for a polynomial $F(t)$ in $t = (t_1, t_2, \ldots, t_r)$, $\deg F(t)$ denotes the degree of $F(t)$ in $t = (t_1, t_2, \ldots, t_r)$ unless otherwise stated.

If we adapt the proof of Proposition 3.3 of [19] to the case of a non-cylindrical ruled submanifold in the Minkowski m-space $\mathbb{L}^m$, we may assume that the generator vector fields $e_1, e_2, \ldots, e_r$ of the rulings of M satisfy

$$e_j' \neq \mathbf{0}$$

on the domain I of α for all $j = 1, 2, \ldots, r$ if M has pointwise 1-type Gauss map of the second kind. Then, we get the components of the metric $\langle \cdot, \cdot \rangle$ on M

$$g_{11} = q, \quad g_{1j} = 0 \quad \text{and} \quad g_{ij} = \varepsilon_i \delta_{ij}$$

for $i, j = 2, 3, \ldots, r+1$.

It is enough for us to consider the case of $q > 0$. Accordingly, Equation (9) gives $\varepsilon = 1$. By definition, we have the Laplacian of the form

$$\Delta = \frac{1}{2q^2}\frac{\partial q}{\partial s}\frac{\partial}{\partial s} - \frac{1}{q}\frac{\partial^2}{\partial s^2} - \frac{1}{2q}\sum_{i=1}^{r} \varepsilon_i \frac{\partial q}{\partial t_i}\frac{\partial}{\partial t_i} - \sum_{i=1}^{r} \varepsilon_i \frac{\partial^2}{\partial t_i^2}. \tag{10}$$

First, we suppose that $e'_1, e'_2, \ldots, e'_r$ are non-null. Then, using the Formula (10), $\Delta G = f(G + \mathbf{C})$ can be expressed as

$$\begin{aligned}(\frac{\partial q}{\partial s})^2(\Phi + \sum_{j=1}^{r}\Psi_j t_j) - \frac{3}{2}q\frac{\partial q}{\partial s}(\Phi' + \sum_{j=1}^{r}\Psi'_j t_j) - \frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(\Phi + \sum_{j=1}^{r}\Psi_j t_j)\\ +q^2(\Phi'' + \sum_{j=1}^{r}\Psi''_j t_j) + \frac{1}{2}q\sum_{i=1}^{r}\varepsilon_i(\frac{\partial q}{\partial t_i})^2(\Phi + \sum_{j=1}^{r}\Psi_j t_j) - \frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\Psi_i\\ -\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial^2 q}{\partial t_i^2}(\Phi + \sum_{j=1}^{r}\Psi_j t_j) + f\{q^3(\Phi + \sum_{j=1}^{r}\Psi_j t_j) + q^{\frac{7}{2}}\mathbf{C}\} = \mathbf{0}.\end{aligned} \tag{11}$$

If we use the indefinite scalar product $\ll \cdot,\cdot \gg$ on $G(r+1,m)$, we have

$$\begin{aligned}&\ll \Phi,\Phi \gg = \tilde{\varepsilon}, \quad \ll \Phi,\Phi' \gg = 0,\\ &\ll \Phi,\Phi'' \gg = -\tilde{\varepsilon}\mu + 2\sum_{k=1}^{r}\tilde{\varepsilon}\varepsilon_k u_k^2 - \sum_{k=1}^{r}\tilde{\varepsilon}\varepsilon_k w_{kk},\\ &\ll \Phi,\Psi_i \gg = \tilde{\varepsilon}u_i, \quad \ll \Phi,\Psi'_i \gg = \tilde{\varepsilon}p_i,\\ &\ll \Phi,\Psi''_i \gg = \tilde{\varepsilon}y_i + 2\sum_{k=1}^{r}\tilde{\varepsilon}\varepsilon_k u_k w_{ik} - \sum_{k=1}^{r}\tilde{\varepsilon}\varepsilon_k u_i w_{kk},\\ &\ll \Psi_i,\Phi' \gg = \tilde{\varepsilon}z_i, \quad \ll \Psi_i,\Psi_j \gg = \tilde{\varepsilon}w_{ij}, \quad \ll \Psi_i,\Psi'_j \gg = \tilde{\varepsilon}\xi_{ij},\end{aligned}$$

where we have put

$$\tilde{\varepsilon} = \varepsilon_1\cdots\varepsilon_r, \mu = \langle\alpha'',\alpha''\rangle, p_i = \langle\alpha', e''_i\rangle, y_i = \langle\alpha', e'''_i\rangle, z_i = \langle\alpha'', e'_i\rangle, \xi_{ij} = \langle e'_i, e''_j\rangle.$$

Then, we get

$$u'_i(s) = p_i(s) + z_i(s) \quad \text{and} \quad w'_{ij} = \xi_{ij} + \xi_{ji}. \tag{12}$$

By taking the indefinite scalar product with the vector Φ to both sides of (11), we obtain

$$\begin{aligned}(\frac{\partial q}{\partial s})^2(1 + \sum_{j=1}^{r}u_j t_j) - \frac{3}{2}q\frac{\partial q}{\partial s}(\sum_{j=1}^{r}p_j t_j) - \frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(1 + \sum_{j=1}^{r}u_j t_j)\\ +q^2(\tilde{\varepsilon}\phi + \sum_{j=1}^{r}\tilde{\varepsilon}\varphi_j t_j) + \frac{1}{2}q\sum_{i=1}^{r}\varepsilon_i(\frac{\partial q}{\partial t_i})^2(1 + \sum_{j=1}^{r}u_j t_j) - \frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}u_i\\ -\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial^2 q}{\partial t_i^2}(1 + \sum_{j=1}^{r}u_j t_j) + f\{q^3(1 + \sum_{j=1}^{r}u_j t_j) + q^{\frac{7}{2}}\gamma(s)\} = 0,\end{aligned} \tag{13}$$

where we have put

$$\gamma(s) = \ll \mathbf{C},\Phi(s) \gg, \ \phi = \ll \Phi,\Phi'' \gg \quad \text{and} \quad \varphi_i = \ll \Phi,\Psi''_i \gg. \tag{14}$$

Let $\bar{e}_{r+1}, \bar{e}_{r+2}, \ldots, \bar{e}_{m-1}$ be the orthonormal vector fields which are normal to M along α. If we apply Lemma 1 to the normal space $T^{\perp}_{\alpha(s)}M$ of M, then there exists an orthonormal frame $\{e_a\}_{a=r+1}^{m-1}$ of the normal space $T^{\perp}_{\alpha(s)}M$ satisfying

$$\langle e'_a(s), e_b(s)\rangle = 0 \tag{15}$$

for all $a, b = r+1, \ldots, m-1$. Then we can put

$$e'_j = u_j\alpha' + \sum_{a=r+1}^{m-1}\varepsilon_a\lambda_a^j e_a, \tag{16}$$

where $\varepsilon_a = \langle e_a, e_a \rangle = \pm 1$ and $\lambda_a^j(s) = \langle e_j', e_a \rangle$ for $a = r+1, \ldots, m-1$. From (16), we get

$$\Psi_j = u_j \Phi + \sum_{a=r+1}^{m-1} \varepsilon_a \lambda_a^j \xi_a, \tag{17}$$

where $\xi_a = e_a \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r$ for $a = r+1, \ldots, m-1$. And, we may put

$$\alpha'' = -\sum_{i=1}^{r} \varepsilon_i u_i e_i - \sum_{a=r+1}^{m-1} \varepsilon_a u_a e_a, \tag{18}$$

where $u_a(s) = \langle \alpha', e_a' \rangle$ for all $a = r+1, \ldots, m-1$.

Suppose that M is not an $(r+1)$-plane, that is, $G \neq -\mathbf{C}$. To deal with (13), we consider the subset

$$M_1 = \{p \in M | q^3(1 + \sum_{j=1}^{r} u_j t_j) + q^{\frac{7}{2}} \gamma(s) = 0\}.$$

Without loss of generality, we may assume that $f \neq 0$ on M_1. Then, on M_1,

$$(1 + \sum_{j=1}^{r} u_j t_j) + \sqrt{q}\gamma(s) = 0, \quad \text{or,} \quad (1 + \sum_{j=1}^{r} u_j t_j)^2 = q\gamma^2(s).$$

By (9) and $\varepsilon = 1$, we see that $\gamma^2(s) = 1$ and hence

$$q = (1 + \sum_{j=1}^{r} u_j t_j)^2$$

which implies that

$$M_1 = \{p \in M | 1 + \gamma(s) = 0\}.$$

Also, it follows from (16) that on M_1,

$$\sum_{a=r+1}^{m-1} \varepsilon_a \lambda_a^k \lambda_a^j = 0 \tag{19}$$

for all $j, k = 1, \ldots, r$.

Lemma 3. *Let M be an $(r+1)$-dimensional non-cylindrical and non-planar ruled submanifold parameterized by (3) in $\mathbb{L}^m$. Let $e_1, e_2, \ldots, e_r$ be the orthonormal generators of the rulings along the base curve α such that e_j' are non-null for all $j = 1, 2, \ldots, r$. If the Gauss map G of M satisfies $\Delta G = f(G + \mathbf{C})$ for some non-zero function f and non-zero constant vector $\mathbf{C}$, then*

$$\gamma(s) = \ll \mathbf{C}, \Phi(s) \gg \neq -1$$

on $\{p \in M | f \neq 0\}$.

Proof. Let $\tilde{I}_1 = \{s \in I | 1 + \gamma(s) = 0\}$. We suppose that the interior $\mathrm{Int}(\tilde{I}_1)$ of $\tilde{I}_1$ is non-empty.

If we put

$$P(t) =(\frac{\partial q}{\partial s})^2(1+\sum_{j=1}^{r}u_jt_j) - \frac{3}{2}q\frac{\partial q}{\partial s}(\sum_{j=1}^{r}p_jt_j) - \frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(1+\sum_{j=1}^{r}u_jt_j)$$
$$+q^2(\tilde{\varepsilon}\phi+\sum_{j=1}^{r}\tilde{\varepsilon}\varphi_jt_j)+\frac{1}{2}q\sum_{i=1}^{r}\varepsilon_i(\frac{\partial q}{\partial t_i})^2(1+\sum_{j=1}^{r}u_jt_j) \tag{20}$$
$$-\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\varepsilon u_i - \frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial^2 q}{\partial t_i^2}(1+\sum_{j=1}^{r}u_jt_j),$$

Equation (13) tells us that $P(t)\equiv 0$ on $\mathrm{Int}(\tilde{I}_1)$. Using $q=(1+\sum_j u_jt_j)^2$ on $\mathrm{Int}(\tilde{I}_1)$, equation $P(t)=0$ yields that

$$3(\sum_{j=1}^{r}u_j't_j)(\sum_{j=1}^{r}(u_j'-p_j)t_j)+(1+\sum_{j=1}^{r}u_jt_j)(\tilde{\varepsilon}\phi+\sum_{j=1}^{r}(\tilde{\varepsilon}\varphi_j-u_j'')t_j)=0$$

which provides

$$\phi=0,\quad \tilde{\varepsilon}\varphi_j=u_j'' \quad \text{and} \quad u_j'(u_j'-p_j)=0 \tag{21}$$

as the coefficients of terms containing t_j^0, t_j^1 and t_j^2, respectively, for $j=1,\dots,r$.

Now, we will proceed with the proof according to the following steps.

Step 1. $u_j'=0$ on $\mathrm{Int}(\tilde{I}_1)$.

We suppose that $u_j'\neq 0$ at some point in $\mathrm{Int}(\tilde{I}_1)$. Then, $u_j'=p_j$ in (21) and hence

$$z_j=\langle\alpha'',e_j'\rangle=\sum_{a=r+1}^{m-1}\varepsilon_a u_a\lambda_a^j=0 \tag{22}$$

because of (12), (16) and (18). Since $\phi=\ll\Phi,\Phi''\gg=-\ll\Phi',\Phi'\gg$ and $\ll\Phi',\Psi_j\gg=\tilde{\varepsilon}z_j$,

$$\ll\Phi',\Phi\gg=\ll\Phi',\Phi'\gg=\ll\Phi',\Phi''\gg=\ll\Phi',\Psi_j\gg=0. \tag{23}$$

Now, we suppose that $\gamma'(s_0)\neq 0$ for some $s_0\in\mathrm{Int}\tilde{I}_1$. Then, at s_0 Equation (11) is rewritten as

$$3(\sum_{j=1}^{r}u_j't_j)^2(\Phi+\sum_{j=1}^{r}\Psi_jt_j)-3(1+\sum_{j=1}^{r}u_jt_j)(\sum_{j=1}^{r}u_j't_j)(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)$$
$$-(1+\sum_{j=1}^{r}u_jt_j)(\sum_{j=1}^{r}u_j''t_j)(\Phi+\sum_{j=1}^{r}\Psi_jt_j)+(1+\sum_{j=1}^{r}u_jt_j)^2(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)$$
$$+(1+\sum_{j=1}^{r}u_jt_j)^2(\sum_{i=1}^{r}\varepsilon_iu_i^2)(\Phi+\sum_{j=1}^{r}\Psi_jt_j)-(1+\sum_{j=1}^{r}u_jt_j)^3(\sum_{i=1}^{r}\varepsilon_iu_i\Psi_i) \tag{24}$$
$$+f(1+\sum_{j=1}^{r}u_jt_j)^4\{(\Phi+\sum_{j=1}^{r}\Psi_jt_j)+(1+\sum_{j=1}^{r}u_jt_j)\mathbf{C}\}=\mathbf{0}$$

and taking the indefinite scalar product with Φ' to (24) gives us the following

$$f=\frac{3(\sum_j u_j't_j)(\sum_j\eta_jt_j)-(1+\sum_j u_jt_j)(\sum_j\vartheta_jt_j)}{\gamma'(1+\sum_j u_jt_j)^4}, \tag{25}$$

where we have put

$$\eta_j(s)=\ll\Phi',\Psi_j'\gg \quad \text{and} \quad \vartheta_j(s)=\ll\Phi',\Psi_j''\gg.$$

Substituting (25) into (24) and then considering the constant terms of the equation obtained in such a way, we get

$$\Phi'' + (\sum_{i=1}^{r} \varepsilon_i u_i^2)\Phi - (\sum_{i=1}^{r} \varepsilon_i u_i \Psi_i) = \mathbf{0}. \tag{26}$$

By straightforward computation, it follows from (19), (21) and (22) that at s_0

$$\Phi'' = \sum_{a=r+1}^{m-1} \varepsilon_a(-u_a' + \sum_{k=1}^{r} \varepsilon_k u_k \lambda_a^k)\xi_a + \sum_{k=1}^{r}\sum_{a=r+1}^{m-1} \varepsilon_a (u_k u_a + (\lambda_a^k)')\Gamma_a^k, \tag{27}$$

where we have put

$$\Gamma_a^k = \alpha' \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_a \wedge e_{k+1} \wedge \cdots \wedge e_r.$$

Combining (17), (26) and (27), we can obtain

$$u_a' = 0 \quad \text{and} \quad (\lambda_a^k)' = -u_k u_a \tag{28}$$

for all $k = 1, \ldots, r$ and $a = r+1, \ldots, m-1$. Using (19), (21) and (28), we also get at s_0

$$\begin{aligned} \Psi_j' =& u_j'\Phi - 2\sum_{a=r+1}^{m-1} \varepsilon_a u_j u_a \xi_a + \sum_{k=1}^{r}\sum_{a=r+1}^{m-1} \varepsilon_a (u_j \lambda_a^k - u_k \lambda_a^j)\Gamma_a^k, \\ \Psi_j'' =& u_j''\Phi + \sum_{a=r+1}^{m-1} \varepsilon_a(-3u_j' u_a + \sum_{k=1}^{r} \varepsilon_k u_k (u_j \lambda_a^k - u_k \lambda_a^j))\xi_a \\ &+ \sum_{k=1}^{r}\sum_{a=r+1}^{m-1} \varepsilon_a (2u_j' \lambda_a^k - u_k' \lambda_a^j + 2u_j u_k u_a)\Gamma_a^k. \end{aligned} \tag{29}$$

Since $\Phi' = -\sum_a \varepsilon_a u_a \xi_a + \sum_k \sum_a \varepsilon_a \lambda_a^k \Gamma_a^k$, the functions $\eta_j(s)$ and $\vartheta_j(s)$ are identically zero on M_2 for all $j = 1, \ldots, r$, with the help of (19), (21), (22) and (29). Thus, we have

$$\ll \Phi', \Phi \gg = \ll \Phi', \Phi' \gg = \ll \Phi', \Phi'' \gg = \ll \Phi', \Psi_j \gg = \ll \Phi', \Psi_j' \gg = \ll \Phi', \Psi_j'' \gg = 0$$

which means that Φ' is orthogonal to all vectors of (24) except the constant vector $\mathbf{C}$, so taking the indefinite scalar product with Φ' to (24) yields that

$$f(1 + \sum_{j=1}^{r} u_j t_j)^5 \ll \Phi', \mathbf{C} \gg = f(1 + \sum_{j=1}^{r} u_j t_j)^5 \gamma' = 0$$

which is a contradiction.

Therefore, we have

$$\gamma' = 0 \tag{30}$$

on $\tilde{I}_1$. Using (30), (24) implies that

$$3(\sum_{j=1}^{r} u_j' t_j)(\sum_{j=1}^{r} \eta_j t_j) = (1 + \sum_{j=1}^{r} u_j t_j)(\sum_{j=1}^{r} \vartheta_j t_j)$$

by taking the indefinite scalar product with Φ'. Thus, we can see that

$$\eta_j = 0 = \vartheta_j \tag{31}$$

as the coefficients of terms containing t_j and t_j^2 for $j = 1, \ldots, r$, by virtue of $u_j' \neq 0$. Equations (23), (30) and (31) indicate that Φ' is orthogonal to all vectors of (24), so the coefficient of Φ' has to be identically zero, which yields that $u_j' = 0$ for $j = 1, 2, \ldots, r$. It contradicts our assumption.

Therefore, we conclude that the functions u_j are constant for all $j = 1, \ldots, r$, that is,

$$\frac{\partial q}{\partial s} = 0$$

on $(\tilde{I}_1)$.

Step 2. An expression for f on $\tilde{I}_1$.

According to Step 1, Equation (24) is simplified as

$$\begin{aligned}(\Phi'' + \sum_{j=1}^{r} \Psi_j'' t_j) + (\sum_{i=1}^{r} \varepsilon_i u_i^2)(\Phi + \sum_{j=1}^{r} \Psi_j t_j) - (1 + \sum_{j=1}^{r} u_j t_j)(\sum_{i=1}^{r} \varepsilon_i u_i \Psi_i) \\ + f(1 + \sum_{j=1}^{r} u_j t_j)^2 \{(\Phi + \sum_{j=1}^{r} \Psi_j t_j) + (1 + \sum_{j=1}^{r} u_j t_j)\mathbf{C}\} = \mathbf{0}.\end{aligned} \tag{32}$$

We repeat taking the indefinite scalar product to Φ' to (32) and then we obtain

$$\begin{aligned}(\sum_{j=1}^{r} \vartheta_j t_j) + (\sum_{i=1}^{r} \varepsilon_i u_i^2)(\sum_{j=1}^{r} z_j t_j) - (1 + \sum_{j=1}^{r} u_j t_j)(\sum_{i=1}^{r} \varepsilon_i u_i z_i) \\ + f(1 + \sum_{j=1}^{r} u_j t_j)^2 \{(\sum_{j=1}^{r} z_j t_j) + (1 + \sum_{j=1}^{r} u_j t_j)\gamma'\} = 0.\end{aligned} \tag{33}$$

If $(\sum_j z_j t_j) + (1 + \sum_j u_j t_j)\gamma' = 0$, then $\gamma' = 0 = z_j$ and hence $\vartheta_j = 0$ in (33) for all $j = 1, \ldots, r$. In this case, Φ'' and Ψ_j'' are given by (27) and

$$\begin{aligned}\Psi_j'' = & \sum_{a=r+1}^{m-1} \varepsilon_a (u_j \sum_{k=1}^{r} \varepsilon_k u_k \lambda_a^k - \sum_{k=1}^{r} \varepsilon_k u_k^2 \lambda_a^j - u_j u_a' + (\lambda_a^j)'')\xi_a \\ & + \sum_{k=1}^{r} \sum_{a=r+1}^{m-1} \varepsilon_a (u_j u_k u_a - 2u_k (\lambda_a^j)' + u_j (\lambda_a^k)')\Gamma_a^k \\ & + \sum_{k=1}^{r} \sum_{a,b=r+1}^{m-1} \varepsilon_a \varepsilon_b (2\langle e_j'', e_a \rangle \lambda_b^k + \langle e_k'', e_b \rangle \lambda_a^j)\Gamma_{ab}^k,\end{aligned} \tag{34}$$

respectively, where we have put

$$\Gamma_{ab}^k = e_a \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_b \wedge e_{k+1} \wedge \cdots \wedge e_r.$$

Together with (27) and (34), (32) yields that the constant vector $\mathbf{C}$ can be expressed as

$$\begin{aligned}\mathbf{C} = & -\tilde{\varepsilon}\Phi + \sum_{a=r+1}^{m-1} \tilde{\varepsilon}\varepsilon_a \ll \mathbf{C}, \xi_a \gg \xi_a + \sum_{k=1}^{r} \sum_{a=r+1}^{m-1} \tilde{\varepsilon}\varepsilon_k \varepsilon_a \ll \mathbf{C}, \Gamma_a^k \gg \Gamma_a^k \\ & + \sum_{k=1}^{r} \sum_{a,b=r+1}^{m-1} \tilde{\varepsilon}\varepsilon_k \varepsilon_a \varepsilon_b \ll \mathbf{C}, \Gamma_{ab}^k \gg \Gamma_{ab}^k\end{aligned} \tag{35}$$

and that the equations containing $\ll \mathbf{C}, \xi_a \gg$ and $\ll \mathbf{C}, \Gamma_a^k \gg$ are given by

$$0 =(-u'_a + \sum_{k=1}^{r} \varepsilon_k u_k \lambda_a^k) + \sum_{j=1}^{r} t_j(u_j \sum_{k=1}^{r} \varepsilon_k u_k \lambda_a^k - \sum_{k=1}^{r} \varepsilon_k u_k^2 \lambda_a^j - u_j u'_a + (\lambda_a^j)'')$$
$$+ (\sum_{i=1}^{r} \varepsilon_i u_i^2)(\sum_{j=1}^{r} t_j \lambda_a^j) - (1 + \sum_{j=1}^{r} u_j t_j)(\sum_{i=1}^{r} \varepsilon_i u_i \lambda_a^i)$$
$$+ f(1 + \sum_{j=1}^{r} u_j t_j)^2 \{(\sum_{j=1}^{r} t_j \lambda_a^j) + \tilde{\varepsilon}(1 + \sum_{j=1}^{r} u_j t_j) \ll \mathbf{C}, \xi_a \gg\},$$

$$0 = (u_k u_a + (\lambda_a^k)') + \sum_{j=1}^{r} t_j(u_j u_k u_a - 2u_k(\lambda_a^j)' + u_j(\lambda_a^k)') + \tilde{\varepsilon}\varepsilon_k f(1 + \sum_{j=1}^{r} u_j t_j)^3 \ll \mathbf{C}, \Gamma_a^k \gg,$$

respectively. Therefore, $\ll \mathbf{C}, \xi_a \gg$ and $\ll \mathbf{C}, \Gamma_a^k \gg$ are of the form

$$\tilde{\varepsilon} \ll \mathbf{C}, \xi_a \gg = \frac{u'_a(1 + \sum_j u_j t_j) - \sum_j t_j(\lambda_a^j)'' - f(1 + \sum_j u_j t_j)^2(\sum_j t_j \lambda_a^j)}{f(1 + \sum_j u_j t_j)^3} \tag{36}$$

and

$$\tilde{\varepsilon}\varepsilon_k \ll \mathbf{C}, \Gamma_a^k \gg = -\frac{A^{ak} + \sum_j B_j^{ak} t_j}{f(1 + \sum_j u_j t_j)^3}, \tag{37}$$

where

$$A^{ak} = \ll \Phi'', \Gamma_a^k \gg = u_k u_a + (\lambda_a^k)' \quad \text{and}$$
$$B_j^{ak} = \ll \Psi_j'', \Gamma_a^k \gg = u_j u_k u_a - 2u_k(\lambda_a^j)' + u_j(\lambda_a^k)'.$$

If $\ll \mathbf{C}, \Gamma_a^k \gg = 0$, then $A^{ak} = 0 = B_j^{ak}$ in (37). By definitions of A^{ak} and B_j^{ak}, $(\lambda_a^j)' = 0$ and hence $u_j u_a = 0$ for all $j = 1, \ldots, r$ and $a = r+1, \ldots, m-1$. Thus, we have $u_a = 0$ and (36) is simplified as

$$\tilde{\varepsilon} \ll \mathbf{C}, \xi_a \gg = -\frac{(\sum_j t_j \lambda_a^j)}{(1 + \sum_j u_j t_j)} \tag{38}$$

for $a = r+1, \ldots, m-1$. Equation (38) implies that $\ll \mathbf{C}, \xi_a \gg = 0$ and $\lambda_a^j = 0$ for $j = 1, \ldots, r$ and $a = r+1, \ldots, m-1$. Thus, $\Psi_j = u_j \Phi$ and hence $G = \Phi$. Also, under these conditions, by computation, we get $\ll \Psi_j'', \Gamma_{ab}^k \gg = 0$ which implies that $\mathbf{C} = -\tilde{\varepsilon}\Phi$ by virtue of (27), (34) and (35). Therefore, the Gauss map G is a constant vector, a contradiction. Therefore, we see that $\ll \mathbf{C}, \Gamma_a^k \gg \neq 0$. Then, it follows from (37) that

$$f(1 + \sum_{j=1}^{r} u_j t_j)^3 = h(s)(A^{ak} + \sum_{j=1}^{r} B_j^{ak} t_j) \tag{39}$$

for some non-vanishing function h of s. Putting (39) into (36), we have

$$\tilde{\varepsilon} \ll \mathbf{C}, \xi_a \gg$$
$$= \frac{u'_a(1 + \sum_j u_j t_j)^2 - (1 + \sum_j u_j t_j)(\sum_j t_j(\lambda_a^j)'') - h(s)(A^{ak} + \sum_j B_j^{ak} t_j)(\sum_j t_j \lambda_a^j)}{h(s)(1 + \sum_j u_j t_j)(A^{ak} + \sum_j B_j^{ak} t_j)}$$

which allows us to have the following equation

$$\begin{aligned} &g(s)h(s)(1 + \sum_{j=1}^{r} u_j t_j)(A^{ak} + \sum_{j=1}^{r} B_j^{ak} t_j) \\ &= u'_a(1 + \sum_{j=1}^{r} u_j t_j)^2 - (1 + \sum_{j=1}^{r} u_j t_j)(\sum_{j=1}^{r} t_j(\lambda_a^j)'') - h(s)(A^{ak} + \sum_{j=1}^{r} B_j^{ak} t_j)(\sum_{j=1}^{r} t_j \lambda_a^j) \end{aligned} \tag{40}$$

for some non-vanishing function g of s. Comparing the coefficients of terms containing t_j^0, t_j^1 and t_j^2 of (40) gives us three equations:

$$g(s)h(s)A^{ak} = u_a',$$
$$g(s)h(s)u_jA^{ak} + g(s)h(s)B_j^{ak} = 2u_ju_a' - (\lambda_a^j)'' - h(s)A^{ak}\lambda_a^j,$$
$$g(s)h(s)u_jB_j^{ak} = u_j^2u_a' - u_j(\lambda_a^j)'' - h(s)B_j^{ak}\lambda_a^j.$$

Combining these equations, we get

$$u_jA^{ak} = B_j^{ak}.$$

Therefore,

$$u_j(u_ku_a + (\lambda_a^k)') = u_ju_ku_a - 2u_k(\lambda_a^j)' + u_j(\lambda_a^k)',$$

that is, $u_k(\lambda_a^j)' = 0$ and hence

$$(\lambda_a^j)' = 0 \tag{41}$$

for all $j, k = 1, \ldots, r$ and $a = r+1, \ldots, m-1$. From (39), we can obtain

$$f = \frac{h(s)A^{ak}}{(1 + \sum_j u_jt_j)^2} = \frac{h(s)u_ku_a}{(1 + \sum_j u_jt_j)^2} \tag{42}$$

and then we have

$$\begin{aligned}(\Phi'' + \sum_{j=1}^r \Psi_j''t_j) + (\sum_{i=1}^r \varepsilon_iu_i^2)(\Phi + \sum_{j=1}^r \Psi_jt_j) - (1 + \sum_{j=1}^r u_jt_j)(\sum_{i=1}^r \varepsilon_iu_i\Psi_i)\\ + h(s)u_ku_a\{(\Phi + \sum_{j=1}^r \Psi_jt_j) + (1 + \sum_{j=1}^r u_jt_j)\mathbf{C}\} = \mathbf{0}\end{aligned} \tag{43}$$

from (32). By regarding (43) as the polynomial in t of degree 1, we get

$$\Phi'' + (\sum_{i=1}^r \varepsilon_iu_i^2)\Phi - (\sum_{i=1}^r \varepsilon_iu_i\Psi_i) + h(s)u_ku_a\Phi + h(s)u_ku_a\mathbf{C} = \mathbf{0},$$

$$\Psi_j'' + (\sum_{i=1}^r \varepsilon_iu_i^2)\Psi_j - u_j(\sum_{i=1}^r \varepsilon_iu_i\Psi_i) + h(s)u_ku_a\Psi_j + h(s)u_ku_au_j\mathbf{C} = \mathbf{0}$$

which produce that

$$\Psi_j'' = u_j\Phi'' - (\sum_{i=1}^r \varepsilon_iu_i^2)(\Psi_j - u_j\Phi) - h(s)u_ku_a(\Psi_j - u_j\Phi)$$

for all j. With the help of (17), (27), (34) and (41), the equation above provides that

$$h(s)u_ku_a = 0$$

which means that $f = 0$ due to (42), a contradiction.

Therefore, we conclude that

$$(\sum_{j=1}^r z_jt_j) + (1 + \sum_{j=1}^r u_jt_j)\gamma' \neq 0$$

on $\tilde{I}_1$ and hence the function f is given by

$$f = -\frac{(\sum_j \vartheta_j t_j) + (\sum_i \varepsilon_i u_i^2)(\sum_j z_j t_j) - (1 + \sum_j u_j t_j)(\sum_i \varepsilon_i u_i z_i)}{(1 + \sum_j u_j t_j)^2((\sum_j z_j t_j) + (1 + \sum_j u_j t_j)\gamma')} \tag{44}$$

from (33).

Step 3. We find the another equation for f on $\tilde{I}_1$.

First, we suppose that $\ll \mathbf{C}, \Gamma_a^k \gg = \ll \mathbf{C}, \Gamma_{ab}^k \gg = 0$ of (35) for all $k = 1, \ldots, r$ and $a, b = r+1, \ldots, m-1$. Then, we have

$$\mathbf{C} = -\tilde{\varepsilon}\Phi + \sum_{a=r+1}^{m-1} \tilde{\varepsilon}\varepsilon_a \ll \mathbf{C}, \xi_a \gg \xi_a$$

and

$$\ll \Phi'', \Gamma_a^k \gg = \ll \Phi'', \Gamma_{ab}^k \gg = \ll \Psi_j'', \Gamma_a^k \gg = \ll \Psi_j'', \Gamma_{ab}^k \gg = 0 \tag{45}$$

for $k = 1, \ldots, r$ and $a, b = r+1, \ldots, m-1$. In this case, Φ'' and Ψ_j'' are given by

$$\begin{aligned} \Phi'' &= \sum_{a=r+1}^{m-1} \varepsilon_a(-u_a' + \sum_{k=1}^{r} \varepsilon_k u_k \lambda_a^k)\xi_a + \sum_{k=1}^{r} \sum_{a=r+1}^{m-1} \varepsilon_a (u_k u_a + (\lambda_a^k)')\Gamma_a^k \\ &\quad - 2\sum_{k=1}^{r} \sum_{a,b=r+1}^{m-1} \varepsilon_a \varepsilon_b u_a \lambda_b^k \Gamma_{ab}^k, \\ \Psi_j'' &= \sum_{a=r+1}^{m-1} \varepsilon_a(\langle e_j''', e_a\rangle + 2u_j \sum_{k=1}^{r} \varepsilon_k u_k \lambda_a^k - \sum_{k=1}^{r} \varepsilon_k u_k^2 \lambda_a^j)\xi_a \\ &\quad + \sum_{k=1}^{r} \sum_{a=r+1}^{m-1} \varepsilon_a (2\langle e_j'', \alpha'\rangle \lambda_a^k - 2u_k \langle e_j'', e_a\rangle + u_j \langle e_k'', e_a\rangle - \langle e_k'', \alpha'\rangle \lambda_a^j)\Gamma_a^k \\ &\quad + \sum_{k=1}^{r} \sum_{a,b=r+1}^{m-1} \varepsilon_a \varepsilon_b (2\langle e_j'', e_a\rangle \lambda_b^k + \langle e_k'', e_b\rangle \lambda_a^j)\Gamma_{ab}^k. \end{aligned} \tag{46}$$

With the help of (46), the first three equations of (45) provide

$$(\lambda_a^k)' = -u_k u_a, \quad u_a \lambda_b^k = u_b \lambda_a^k, \quad \text{and} \quad u_a = 0 \tag{47}$$

for all $k = 1, \ldots, r$ and $a, b = r+1, \ldots, m-1$.

Taking the indefinite scalar product with ξ_a to (32) gives us the equation containing $\ll \mathbf{C}, \xi_a \gg$ in the following

$$\begin{aligned} &\ll \Phi'', \xi_a \gg + \sum_{j=1}^{r} t_j \ll \Psi_j'', \xi_a \gg + (\sum_{i=1}^{r} \varepsilon_i u_i^2) \sum_{j=1}^{r} t_j \ll \Psi_j, \xi_a \gg \\ &- (1 + \sum_{j=1}^{r} u_j t_j)(\sum_{i=1}^{r} \varepsilon_i u_i \ll \Psi_j, \xi_a \gg) \\ &+ f(1 + \sum_{j=1}^{r} u_j t_j)^2 \{(\sum_{j=1}^{r} t_j \ll \Psi_j, \xi_a \gg) + (1 + \sum_{j=1}^{r} u_j t_j) \ll \mathbf{C}, \xi_a \gg\} = 0. \end{aligned} \tag{48}$$

Using (46) and (47), Equation (48) is rewritten as

$$f(1 + \sum_{j=1}^{r} u_j t_j)^2 \{(\sum_{j=1}^{r} \lambda_a^j t_j) + \tilde{\varepsilon}(1 + \sum_{j=1}^{r} u_j t_j) \ll \mathbf{C}, \xi_a \gg\} = 0$$

which gives

$$(\sum_{j=1}^{r}\lambda_a^j t_j)+\tilde{\varepsilon}(1+\sum_{j=1}^{r}u_j t_j)\ll \mathbf{C},\xi_a\gg=0$$

and hence

$$\lambda_a^j=0 \quad \text{and} \quad \ll \mathbf{C},\xi_a\gg=0$$

for all $j=1,\ldots,r$ and $a=r+1,\ldots,m-1$. Thus, $G=\Phi=-\tilde{\varepsilon}\mathbf{C}$, a contradiction. Therefore, we conclude that $\ll \mathbf{C},\Gamma_a^k\gg\neq 0$ or $\ll \mathbf{C},\Gamma_{ab}^k\gg\neq 0$ for some k, a and b. Now we assume that $\ll \mathbf{C},\Gamma_a^k\gg\neq 0$. Then, taking the indefinite scalar product with Γ_a^k to (32), we obtain

$$\ll \Phi'',\Gamma_a^k\gg+\sum_{j=1}^{r}t_j\ll \Psi_j'',\Gamma_a^k\gg+f(1+\sum_{j=1}^{r}u_j t_j)^3\ll \mathbf{C},\Gamma_a^k\gg=0,$$

or, equivalently,

$$f=-\frac{A^{ak}+\sum_j B_j^{ak}t_j}{(1+\sum_j u_j t_j)^3\ll \mathbf{C},\Gamma_a^k\gg}. \tag{49}$$

Comparing two Equations (44) and (49) regarding the function f, we get

$$\begin{aligned}&(A^{ak}+\sum_{j=1}^{r}B_j^{ak}t_j)\{(\sum_{j=1}^{r}z_j t_j)+(1+\sum_{j=1}^{r}u_j t_j)\gamma')\}\\&=(1+\sum_{j=1}^{r}u_j t_j)\ll \mathbf{C},\Gamma_a^k\gg\{(\sum_{j=1}^{r}\vartheta_j t_j)+(\sum_{i=1}^{r}\varepsilon_i u_i^2)(\sum_{j=1}^{r}z_j t_j)-(1+\sum_{j=1}^{r}u_j t_j)(\sum_{i=1}^{r}\varepsilon_i u_i z_i)\}\end{aligned}$$

which provides that

$$A^{ak}\gamma'=-\ll \mathbf{C},\Gamma_a^k\gg(\sum_{i=1}^{r}\varepsilon_i u_i z_i),$$

$$A^{ak}z_j+\gamma' u_j A^{ak}+\gamma' B_j^{ak}=\ll \mathbf{C},\Gamma_a^k\gg(\vartheta_j+z_j(\sum_{i=1}^{r}\varepsilon_i u_i^2)-2u_j(\sum_{i=1}^{r}\varepsilon_i u_i z_i)),$$

$$z_j B_j^{ak}+\gamma' u_j B_j^{ak}=u_j\ll \mathbf{C},\Gamma_a^k\gg(\vartheta_j+z_j(\sum_{i=1}^{r}\varepsilon_i u_i^2)-u_j(\sum_{i=1}^{r}\varepsilon_i u_i z_i))$$

as the coefficients of terms containing t_j^0, t_j and t_j^2 for $j=1,\ldots,r$. Combining these three equations above, we have

$$z_j(B_j^{ak}-u_j A^{ak})=0$$

for all $j,k=1,\ldots,r$ and $a,b=r+1,\ldots,m-1$. If $z_j=0$ for some j, then we can see that $\gamma'=0$ by applying the same arguments used to show that $\gamma'=0$ on $\tilde{I}_1$. But, it contradicts $(\sum_j z_j t_j)+(1+\sum_j u_j t_j)\gamma'\neq 0$. Therefore, we have

$$B_j^{ak}=u_j A^{ak}$$

and hence the function f becomes

$$f=-\frac{A^{ak}}{\ll \mathbf{C},\Gamma_a^k\gg(1+\sum_j u_j t_j)^2} \tag{50}$$

for all $k=1,\ldots,r$ and $a=r+1,\ldots,m-1$.

Step 4. We compare the equations for f obtained in Steps 2 and 3.

Putting (50) into (32) and then considering the coefficients of terms containing t_j^0 and t_j in the equation obtained in such a way, we get

$$\Psi_j'' = u_j\Phi'' + (\frac{A^{ak}}{\ll \mathbf{C}, \Gamma_a^k \gg} - \sum_{i=1}^{r} \varepsilon_i u_i^2)(\sum_{a=r+1}^{m-1} \varepsilon_a \lambda_a^j \xi_a) \tag{51}$$

for $j = 1, \ldots, r$. With the help of (46), Equation (51) gives that $\ll \Psi_j'', \Gamma_{ab}^k \gg = u_j \ll \Phi'', \Gamma_{ab}^k \gg$, so we see that $\ll \Psi_j'', \Gamma_{ab}^k \gg = 0$ if and only if $\ll \Phi'', \Gamma_{ab}^k \gg = 0$.

If $\ll \Phi'', \Gamma_{ab}^k \gg = 0$, then $u_a \lambda_b^k = u_b \lambda_a^k$ because of (46). In this case,

$$\begin{aligned} u_a z_j &= u_a \sum_{b=r+1}^{m-1} \varepsilon_b u_b \lambda_b^j = \sum_{b=r+1}^{m-1} \varepsilon_b u_b (u_a \lambda_b^j) = \sum_{b=r+1}^{m-1} \varepsilon_b u_b (u_b \lambda_a^j) \\ &= \lambda_a^j \sum_{b=r+1}^{m-1} \varepsilon_b u_b^2 = -\lambda_a^j \phi = 0 \end{aligned}$$

which means that $u_a = 0$ or $z_j = 0$ for $j = 1, \ldots, r$ and $a = r+1, \ldots, m-1$. The case of $u_a = 0$ also guarantees $z_j = 0$, a contradiction. Therefore, we see that $\ll \Psi_j'', \Gamma_{ab}^k \gg \neq 0$ and $\ll \Phi'', \Gamma_{ab}^k \gg \neq 0$ for all $k = 1, \ldots, r$ and $a, b = r+1, \ldots, m-1$. By computation, equation $B_j^{ak} = u_j A^{ak}$ gives us

$$2u_k(\lambda_a^j)' = (\sum_{b=r+1}^{m-1} \varepsilon_b u_b \lambda_b^k) \lambda_a^j \tag{52}$$

and equation $\ll \Psi_j'', \Gamma_{ab}^k \gg = u_j \ll \Phi'', \Gamma_{ab}^k \gg$ provides

$$2(\lambda_a^j)'\lambda_b^k - u_k u_b \lambda_a^j + (\lambda_b^k)'\lambda_a^j - 2(\lambda_b^j)'\lambda_a^k + u_k u_a \lambda_b^j - (\lambda_a^k)'\lambda_b^j = 0 \tag{53}$$

for all j, k, a and b. In particular, by replacing k with j in (53), we have

$$(\lambda_a^j)'\lambda_b^j - (\lambda_b^j)'\lambda_a^j + u_j(u_a \lambda_b^j - u_b \lambda_a^j) = 0$$

which implies that

$$\ll \Phi'', \Gamma_{ab}^k \gg = u_a \lambda_b^j - u_b \lambda_a^j = 0$$

by virtue of (52). This is a contradiction.

According to Steps 1, 2, 3 and 4, we can conclude that the subset $\tilde{I}_1$ is empty, that is, we may assume that $1 + \gamma \neq 0$ on M. $\square$

By Lemma 3, we can see that the function f of (13) is a rational function in t with functions in s as coefficients of the form

$$f(t) = -\frac{P(t)}{q^3(1 + \sum_j u_j t_j) + q^{\frac{7}{2}} \gamma(s)}. \tag{54}$$

If we substitute (54) into (11) and multiply $(1 + \sum_j u_j t_j)$ by the equation obtained in such a way, then we have

$$
\begin{aligned}
&(1+\sum_{j=1}^{r}u_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)+q^2(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\Psi_i\}\\
&-(\Phi+\sum_{j=1}^{r}\Psi_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\sum_{j=1}^{r}p_jt_j)+q^2(\tilde{\varepsilon}\phi+\sum_{j=1}^{r}\tilde{\varepsilon}\varphi_jt_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}u_i\}\\
=&-q^{\frac{1}{2}}\gamma(s)\{(\frac{\partial q}{\partial s})^2(\Phi+\sum_{j=1}^{r}\Psi_jt_j)-\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)-\frac{1}{2}q(\frac{\partial^2 q}{\partial s^2})(\Phi+\sum_{j=1}^{r}\Psi_jt_j)\\
&+q^2(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)+\frac{1}{2}q\sum_{i=1}^{r}\varepsilon_i(\frac{\partial q}{\partial t_i})^2(\Phi+\sum_{j=1}^{r}\Psi_jt_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\Psi_i\\
&-\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i(\frac{\partial^2 q}{\partial t_i^2})(\Phi+\sum_{j=1}^{r}\Psi_jt_j)\}\ +\ q^{\frac{1}{2}}\mathbf{C}P(t).
\end{aligned}
\tag{55}
$$

Next, we will show that the function q is independent of the parameter s and it is a form of perfect square expression in t of degree 2.

We suppose that $q^{\frac{1}{2}}$ is not a polynomial in t. Then we have

$$
\begin{aligned}
&(1+\sum_{j=1}^{r}u_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)+q^2(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\Psi_i\}\\
&-(\Phi+\sum_{j=1}^{r}\Psi_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\sum_{j=1}^{r}p_jt_j)+q^2(\tilde{\varepsilon}\phi+\sum_{j=1}^{r}\tilde{\varepsilon}\varphi_jt_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}u_i\}=\mathbf{0}.
\end{aligned}
\tag{56}
$$

By following the same argument to prove Lemma 3.4 in [19], (56) implies that

$$\frac{\partial q}{\partial s}=0.$$

Then, we deduce from (55) the following

$$
\begin{aligned}
&\gamma(s)\{q(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)+\frac{1}{2}\sum_{i=1}^{r}\varepsilon_i((\frac{\partial q}{\partial t_i})^2-\frac{\partial^2 q}{\partial t_i^2})(\Phi+\sum_{j=1}^{r}\Psi_jt_j)-\frac{1}{2}q\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}\Psi_i\}\\
&=\mathbf{C}\{q(\tilde{\varepsilon}\phi+\sum_{j=1}^{r}\tilde{\varepsilon}\varphi_jt_j)+\frac{1}{2}\sum_{i=1}^{r}\varepsilon_i((\frac{\partial q}{\partial t_i})^2-\frac{\partial^2 q}{\partial t_i^2})(1+\sum_{j=1}^{r}u_jt_j)-\frac{1}{2}q\sum_{i=1}^{r}\varepsilon_i\frac{\partial q}{\partial t_i}u_i\},
\end{aligned}
$$

or,

$$\frac{1}{2}\sum_{i=1}^{r}\varepsilon_i(\frac{\partial q}{\partial t_i})^2\{\gamma(s)\Phi-\mathbf{C}+\sum_{j=1}^{r}(\gamma(s)\Psi_j-u_j\mathbf{C})t_j\}=q(t)\Gamma(t),\tag{57}$$

where $\Gamma(t)$ is a polynomial in t such that deg $\Gamma=1$ with vector functions of s as coefficients. Considering the degrees of (57), we see that

$$\sum_{i=1}^{r}\varepsilon_i(\frac{\partial q}{\partial t_i})^2=cq(t)\tag{58}$$

for some constant c.

Suppose that there exist $j_1,\ldots,j_l\in\{1,\ldots,r\}$ such that $(\frac{\partial q}{\partial t_{j_k}})^2$ are not a multiple of $q(t)$ for $k=1,\ldots,l$. By (58), we get

$$\sum_{k=1}^{l}\varepsilon_{j_k}(\frac{\partial q}{\partial t_{j_k}})^2=c_1q(t)\tag{59}$$

for some constant c_1. By hypothesis, we can put

$$(\frac{\partial q}{\partial t_{j_k}})^2 = c_{j_k} q(t) + r_{j_k}(t)$$

for some constants c_{j_k} and polynomials $r_{j_k}(t)$ in t with $\deg r_{j_k}(t) \leq 1$ for $k = 1, \ldots, l$. Then, $\sum_{k=1}^{l} \varepsilon_{j_k} r_{j_k}(t)$ has to be a multiple of $q(t)$ because of (59), a contradiction. Thus, we have

$$(\frac{\partial q}{\partial t_i})^2 = 4u_i^2 q(t) \tag{60}$$

which yields

$$w_{ij} = u_i u_j,$$

by comparing the both sides of (60) for all $i, j = 1, \ldots, r$. It contradicts that $q^{\frac{1}{2}}$ is not a polynomial. Therefore, we have

$$q = (1 + \sum_{j=1}^{r} u_j t_j)^2 \tag{61}$$

for all s and t.

Since $1 + \gamma(s) \neq 0$ on M, the rational function f defined by (54) becomes

$$f = \frac{P(t)}{(1 + \sum_j u_j t_j)^7 (1 + \gamma(s))}.$$

Lemma 4. *Let M be an $(r+1)$-dimensional non-cylindrical and non-planar ruled submanifold parameterized by (3) in $\mathbb{L}^m$. Let $e_1, e_2, \ldots, e_r$ be the orthonormal generators of the rulings along the base curve α such that e_j' are non-null for all $j = 1, 2, \ldots, r$. If M has pointwise 1-type Gauss map of the second kind, then we have*

$$e_j' = u_j \alpha'$$

for all $j = 1, 2, \ldots, r$.

Proof. If M is Lorentzian, it is obvious. We now suppose that M is space-like. In this case,

$$\varepsilon_j = 1 \quad \text{and} \quad q = (1 + \sum_{j=1}^{r} u_j t_j)^2$$

for all $j = 1, \ldots, r$. Therefore, (55) can be rewritten as

$$3(\sum_{j=1}^{r} u_j' t_j)(\sum_{j=1}^{r} (p_j + \gamma(s) u_j') t_j)(\Phi + \sum_{j=1}^{r} \Psi_j t_j) = (1 + \sum_{j=1}^{r} u_j t_j) \Gamma_1(t), \tag{62}$$

where we have put

$$\begin{aligned}
\Gamma_1(t) =& 3(\sum u_j' t_j)(\Phi' + \sum \Psi_j' t_j) - (1 + \sum u_j t_j)(\Phi'' + \sum \Psi_j'' t_j) \\
&+ (1 + \sum u_j t_j)^2 (\sum u_i \Psi_i) + (\phi + \sum \varphi_j t_j)(\Phi + \sum \Psi_j t_j) \\
&- (1 + \sum u_j t_j)(\sum u_i^2)(\Phi + \sum \Psi_j t_j) + 3\gamma(\sum u_j' t_j)(\Phi' + \sum \Psi_j' t_j) \\
&+ \gamma(\sum u_j'' t_j)(\Phi + \sum \Psi_j t_j) - \gamma(1 + \sum u_j t_j)(\Phi'' + \sum \Psi_j'' t_j) \\
&- \gamma(1 + \sum u_j t_j)(\sum u_i^2)(\Phi + \sum \Psi_j t_j) + \gamma(1 + \sum u_j t_j)^2 (\sum u_i \Psi_i) \\
&+ 3(\sum u_j' t_j)^2 \mathbf{C} - 3(\sum u_j' t_j)(\sum p_j t_j)\mathbf{C} - (1 + \sum u_j t_j)(\sum u_j'' t_j)\mathbf{C} \\
&+ (1 + \sum u_j t_j)(\phi + \sum \varphi_j t_j)\mathbf{C}.
\end{aligned} \tag{63}$$

According to (63), we may have put

$$\Gamma_1(t) = \Lambda_0(s) + \sum_{j=1}^{r} \Lambda_j(s)t_j + \sum_{j,k=1}^{r} \Lambda_{jk}(s)t_jt_k, \tag{64}$$

where Λ_0, Λ_j and Λ_{jk} are vector functions of s for $j, k = 1, \ldots, r$. Then, by considering the degrees of polynomials (62) and (64) in t, we can see that

$$\Lambda_0(s) = \mathbf{0} \quad \text{and} \quad \Lambda_j(s) = \mathbf{0}$$

which implies that

$$3u_j'(p_j + \gamma(s)u_j')\Phi = \Lambda_{jj} \quad \text{and} \quad 3u_j'(p_j + \gamma(s)u_j')\Psi_j = u_j\Lambda_{jj}$$

for all $j = 1, \ldots, r$. From the above two equations, we have

$$u_j'(p_j + \gamma(s)u_j')(\Psi_j - u_j\Phi) = \mathbf{0}. \tag{65}$$

If $\Psi_j - u_j\Phi \equiv \mathbf{0}$, then $\lambda_a^j = 0$ in (17), that is, $e_j' = u_j\alpha'$ for all $j = 1, \ldots, r$ and $a = r+1, \ldots, m-1$. Now, we consider $J_1 = \{s \in I | \Psi_j - u_j\Phi \neq \mathbf{0}\}$ and suppose that $J_1 \neq \emptyset$. Then, on J_1,

$$u_j'(p_j + \gamma(s)u_j') = 0 \tag{66}$$

and hence

$$\Lambda_{jj} = \mathbf{0}.$$

With the help of (63) and (64), the relations $\Lambda_0 = \Lambda_j = \Lambda_{jj} = \mathbf{0}$ provide us with the following results

$$\phi\mathbf{C} = (1+\gamma)\{\Phi'' - \sum u_i\Psi_i + (\sum u_i^2)\Phi\} - \phi\Phi, \tag{67}$$

$$3u_j'(u_j' - p_j)\mathbf{C} = -3(1+\gamma)u_j'(\Psi_j' - u_j\Phi') + (\Psi_j - u_j\Phi)(u_j\phi - \gamma u_j'' - \varphi_j) \tag{68}$$

for $j = 1, \ldots, r$. Considering the orthogonality of vectors of the right sides in (67) and (68), we can see that the $p_j = \ll \Psi_j', \Phi \gg$ must be zero. Therefore, it follows from (66) that

$$(u_j')^2\gamma = 0. \tag{69}$$

Case 1. If $\gamma \neq 0$ on some open interval J_2 $(\subset J_1)$, then $u_j' = 0$ on J_2. Then, (55) is simplified as

$$\begin{aligned}
&(1+\sum u_jt_j)(\Phi'' + \sum \Psi_j''t_j) - (1+\sum u_jt_j)^2(\sum u_i\Psi_i) \\
&- (\phi + \sum \varphi_jt_j)(\Phi + \sum \Psi_jt_j) + (1+\sum u_jt_j)(\sum u_i^2)(\Phi + \sum \Psi_jt_j) \\
=& -\gamma(1+\sum u_jt_j)(\Phi'' + \sum \Psi_j''t_j) - \gamma(1+\sum u_jt_j)(\sum u_i^2)(\Phi + \sum \Psi_jt_j) \\
&+ \gamma(1+\sum u_jt_j)^2(\sum u_i\Psi_i) + (1+\sum u_jt_j)(\phi + \sum \varphi_jt_j)\mathbf{C},
\end{aligned} \tag{70}$$

or,

$$(\phi + \sum_{j=1}^{r} \varphi_jt_j)(\Phi + \sum_{j=1}^{r} \Psi_jt_j) = (1 + \sum_{j=1}^{r} u_jt_j)\Gamma_2(t), \tag{71}$$

where Γ_2 is a polynomial in t with vector functions of s as coefficients.

If $\Gamma_2(t) = \{a(s) + \sum_j b_j(s)t_j\}\Upsilon(s)$ for some functions a, b_j of s and a vector Υ of s, then Equation (71) gives us

$$\phi\Phi = a\Upsilon, \tag{72}$$

$$\phi\Psi_j + \varphi_j\Phi = au_j\Upsilon + b_j\Upsilon, \tag{73}$$

$$\varphi_j\Psi_j = u_jb_j\Upsilon \tag{74}$$

as the coefficients of terms containing t_j^0, t_j^1 and t_j^2 for $j = 1, \ldots, r$. Putting (72) into (73) and substituting the equation obtained in such a way into (74), we get

$$(\varphi_j - u_j\phi)(\Psi_j - u_j\Phi) = \mathbf{0}$$

which implies that $\varphi_j - u_j\phi = 0$ on J_2.

If Γ_2 is of the form $\Gamma_2 = \Upsilon_0(s) + \sum_j \Upsilon_j(s)t_j$ for some vectors Υ_0 and Υ_j along s, we also have the only possible case of $(\phi + \sum \varphi_j t_j) = \phi(1 + \sum u_j t_j)$.

Then, the condition of $\varphi_j = u_j\phi$ renders (70) simple as follows

$$\begin{aligned}(1+\gamma)\{\Phi'' + \sum \Psi_j'' t_j - (\sum u_i\Psi_i)(1 + \sum u_j t_j) + (\sum u_i^2)(\Phi + \sum \Psi_j t_j)\} \\ = \phi\{\Phi + \sum \Psi_j t_j + (1 + \sum u_j t_j)\mathbf{C}\}.\end{aligned} \tag{75}$$

Here, we may assume that $\phi \neq 0$. If not, that is, $\phi = 0 = \varphi_j$, it follows from (75) that

$$\Phi'' + \sum \Psi_j'' t_j = (\sum u_i\Psi_i)(1 + \sum u_j t_j) - (\sum u_i^2)(\Phi + \sum \Psi_j t_j) \tag{76}$$

because of $1 + \gamma \neq 0$. By computations, $u_j' = 0$ and (76) implies that $\Delta G = 0$. According to Theorem 3.4 in [9], we can see that it is part of an $(r+1)$-plane in $\mathbb{L}^m$.

Considering the constant terms with respect to t and the coefficients of terms containing t_j of (75), we have

$$(1+\gamma)\{\Phi'' - \sum u_i\Psi_i + (\sum u_i^2)\Phi\} = \phi(\Phi + \mathbf{C}), \tag{77}$$

$$(1+\gamma)\{\Psi_j'' - u_j(\sum u_i\Psi_i) + (\sum u_i^2)\Psi_j\} = \phi(\Psi_j + u_j\mathbf{C}). \tag{78}$$

Differentiating (17) with respect to s gives

$$\Psi_j'' = u_j\Phi'' + \sum_{a=r+1}^{m-1} \varepsilon_a(\lambda_a^j)''\xi_a + 2\sum_{a=r+1}^{m-1} \varepsilon_a(\lambda_a^j)'\xi_a' + \sum_{a=r+1}^{m-1} \varepsilon_a\lambda_a^j\xi_a''. \tag{79}$$

Here,

$$\begin{aligned}\xi_a' &= e_a' \wedge e_1 \wedge \cdots \wedge e_r + \sum_{i=1}^{r} e_a \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r, \\ \xi_a'' &= e_a'' \wedge e_1 \wedge \cdots \wedge e_r + 2\sum_{i=1}^{r} e_a' \wedge e_1 \wedge \cdots \wedge e_i' \wedge \cdots \wedge e_r \\ &\quad + \sum_{k,l=1}^{r} e_a \wedge e_1 \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_l' \wedge \cdots \wedge e_r + \sum_{i=1}^{r} e_a \wedge e_1 \wedge \cdots \wedge e_i'' \wedge \cdots \wedge e_r.\end{aligned} \tag{80}$$

In (80), we can see that the vector $e_a \wedge e_1 \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_l' \wedge \cdots \wedge e_r$ of ξ_a'' is orthogonal to ξ_a and other vectors in (80) except the vectors having the same form. Note that

$$\begin{aligned}&\ll e_a \wedge e_1 \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_l' \wedge \cdots \wedge e_r,\ e_b \wedge e_1 \wedge \cdots \wedge e_k' \wedge \cdots \wedge e_l' \wedge \cdots \wedge e_r \gg \\ &= \begin{vmatrix} 0 & \lambda_a^k & \lambda_a^l \\ \lambda_b^k & u_k^2 & u_ku_l \\ \lambda_b^l & u_ku_l & u_l^2 \end{vmatrix} = u_l^2\lambda_a^k\lambda_b^k + u_ku_l(\lambda_a^k\lambda_b^l + \lambda_a^l\lambda_b^k) - u_k^2\lambda_a^l\lambda_b^l\end{aligned} \tag{81}$$

for $k, l = 1, \ldots, r$ and $a, b = r+1, \ldots, m-1$.

We multiply u_j by (77) and then compare (78) and the equation obtained in such a way. Then, we can obtain

$$\begin{aligned}\phi\sum_{a=r+1}^{m-1}\varepsilon_a\lambda_a^j\xi_a=(1+\gamma(s))\{&\sum_{a=r+1}^{m-1}\varepsilon_a(\lambda_a^j)''\xi_a+2\sum_{a=r+1}^{m-1}\varepsilon_a(\lambda_a^j)'\xi_a'\\&+\sum_{a=r+1}^{m-1}\varepsilon_a\lambda_a^j\xi_a''+(\sum_{i=1}^{r}u_i^2)\sum_{a=r+1}^{m-1}\varepsilon_a\lambda_a^j\xi_a\}\end{aligned}\tag{82}$$

with the help of (17) and (79). By taking the wedge product with e_k to (82) for some k, (80) and (82) induce the following

$$\begin{aligned}\mathbf{0}=&2\sum_{a=r+1}^{m-1}\varepsilon_a(\lambda_a^j)'e_a\wedge e_1\wedge\cdots\wedge e_k'\wedge\cdots\wedge e_r\wedge e_k\\&+2\sum_{a=r+1}^{m-1}\varepsilon_a\lambda_a^j e_a'\wedge e_1\wedge\cdots\wedge e_k'\wedge\cdots\wedge e_r\wedge e_k\\&+\sum_{a=r+1}^{m-1}\varepsilon_a\lambda_a^j e_a\wedge e_1\wedge\cdots\wedge e_k''\wedge\cdots\wedge e_r\wedge e_k\\&+\sum_{a=r+1}^{m-1}\varepsilon_a\lambda_a^j(\sum_{i\neq k}^{r}e_a\wedge e_1\wedge\cdots\wedge e_k'\wedge\cdots\wedge e_i'\wedge\cdots\wedge e_r\wedge e_k).\end{aligned}\tag{83}$$

Again, taking the wedge product with e_i to (83) for $i\neq k$, we get

$$\varepsilon_a\lambda_a^j e_a\wedge e_1\wedge\cdots\wedge e_k'\wedge\cdots\wedge e_i'\wedge\cdots\wedge e_r\wedge e_k\wedge e_i=\mathbf{0}\tag{84}$$

for all $j=1,\ldots,r$ and $a=r+1,\ldots,m-1$. If $\lambda_a^j\neq 0$ for some j and a, then putting (16) into (84) implies that

$$u_k\lambda_b^i-u_i\lambda_b^k=0\tag{85}$$

for $b=r+1,\ldots,m-1$. Here, we note that $(e_k')^\perp\wedge(e_i')^\perp=\mathbf{0}$ because of $w_{ij}=u_iu_k$. Using (85), we can see that the value of (81) becomes zero, which means that the coefficients of ξ_a'' in (82) must be identically zero by the orthogonality of vectors. Therefore, we have $\lambda_a^j=0$ and hence $e_j'=u_j\alpha'$ on J_2, for all j and a.

Case 2. We consider $A=\{s\in I|\gamma(s)=0\}$ and suppose that $Int(A)\neq\emptyset$. Then, on $Int(A)$,

$$0=\ll\Phi,\mathbf{C}\gg=\ll\Phi',\mathbf{C}\gg\tag{86}$$

and Equation (68) is simplified as

$$3(u_j')^2\mathbf{C}=-3u_j'(\Psi_j'-u_j\Phi')+(\Psi_j-u_j\Phi)(u_j\phi-\varphi_j).\tag{87}$$

With the help of (86), taking the indefinite scalar product with Φ' to (87) gives

$$0=-3u_j'(\ll\Psi_j',\Phi'\gg-u_j\ll\Phi',\Phi'\gg)+\ll\Psi_j,\Phi'\gg(u_j\phi-\varphi_j)$$

which yields

$$-2u_j'(u_j\phi-\varphi_j)=0$$

because of $\ll\Psi_j',\Phi'\gg=-\varphi_j$, $\ll\Phi',\Phi'\gg=-\phi$ and $\ll\Psi_j,\Phi'\gg=u_j'$. If $u_j'\neq 0$, then $\varphi_j=u_j\phi$, which implies that $e_j'=u_j\alpha'$ by applying the same arguments used in Case 1. If $u_j'=0$ on $Int(A)$, then, by continuity, $u_j'=0$ on A and hence $e_j'=u_j\alpha'$ on A for the same reasons as Case 1.

According to Cases 1 and 2, we can conclude that

$$e_j' = u_j \alpha'$$

on J_1 and hence on M. This proof is complete. $\square$

From Lemma 4, the Gauss map G is given by

$$G = \Phi$$

and thus $\Delta G = f(G + \mathbf{C})$ yields

$$\frac{1}{2q^2}\frac{\partial q}{\partial s}\Phi' - \frac{1}{q}\Phi'' = f(\Phi + \mathbf{C}). \tag{88}$$

Taking the indefinite scalar product to (88) with Φ, we obtain

$$-\frac{1}{q}\phi(s) = f(\tilde{\varepsilon} + \gamma(s)). \tag{89}$$

Suppose that $\phi(s) \equiv 0$ on I. Then we have two cases concerning Φ'.

If $\Phi' \equiv \mathbf{0}$ on I, then the Gauss map G is a constant vector field and hence M is an open part of an $(r+1)$-plane in $\mathbb{L}^m$.

Now, we suppose that Φ' is null on some interval U. Then the normal part of α'' of (18) has to be null on U as well. Therefore, we have $\varepsilon_j = 1$ for $j = 1, \ldots, r$. By (89), we get

$$f(1 + \gamma(s)) = 0.$$

Since $1 + \gamma(s) \neq 0$, we see that $f = 0$ on U. Then, Equation (88) is rewritten as

$$(\sum_{j=1}^{r} \varepsilon u_j' t_j)\Phi' - (1 + \sum_{j=1}^{r} \varepsilon u_j t_j)\Phi'' = \mathbf{0}$$

which yields

$$\Phi'' = \mathbf{0} \quad \text{and} \quad u_j' = 0$$

for all $j = 1, \ldots, r$. By the definition of Φ and Lemma 4, we have

$$\mathbf{0} = \Phi'' = \alpha''' \wedge e_1 \wedge \cdots \wedge e_r + \sum_{k=1}^{r} \alpha'' \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k' \wedge e_{k+1} \wedge \cdots \wedge e_r. \tag{90}$$

Taking the wedge product with e_k to (90) for some k, we obtain

$$\begin{aligned} \mathbf{0} &= \alpha'' \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k' \wedge e_{k+1} \wedge \cdots \wedge e_r \wedge e_k \\ &= \alpha'' \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge u_k \alpha' \wedge e_{k+1} \wedge \cdots \wedge e_r \wedge e_k. \end{aligned}$$

Without loss of generality, we may assume that u_k is a non-zero constant. So we have

$$\alpha'' \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge \alpha' \wedge e_{k+1} \wedge \cdots \wedge e_r \wedge e_k = \mathbf{0}$$

which means that α'' is tangent to M, a contradiction.

Therefore, we conclude that if $\phi(s) \equiv 0$ on I, then $\Phi' \equiv \mathbf{0}$ and M is part of an $(r+1)$-plane in $\mathbb{L}^m$.

We now suppose that the open subset $J = \{s \in I | \phi(s) \neq 0\}$ is not empty. Then, we may put

$$f = -\frac{\phi(s)}{q(\tilde{\varepsilon} + \gamma(s))}. \tag{91}$$

Using $q = (1 + \sum_i \varepsilon u_i t_i)^2$ and putting (91) into (88), we have

$$(\sum_{j=1}^{r} u_j' t_j)\Phi' - (1 + \sum_{j=1}^{r} u_j t_j)\Phi'' = -\frac{\phi}{(\tilde{\varepsilon} + \gamma(s))}(1 + \sum_{j=1}^{r} u_j t_j)(\Phi + \mathbf{C}). \tag{92}$$

In (92), considering the constant terms with respect to t and the coefficients of terms containing t_j, we see

$$\Phi'' = \frac{\phi}{(\tilde{\varepsilon} + \gamma(s))}(\Phi + \mathbf{C}) \tag{93}$$

and hence

$$u_j' \Phi' = \mathbf{0}$$

for all $j = 1, 2, \ldots, r$. Since $\Phi'(s) \neq \mathbf{0}$, u_j are constant for all $j = 1, 2, \ldots, r$ and $s \in J$. Together with Lemma 4, we have

Lemma 5. *Let M be an $(r+1)$-dimensional non-cylindrical and non-planar ruled submanifold parameterized by (3) in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the second kind. Let $e_1, e_2, \ldots, e_r$ be the orthonormal generators of the rulings along the base curve α. If e_j' are non-null for all $j = 1, 2, \ldots, r$, then the functions*

$$u_j(s) = \langle \alpha', e_j' \rangle \ \text{ and } \ w_{ij}(s) = \langle e_i', e_j' \rangle$$

are constant functions on the open interval $J = \{s \in I | \phi(s) \neq 0\}$ for all $i, j = 1, 2, \ldots, r$, where $\phi(s) = \ll \Phi(s), \Phi''(s) \gg$.

Furthermore, for the case that e_j' are non-null for all j, we have

Lemma 6. *Let M be an $(r+1)$-dimensional non-cylindrical and non-planar ruled submanifold parameterized by (3) in $\mathbb{L}^m$. We suppose that e_j' are non-null for all $j = 1, 2, \ldots, r$. If M has pointwise 1-type Gauss map of the second kind, we can choose an orthonormal frame $\{e_a\}_{a=r+1}^{m-1}$ of the normal space $(T_{\alpha(s)}M)^\perp$ of M along α satisfying*

$$e_a' \wedge \alpha'(s) = \mathbf{0}$$

for all $a = r+1, \ldots, m-1$.

Proof. It is sufficient to see Lemma 3.5 in [20]. □

Proposition 1. *Let M be an $(r+1)$-dimensional non-cylindrical and non-planar ruled submanifold parameterized by (3) in $\mathbb{L}^m$. Let $e_1, e_2, \ldots, e_r$ be the orthonormal generators of the rulings along the base curve α such that e_j' are non-null for all $j = 1, 2, \ldots, r$. If M has pointwise 1-type Gauss map of the second kind, then the parametrization of M is given by*

$$x(s, t_1, \ldots, t_r) = t_1 \beta(s) + \sum_{i=2}^{r} t_i \mathbf{a}_i + \mathbf{D}, \tag{94}$$

where $\beta(s)$ is part of a circle or a hyperbola in $\mathbb{L}^m$ with $\mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_r$ orthonormal constant vectors satisfying $\langle \beta'(s), \mathbf{a}_i \rangle = \langle \beta(s), \mathbf{a}_i \rangle = 0$, $\mathbf{D}$ a constant vector and $t_i \in I_i$ for some open intervals I_i and $i = 2, \ldots, r$.

Proof. Suppose $\phi(s) \equiv 0$ on the whole domain I of α. In this case, we showed that M is part of an $(r+1)-$plane in $\mathbb{L}^m$. Clearly, a plane can be parameterized as (94) for some suitable constant vectors $\mathbf{a}_2, \ldots, \mathbf{a}_r$.

Now, we suppose that M is not part of an $(r+1)-$plane, that is, $J = \{s \in I | \phi(s) \neq 0\}$ is not empty. Note that

$$\ll \Phi', \Phi'' \gg = -\frac{\phi'}{2}, \quad \ll \Phi', \Phi \gg = 0 \quad \text{and} \quad \phi = -\tilde{\varepsilon} \sum_{a=r+1}^{m-1} \varepsilon_a u_a^2. \tag{95}$$

Then, Equation (93) implies

$$-\frac{\phi'}{2} = \frac{\phi}{\tilde{\varepsilon} + \gamma} \ll \Phi', \mathbf{C} \gg = \frac{\phi}{\tilde{\varepsilon} + \gamma} \gamma',$$

or, equivalently,

$$-\frac{1}{2} \frac{\phi'}{\phi} = \frac{(\tilde{\varepsilon} + \gamma)'}{(\tilde{\varepsilon} + \gamma)} \tag{96}$$

on J. Equation (96) yields

$$\frac{1}{\sqrt{|\phi|}} = \tilde{\lambda} |\tilde{\varepsilon} + \gamma|$$

for some positive constant $\tilde{\lambda}$. Therefore, from (93) we get

$$\Phi'' = \lambda \sqrt{|\phi|^3} (\Phi + \mathbf{C})$$

for some non-zero constant λ.

Meanwhile, according to Lemma 6, we can put

$$\alpha'' = -\sum_{i=1}^{r} \varepsilon_i u_i e_i - \sum_{a=r+1}^{m-1} \varepsilon_a u_a e_a \quad \text{and} \quad e_a' = \varepsilon u_a \alpha' \tag{97}$$

for all $a = r+1, \ldots, m-1$. Since $\frac{1}{\lambda \sqrt{|\phi|^3}} \Phi'' - \Phi$ is constant, by straightforward computation, we have

$$\begin{aligned} \mathbf{0} = & \sum_{a=r+1}^{m-1} \varepsilon_a \{ (\frac{u_a'}{\lambda \sqrt{|\phi|^3}})' u_a - u_a (\frac{\varepsilon \mu}{\lambda \sqrt{|\phi|^3}} + 1) \} \xi_a \\ & + \sum_{j=1}^{r} \sum_{a=r+1}^{m-1} \varepsilon \varepsilon_a u_j \{ (\frac{1}{\lambda \sqrt{|\phi|^3}})' u_a + (\frac{2}{\lambda \sqrt{|\phi|^3}}) u_a' \} e_a \wedge e_1 \wedge \cdots \wedge e_{j-1} \wedge \alpha' \wedge e_{j+1} \wedge \cdots \wedge e_r, \end{aligned} \tag{98}$$

where $\mu = \langle \alpha'', \alpha'' \rangle$. By the orthogonality of the vectors α', e_j and e_a for all $j = 1, \ldots, r$ and $a = r+1, \ldots, m-1$, (98) yields

$$(\frac{1}{\lambda \sqrt{|\phi|^3}})' u_a + (\frac{2}{\lambda \sqrt{|\phi|^3}}) u_a' = 0 \tag{99}$$

for all $a = r+1, \ldots, m-1$. Since $\phi \neq 0$ on J and $\phi = -\tilde{\varepsilon} \sum_a \varepsilon_a u_a^2$, there exists a non-zero function u_b for some $b = r+1, \ldots, m-1$. Then, (99) implies

$$\frac{3}{4} \frac{|\phi|'}{|\phi|} = \frac{u_b'}{u_b}.$$

So we can see that

$$|\phi|^{\frac{3}{4}} = \lambda_b u_b \quad \text{or,} \quad u_b^2 = \frac{1}{\lambda_b^2} |\phi|^{\frac{3}{2}}$$

for some non-zero real number λ_b. By (95), we have

$$\phi = c |\phi|^{\frac{3}{2}}$$

for some negative constant c, which means that the function ϕ is constant and hence the functions u_a are constant for all $a = r+1, \ldots, m-1$ by virtue of (99). By continuity, the interval J is the whole domain I of α. Furthermore, (97) implies

$$\alpha''' = -\mu\alpha' \tag{100}$$

for the constant $\mu = \sum_i \varepsilon_i u_i^2 + \sum_a \varepsilon_a u_a^2$. By Lemma 2, we can see that the curve α is contained in a 2-dimensional subspace of $\mathbb{L}^m$. Equation (100) gives that the curvature is non-zero constant and hence the plane curve α is part of a circle or a hyperbola.

Considering Lemmas 4–6, we may put

$$\alpha(s) = \frac{\varepsilon}{u_1}(e_1 - \mathbf{a}_1) \quad \text{and} \quad e_i(s) = \frac{u_i}{u_1}e_1(s) + \mathbf{b}_i \tag{101}$$

for some constant vectors $\mathbf{a}_1$ and $\mathbf{b}_i$ for $i = 2, \ldots, r$ such that $e_1(s)$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\ldots$, $\mathbf{b}_r$ are linearly independent for each s. By applying Gram-Schmidt's orthogonalization, we get orthonormal constant vectors $\mathbf{a}_2, \ldots, \mathbf{a}_r$ from $\mathbf{b}_2, \ldots, \mathbf{b}_r$. $\langle e_1(s), \mathbf{b}_i\rangle$ are constant and thus $\langle e_1(s), \mathbf{a}_i\rangle$ are also constant for all $i = 2, \ldots, r$.

We put $v_i = \langle e_1(s), \mathbf{a}_i\rangle$ for all $i = 2, \ldots, r$. Define

$$\beta_1(s) = e_1(s) - \sum_{r=2}^{r} \tau_i v_i \mathbf{a}_i,$$

where $\tau_i = \langle \mathbf{a}_i, \mathbf{a}_i\rangle (= \pm 1)$. Then $\langle \beta_1(s), \beta_1(s)\rangle = \varepsilon_1 - \sum_{i=2}^{r} \tau_i v_i^2$ is a non-zero constant since $e_1(s)$, $\mathbf{a}_2$, $\ldots$, $\mathbf{a}_r$ are linearly independent. Take $\beta(s) = \frac{\beta_1(s)}{||\beta_1(s)||}$, where $||\beta_1(s)|| = \sqrt{|\langle \beta_1, \beta_1\rangle|}$. After appropriate change of parameters $t_1, t_2, \ldots, t_r$, the parametrization of (3) for M can be reduced to

$$x(s, \bar{t}_1, \bar{t}_2, \ldots, \bar{t}_r) = \bar{t}_1\beta(s) + \sum_{i=2}^{r} \bar{t}_i \mathbf{a}_i + \mathbf{D}$$

for some constant vector $\mathbf{D}$.

If α is a circle, we can see that the trace of position vectors of $\beta(s)$ is a circle on the unit sphere by virtue of the first equation of (101). □

Note that if $q < 0$, that is, α is time-like, then we can see that α is part of a hyperbola in $\mathbb{L}^m$ by applying the same arguments developed in the proof of Proposition 1. Therefore, we can also obtain the parametrization (94) for M such that the curve β is part of a hyperbola.

We now consider the case that some of the generators of rulings have null derivatives. Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold parameterized by (3) in $\mathbb{L}^m$. Again, if we use Proposition 3.3 of [19], we may assume that $e_i' \neq \mathbf{0}$ for all $i = 1, \ldots, r$.

Case 3. Suppose that e_i' are null for all $i = 1, \ldots, r$. We then have three possible cases according to the degree of q.

Subcase 3.1. Let $\deg q(t) = 0$, that is, e_i' are null with $e_i'(s) \wedge e_l'(s) = \mathbf{0}$ for i, $l = 1, 2, \ldots, r$ and $u_j = \langle \alpha'(s), e_j'(s)\rangle = 0$ for $j = 1, 2, \ldots, r$. Note that $\varepsilon_i = 1$ for all $i = 1, 2, \ldots, r$. Then M has the Gauss map of the form

$$G = \Phi + \sum_{j=1}^{r} t_j \Psi_j.$$

Therefore, $\Delta G = f(G + \mathbf{C})$ implies

$$\Phi'' = -f(\Phi + \mathbf{C}) \quad \text{and} \quad \Psi_j'' = -f\Psi_j \tag{102}$$

for all $j = 1, 2, \ldots, r$. Equation (102) shows that the function f depends on the parameter s only. From $e_i'(s) \wedge e_j'(s) = \mathbf{0}$, we can put

$$e_j' = \sigma_j e_1', \tag{103}$$

where σ_j are non-vanishing functions of s for all $i, j = 1, 2, \ldots, r$. Also, $e_j'(s) \wedge e_j''(s) = \mathbf{0}$ follows from $e_i'(s) \wedge e_j'(s) = \mathbf{0}$ and hence we have

$$e_j'' = h_j e_j' \tag{104}$$

for some functions h_j of s and $j = 1, 2, \ldots, r$. By (18) and (104), we see

$$0 = u_j' = \langle \alpha'', e_j' \rangle = -\sigma_j \sum_{a=r+1}^{m-1} \varepsilon_a u_a \lambda_a^1$$

which implies

$$\sum_{a=r+1}^{m-1} \varepsilon_a u_a \lambda_a^1 = 0. \tag{105}$$

By straightforward computation, equation $\Psi_j'' = -f\Psi_j$ of (102) provides that

$$\sigma_j h_j = \sigma_j' + \sigma_j h_1 \quad \text{and} \quad -f = h_j^2 + h_j' \tag{106}$$

with the help of (103) and (104).

Now, on the non-empty open interval $I_0 = \{s \in I | f(s) \neq 0\}$, the first equation of (102) implies

$$f'\Phi'' = f\Phi''' + f^2\Phi'. \tag{107}$$

In this case, we recall that

$$\phi = \ll \Phi, \Phi'' \gg = - \ll \Phi', \Phi' \gg = -\langle \alpha'', \alpha'' \rangle = -\sum_{a=r+1}^{m-1} \varepsilon_a u_a^2 \tag{108}$$

because of $\ll \Phi, \Phi' \gg = 0$. From the definition of Φ, we get

$$\Phi' = -\sum_{a=r+1}^{m-1} \varepsilon_a u_a \xi_a + \sum_{k=1}^{r} \sigma_k \Omega_1^k, \tag{109}$$

$$\Phi'' = \phi\Phi - \sum_{a=r+1}^{m-1} \varepsilon_a u_a' \xi_a + \sum_{k=1}^{r} \sigma_k h_k \Omega_1^k - 2\sum_{k=1}^{r} \sum_{a=r+1}^{m-1} \varepsilon_a u_a \sigma_k \Omega_{a,1}^k, \tag{110}$$

$$\begin{aligned} \Phi''' = &\frac{3}{2}\phi'\Phi - \sum_{a=r+1}^{m-1} \varepsilon_a (u_a \phi + u_a'') \xi_a + \sum_{k=1}^{r} (3\sigma_k \phi - f\sigma_k) \Omega_1^k \\ &- \sum_{k=1}^{r} \sum_{a=r+1}^{m-1} \varepsilon_a (3u_a' \sigma_k + 3u_a \sigma_k h_k) \Omega_{a,1}^k \end{aligned} \tag{111}$$

with the aid of (18), (103)–(105) and (108), where we have put

$$\Omega_1^k = \alpha' \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_1' \wedge e_{k+1} \wedge \cdots \wedge e_r,$$

$$\Omega_{a,1}^k = e_a \wedge e_1 \wedge \cdots \wedge e_{k-1} \wedge e_1' \wedge e_{k+1} \wedge \cdots \wedge e_r.$$

Considering (107) and (109)–(111), we have the following:

$$f'\phi = \frac{3}{2} f \phi', \tag{112}$$

$$f'\sigma_k h_k = 3f\sigma_k\phi, \tag{113}$$

$$2f'u_a\sigma_k = f(3u_a'\sigma_k + 3u_a\sigma_k h_k) \tag{114}$$

as the coefficients of vectors Φ, Ω_1^k and $\Omega_{a,1}^k$, respectively, for all $k = 1, \ldots, r$ and $a = r+1, \ldots, m-1$. Note that σ_k are non-vanishing for all $k = 1, \ldots, r$. Multiplying (114) by $\varepsilon_a u_a$ and adding the equations obtained in such a way together with respect to a , we obtain

$$2f' \sum_{a=r+1}^{m-1} \varepsilon_a u_a^2 = 3f \sum_{a=r+1}^{m-1} \varepsilon_a u_a u_a' + 3fh_k \sum_{a=r+1}^{m-1} \varepsilon_a u_a^2,$$

or,

$$2f'\phi = \frac{3}{2}f\phi' + 3f\phi h_k \tag{115}$$

for all $k = 1, 2, \ldots, r$. By (112), (115) yields that

$$h_k = \frac{1}{2}\frac{\phi'}{\phi}. \tag{116}$$

By putting (112) into (113) and then considering (116), we have

$$h_k^2 = \phi.$$

Thus, $\mu = \langle \alpha'', \alpha'' \rangle$ is non-positive because of (108). Since e_j' are null and $\langle \alpha'', e_j' \rangle = 0$ for $j = 1, \ldots, r$, the vector α'' can not be time-like and thus $\phi = 0$. Therefore, in (113), we see that

$$f'\sigma_k h_k = 0$$

for all $k = 1, \ldots, r$. If $f' \neq 0$, then $h_k = 0$ and then f is vanishing because of (106), a contradiction on I_0. Therefore, $f' = 0$ on I_0 and hence f is a non-zero constant function on M by continuity. This means that G is of 1-type in the usual sense. For ruled submanifolds with finite-type Gauss map, see [10].

Subcase 3.2. Let $\deg q(t) = 1$. In this case, $\langle \alpha'(s), e_i'(s) \rangle \neq 0$ for some i $(1 \leq i \leq r)$ and the null vector fields e_i' satisfy $e_i' \wedge e_l' = 0$ for $i, l = 1, 2, \ldots, r$. We note that $\tilde{\varepsilon} = 1$ and $\varepsilon_i = 1$ for all $i = 1, 2, \ldots, r$. Thus, $\Delta G = f(G + \mathbf{C})$ implies

$$\begin{aligned}
&(\frac{\partial q}{\partial s})^2(1 + \sum_{j=1}^r u_j t_j) - \frac{3}{2}q\frac{\partial q}{\partial s}(\sum_{j=1}^r p_j t_j) - \frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(1 + \sum_{j=1}^r u_j t_j) \\
&+q^2(\phi + \sum_{j=1}^r \varphi_j t_j) + \frac{1}{2}q\sum_{i=1}^r(\frac{\partial q}{\partial t_i})^2(1 + \sum_{j=1}^r u_j t_j) - \frac{1}{2}q^2\sum_{i=1}^r \frac{\partial q}{\partial t_i}u_i \\
&+f\{q^3(1 + \sum_{j=1}^r u_j t_j) + q^{\frac{7}{2}}\gamma(s)\} = 0.
\end{aligned} \tag{117}$$

Note that $q^3(1 + \sum_j u_j t_j) + q^{\frac{7}{2}}\gamma(s) \neq 0$ because of $\deg q = 1$. Therefore, using the function f obtained from (117), we repeat the same process to get (55). Then, we have the following equation

$$\begin{aligned}
&(1+\sum_{j=1}^{r}u_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)+q^2(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}\Psi_i\}\\
&-(\Phi+\sum_{j=1}^{r}\Psi_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\sum_{j=1}^{r}p_jt_j)+q^2(\phi+\sum_{j=1}^{r}\varphi_jt_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}u_i\}\\
=&-q^{\frac{1}{2}}\gamma(s)\{(\frac{\partial q}{\partial s})^2(\Phi+\sum_{j=1}^{r}\Psi_jt_j)-\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)-\frac{1}{2}q(\frac{\partial^2 q}{\partial s^2})(\Phi+\sum_{j=1}^{r}\Psi_jt_j)\\
&+q^2(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)+\frac{1}{2}q\sum_{i=1}^{r}(\frac{\partial q}{\partial t_i})^2(\Phi+\sum_{j=1}^{r}\Psi_jt_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}\Psi_i\}\\
&+q^{\frac{1}{2}}\mathbf{C}P(t),
\end{aligned} \tag{118}$$

where we have put

$$\begin{aligned}
P(t)=&(\frac{\partial q}{\partial s})^2(1+\sum_{j=1}^{r}u_jt_j)-\frac{3}{2}q\frac{\partial q}{\partial s}(\sum_{j=1}^{r}p_jt_j)-\frac{1}{2}q\frac{\partial^2 q}{\partial s^2}(1+\sum_{j=1}^{r}u_jt_j)\\
&+q^2(\phi+\sum_{j=1}^{r}\varphi_jt_j)+\frac{1}{2}q\sum_{i=1}^{r}(\frac{\partial q}{\partial t_i})^2(1+\sum_{j=1}^{r}u_jt_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}u_i.
\end{aligned}$$

Since $\deg q=1$, the left side of (118) has to be vanishing, that is,

$$\begin{aligned}
&(1+\sum_{j=1}^{r}u_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)+q^2(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}\Psi_i\}\\
&-(\Phi+\sum_{j=1}^{r}\Psi_jt_j)\{-\frac{3}{2}q(\frac{\partial q}{\partial s})(\sum_{j=1}^{r}p_jt_j)+q^2(\phi+\sum_{j=1}^{r}\varphi_jt_j)-\frac{1}{2}q^2\sum_{i=1}^{r}\frac{\partial q}{\partial t_i}u_i\}=\mathbf{0}.
\end{aligned} \tag{119}$$

Using $q=1+\sum_i 2u_it_i$ and $\frac{\partial q}{\partial s}=\sum_i 2u_i't_i$, Equation (119) can be expressed as

$$(\Phi'+\sum_{j=1}^{r}\Psi_j't_j)(1+\sum_{i=1}^{r}u_it_i)-(\sum_{i=1}^{r}p_it_i)(\Phi+\sum_{j=1}^{r}\Psi_jt_j)=qW(t) \tag{120}$$

for some vector $W(t)$. In [29], it was proved that Equation (120) implies

$$\frac{\partial q}{\partial s}=0.$$

Therefore, (119) is simplified as

$$\begin{aligned}
&(1+\sum_{j=1}^{r}u_jt_j)(\Phi''+\sum_{j=1}^{r}\Psi_j''t_j-\sum_{i=1}^{r}u_i\Psi_i)\\
&-(\Phi+\sum_{j=1}^{r}\Psi_jt_j)(\phi+\sum_{j=1}^{r}\varphi_jt_j-\sum_{i=1}^{r}u_i^2)=\mathbf{0}
\end{aligned} \tag{121}$$

which furnishes us with three equations as follows:

$$\Phi''-\sum_{i=1}^{r}u_i\Psi_i-\phi\Phi+(\sum_{i=1}^{r}u_i^2)\Phi=\mathbf{0}, \tag{122}$$

$$\Psi_j'' + u_j\Phi'' - u_j(\sum_{i=1}^{r} u_i\Psi_i) - \varphi_j\Phi - \phi\Psi_j + (\sum_{i=1}^{r} u_i^2)\Psi_j = \mathbf{0}, \tag{123}$$

$$u_j\Psi_j'' - \varphi_j\Psi_j = \mathbf{0} \tag{124}$$

for all $j = 1, 2, \ldots, r$. Combining (122)–(124), we get

$$(\Psi_j - u_j\Phi)(\varphi_j - u_j\phi + u_j\sum_{i=1}^{r} u_i^2) = \mathbf{0}$$

for all $j = 1, 2, \ldots, r$. By the characters of e_j' and α', we see that the functions $\lambda_a^j(s)$ of (16) are non-vanishing for all s and it is impossible to have $\Psi_j = u_j\Phi$. Thus, we have

$$\varphi_j - u_j\phi + u_j\sum_{i=1}^{r} u_i^2 \equiv 0 \tag{125}$$

for all $j = 1, 2, \ldots, r$.

Meanwhile, we note that $e_j' \wedge e_i' = \mathbf{0}$ for all $i, j = 1, \ldots, r$. Then, we can put

$$e_i' = f_i e_{j_0}' \tag{126}$$

for some j_0 with $u_{j_0} \neq 0$, where f_i are non-vanishing functions for all $i = 1, \ldots, r$. From the definition of u_i, we have

$$u_i = f_i u_{j_0} \tag{127}$$

which implies that f_i are non-zero constant for all $i = 1, \ldots, r$. Indeed, we see that $u_i \neq 0$ for all $i = 1, \ldots, r$. By (14) and (126), we also obtain

$$\varphi_i = f_i\varphi_{j_0} \tag{128}$$

for all $i = 1, \ldots, r$. Thus, the following vector and function of (121) are induced as

$$\Phi'' + \sum_{j=1}^{r}\Psi_j''t_j - \sum_{i=1}^{r} u_i\Psi_i = \frac{\varphi_{j_0}}{u_{j_0}}(\Phi + \sum_{j=1}^{r}\Psi_j t_j) \tag{129}$$

and

$$\phi + \sum_{j=1}^{r}\varphi_j t_j - \sum_{i=1}^{r} u_i^2 = \frac{\varphi_{j_0}}{u_{j_0}}(1 + \sum_{j=1}^{r} u_j t_j) \tag{130}$$

by virtue of (122), (124), (125), (127) and (128). Using $\frac{\partial q}{\partial s} = 0$ and (119), and substituting (129) and (130) into (118), Equation (118) is rewritten as

$$(\frac{\varphi_{j_0}}{u_{j_0}}q + 2\sum_{i=1}^{r} u_i^2)\{\gamma(s)(\Phi + \sum_{j=1}^{r}\Psi_j t_j) - \mathbf{C}(1 + \sum_{j=1}^{r} u_j t_j)\} = \mathbf{0}. \tag{131}$$

We note that $\frac{\varphi_{j_0}}{u_{j_0}}q + 2\sum_i u_i^2$ of (131) is non-vanishing for all s. If this not the case, since $q = 1 + \sum_i 2u_i t_i$ is a polynomial in t of degree 1 and $\sum_i u_i^2$ is constant with respect to t, φ_{j_0} has to be vanishing for s and hence $\sum_i u_i^2 = 0$, a contradiction to $u_i \neq 0$ for all i. Therefore, we have

$$\gamma(s)(\Phi + \sum_{j=1}^{r}\Psi_j t_j) = \mathbf{C}(1 + \sum_{j=1}^{r} u_j t_j),$$

or, equivalently,

$$\gamma(s)\Phi = \mathbf{C} \quad \text{and} \quad \gamma(s)\Psi_j = u_j\mathbf{C} \tag{132}$$

for all $j = 1, \ldots, r$. Differentiating '$\gamma(s)\Phi = \mathbf{C}$' of (132) with respect to s and taking the indefinite scalar product with Φ to the equation obtained in such a way, we get

$$\gamma'(s) = 0$$

which implies that γ is a non-zero constant function for all s. If not, that is, $\gamma = 0$, the vector $\mathbf{C}$ is zero, a contradiction. Therefore, (132) yields

$$\Psi_j = u_j\Phi \quad \text{and hence} \quad e_j' = u_j\alpha'$$

for all $j = 1, \ldots, r$. This is also a contradiction to the characters of e_j' and α'.

Consequently, we can conclude that there is no ruled submanifold with deg $q = 1$ which has pointwise 1-type Gauss map of the second kind.

Subcase 3.3. Let deg $q(t) = 2$. In this case, we can easily obtain the same conclusion such as Lemma 4 by referring to the case that $e_1', e_2', \ldots, e_r'$ are non-null. But this is impossible according to the characters of the vectors α' and e_j'. Therefore, we see that no ruled submanifold in $\mathbb{L}^m$ with deg $q = 2$ has pointwise 1-type Gauss map of the second kind.

Case 4. Suppose that $e_{j_1}', \ldots, e_{j_k}'$ are null for $j_1 < j_2 < \cdots < j_k \in \{1, 2, \ldots, r\}$ and e_i' are non-null for $i \neq j_l, l = 1, \ldots, k$.

In this case, deg q = 2. If we follow a similar argument for the case that $e_1', e_2', \ldots, e_r'$ are non-null, for the same reason as in Subcase 3.3, we can conclude that there is no ruled submanifold in $\mathbb{L}^m$ with pointwise 1-type Gauss map of the second kind under these assumptions.

Until now, we have considered the necessary conditions for ruled submanifolds to have a pointwise 1-type Gauss map of the second kind. That is, if the ruled submanifold M in $\mathbb{L}^m$ parameterized by (3) has a pointwise 1-type Gauss map of the second kind, then according to the characters of e_i', M is part of a product manifold of a right cone (or a hyperbolic cone) and a plane, or M has a 1-type Gauss map in the usual sense. Conversely, by straightforward computations, we can see that the Gauss maps of these ruled submanifolds are of pointwise 1-type of the second kind.

Therefore, we have

Theorem 2. *Let M be an $(r+1)$-dimensional non-cylindrical ruled submanifold with non-degenerate rulings in the Minkowski m-space $\mathbb{L}^m$. Then, M has a pointwise 1-type Gauss map G of the second kind if and only if M is one of the following:*
(1) *M has a 1-type Gauss map in the usual sense, i.e., the Gauss map G satisfies $\Delta G = \lambda G + \mathbf{C}$ for some non-zero $\lambda \in \mathbb{R}$ and some constant vector $\mathbf{C}$.*
(2) *M is part of a product manifold of a right cone and a plane of the form $\mathbb{C}_S \times \mathbb{R}^{r-1}$ or $\mathbb{C}_S \times \mathbb{L}^{r-1}$.*
(3) *M is part of a product manifold of a hyperbolic cone and a plane $\mathbb{C}_H \times \mathbb{R}^{r-1}$.*
(3) *M is part of an $(r+1)$-plane in $\mathbb{L}^m$.*

4. Generalized Null Scrolls in $\mathbb{L}^m$

Let M be an $(r+1)$-dimensional ruled submanifold in $\mathbb{L}^m$ with degenerate rulings $E(s,r)$ along a regular curve with a parametrization $\tilde{x}(s,t)$, where $t = (t_1, t_2, \ldots, t_r)$. Since $E(s,r)$ is degenerate, it can be spanned by a degenerate frame $\{B(s) = e_1(s), e_2(s), \ldots, e_r(s)\}$ such that

$$\langle B(s), B(s)\rangle = \langle B(s), e_i(s)\rangle = 0, \quad \langle e_i(s), e_j(s)\rangle = \delta_{ij}, \quad i, j = 2, 3, \ldots, r.$$

Without loss of generality as was shown in Lemma 1, we may assume that

$$\langle e_i'(s), e_j(s)\rangle = 0, \ \ i, j = 2, 3, \cdots, r.$$

Since the tangent space of M at $\tilde{x}(s,t)$ is non-degenerate and contains the degenerate ruling $E(s,r)$, there exists a tangent vector field A to M which satisfies

$$\langle A(s,t), A(s,t)\rangle = 0, \ \ \langle A(s,t), B(s)\rangle = -1, \ \ \langle A(s,t), e_i(s)\rangle = 0, \ \ i = 2, 3, \ldots, r$$

at $\tilde{x}(s,t)$.

Let $\alpha(s)$ be an integral curve of the vector field A on M. Then we can define another parametrization x of M as follows:

$$x(s, t_1, t_2, \ldots, t_r) = \alpha(s) + \sum_{i=1}^{r} t_i e_i(s),$$

where $\alpha'(s) = A(s)$. A ruled submanifold defined as above is called a *generalized null scroll*. We refer to two lemmas for later use.

Lemma 7 ([7])**.** *We may assume that* $\langle A(s), B'(s)\rangle = 0$ *for all* s.

Lemma 8 ([8])**.** *Let* M *be a ruled submanifold with degenerate rulings. Then, the following are equivalent.*
(1) M *is minimal*
(2) B' *is tangent to* M.

If we put $P = \langle x_s, x_s\rangle$ and $Q = -\langle x_s, x_{t_1}\rangle$, Lemma 7 implies

$$P(s,t) = 2\sum_{i=2}^{r} u_i(s) t_i + \sum_{i,j=1}^{r} w_{ij}(s) t_i t_j,$$

$$Q(s,t) = 1 + \sum_{i=2}^{r} v_i(s) t_i,$$

where $v_i(s) = \langle B'(s), e_i(s)\rangle$, $u_i(s) = \langle A(s), e_i'(s)\rangle$ and $w_{ij}(s) = \langle e_i'(s), e_j'(s)\rangle$ for $i, j = 1, 2, \ldots, r$.

Note that P and Q are polynomials in $t = (t_1, t_2, \ldots, t_r)$ with functions in s as coefficients. Then the Laplacian Δ of M can be expressed as follows:

$$\begin{aligned}\Delta = \frac{1}{Q^2}\{&\frac{\partial \bar{P}}{\partial t_1}\frac{\partial}{\partial t_1} - 2Q\sum_{i=2}^{r} v_i \frac{\partial}{\partial t_i} + 2Q\frac{\partial^2}{\partial s \partial t_1} + \bar{P}\frac{\partial^2}{\partial t_1^2} \\ &- 2Q\sum_{i=2}^{r} v_i t_1 \frac{\partial^2}{\partial t_1 \partial t_i} - Q^2 \sum_{i=2}^{r}\frac{\partial^2}{\partial t_i^2}\},\end{aligned}$$

where $\bar{P} = P - t_1^2 \sum_{i=2}^{r} v_i^2$.

By definition of the indefinite scalar product $\ll, \gg$ on $G(r+1, m)$, we may put

$$\ll x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r}, x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r} \gg = -Q^2.$$

Then the Gauss map G is given by

$$
\begin{aligned}
G =&\frac{1}{|Q|}x_s \wedge x_{t_1} \wedge x_{t_2} \wedge \cdots \wedge x_{t_r} \\
=&\frac{1}{|Q|}\{A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + t_1 B' \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ \sum_{i=2}^{r} t_i e_i' \wedge B \wedge e_2 \wedge \cdots \wedge e_r\}.
\end{aligned}
$$

In [9], the authors proved the following theorem.

Theorem 3 ([9])**.** *Let M be a generalized null scroll in $\mathbb{L}^m$. Then, the following are equivalent.*
(1) M is minimal.
(2) M has a harmonic Gauss map.

We now suppose that a generalized null scroll M has a pointwise 1-type Gauss map $\Delta G = f(G + \mathbf{C})$. Without loss of generality, we may assume that $Q > 0$. Then by straightforward computation, we get

$$
\begin{aligned}
&\frac{2}{Q^3}\sum_{h=r+1}^{m-1}\{(\sum_{i=1}^{r}\langle B', e_i'\rangle t_i - \sum_{i=2}^{r} v_i' t_i) v_h + v_h' Q\} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ \frac{2}{Q^2}\sum_{h=r+1}^{m-1} v_h^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ \frac{2}{Q^2}\sum_{i=2}^{r}\sum_{h=r+1}^{m-1} v_i v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge A \wedge e_{i+1} \wedge \cdots \wedge e_r \\
&+ \frac{2}{Q^2}\sum_{i=2}^{r}\sum_{h,l=r+1}^{m-1} v_h \lambda_l^i e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{i-1} \wedge e_l \wedge e_{i+1} \wedge \cdots \wedge e_r \\
=&f[\frac{1}{Q}\{(1 + \sum_{i=2}^{r} t_i v_i) A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \sum_{h=r+1}^{m-1}(t_1 v_h + \sum_{i=2}^{r}\lambda_h^i t_i) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r\} + \mathbf{C}] \\
=&f A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \frac{f}{Q}\sum_{h=r+1}^{m-1}(t_1 v_h + \sum_{i=2}^{r}\lambda_h^i t_i) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r + f\mathbf{C},
\end{aligned}
\tag{133}
$$

where we have put

$$
B' = \sum_{i=2}^{m-1} v_i e_i \quad \text{and} \quad e_j' = v_j A - u_j B + \sum_{l=r+1}^{m-1}\lambda_l^j e_l \tag{134}
$$

for $j = 2, \ldots, r$ and $l = r+1, \ldots, m-1$.

Now, we note that Q is constant with respect to t_1. Then, by differentiating (133) with respect to t_1, we get

$$
\begin{aligned}
&\frac{2}{Q^3}\langle B', B'\rangle \sum_{h=r+1}^{m-1} v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
=&f_{t_1} A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \frac{f_{t_1}}{Q}\sum_{h=r+1}^{m-1}(t_1 v_h + \sum_{i=2}^{r}\lambda_h^i t_i) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ \frac{f}{Q}\sum_{h=r+1}^{m-1} v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r + f_{t_1}\mathbf{C}.
\end{aligned}
\tag{135}
$$

Case 5. $f_{t_1} \equiv 0$ on M.

Equation (135) implies that

$$f = \frac{2}{Q^2}\langle B', B'\rangle = \frac{2w_{11}}{Q^2}. \tag{136}$$

Putting (136) into (133), we obtain the following polynomial in t of degree 1 with functions of s as coefficients

$$\begin{aligned}
&\sum_{h=r+1}^{m-1}\{(\sum_{i=1}^{r}\langle B', e_i'\rangle t_i - \sum_{j=2}^{r} v_j' t_j)v_h + v_h' Q\} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ Q\sum_{h=r+1}^{m-1} v_h^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ Q\sum_{j=2}^{r}\sum_{h=r+1}^{m-1} v_j v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge A \wedge e_{j+1} \wedge \cdots \wedge e_r \\
&+ Q\sum_{j=2}^{r}\sum_{h,l=r+1}^{m-1} v_h \lambda_l^j e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge e_l \wedge e_{j+1} \wedge \cdots \wedge e_r \\
=& w_{11} Q A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + w_{11}\sum_{h=r+1}^{m-1}(t_1 v_h + \sum_{j=2}^{r}\lambda_h^j t_j) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ w_{11} Q \mathbf{C}.
\end{aligned} \tag{137}$$

Comparing the constant terms with respect to t of (137) and using $w_{11} = \sum_{j=2}^{r} v_j^2 + \sum_{l=r+1}^{m-1} v_l^2$, we have

$$\begin{aligned}
\bar{\varepsilon} w_{11}\mathbf{C} =& \sum_{h=r+1}^{m-1} v_h' e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r - \sum_{j=2}^{r} v_j^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ \sum_{j=2}^{r}\sum_{h=r+1}^{m-1} v_j v_h e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge A \wedge e_{j+1} \wedge \cdots \wedge e_r \\
&+ \sum_{j=2}^{r}\sum_{h,l=r+1}^{m-1} v_h \lambda_l^j e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge e_l \wedge e_{j+1} \wedge \cdots \wedge e_r.
\end{aligned} \tag{138}$$

From (136) we see that $w_{11}(s) \neq 0$.

Differentiating (137) with respect to t_j $(j = 2, \ldots, r)$, with the aid of (138) we get

$$\sum_{h=r+1}^{m-1}(\langle B', e_j'\rangle v_h - v_j' v_h - w_{11}\lambda_h^j) e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r = \mathbf{0},$$

which implies

$$v_h \sum_{p=r+1}^{m-1} v_p \lambda_p^j - v_j' v_h - w_{11}\lambda_h^j = 0 \tag{139}$$

as the coefficient of the vector $e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r$ for all $j = 2, \ldots, r$ and $h = r+1, \ldots, m-1$.

If $v_h = 0$ for all h, (134) implies that B' is tangent to M. With the help of Lemma 8 and Theorem 3, we can see that M is minimal and hence the Gauss map G of M is harmonic. In this case, G can be chosen as the constant vector $-\mathbf{C}$. That is, M is part of a Lorentzian $(r+1)$-plane in $\mathbb{L}^m$.

Now, we suppose that $v_h \neq 0$ for some $h \in \{r+1, \ldots, m-1\}$. If we put

$$e_h' = v_h A - u_h B - \sum_{j=2}^{r}\lambda_h^j e_j \tag{140}$$

in the same manner as (15) by virtue of Lemma 1, differentiating (138) with respect to s provides

$$
\begin{aligned}
w'_{11}\mathbf{C} = & \sum_{h=r+1}^{m-1}\{v''_h + (\sum_{j=2}^{r} v_j^2)u_h - \sum_{j=2}^{r} v_j u_j v_h + \sum_{j=2}^{r}\sum_{p=r+1}^{m-1} v_p \lambda_p^j \lambda_h^j - \sum_{j=2}^{r}\sum_{p=r+1}^{m-1} (\lambda_p^j)^2 v_h\} \\
& \qquad e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
& + (\sum_{p=r+1}^{m-1} v_p v'_p - 2\sum_{j=2}^{r} v_j v'_j + \sum_{j=2}^{r}\sum_{p=r+1}^{m-1} v_j \lambda_p^j v_p) A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
& + \sum_{h,p=r+1}^{m-1} (v'_h v_p - \sum_{j=2}^{r} v_j v_h \lambda_p^j) e_h \wedge e_p \wedge e_2 \wedge \cdots \wedge e_r \\
& - \sum_{j=2}^{r}\sum_{h=r+1}^{m-1}\{v_j v'_h + (\sum_{k=1}^{r} v_k^2)\lambda_h^j + (v_j v_h)' - (\sum_{p=r+1}^{m-1} v_p^2)\lambda_h^j + \sum_{p=r+1}^{m-1} v_p \lambda_p^j v_h\} \\
& \qquad A \wedge B \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge e_h \wedge e_{j+1} \wedge \cdots \wedge e_r \\
& + \sum_{j=2}^{r}\sum_{h,p=r+1}^{m-1}\{v'_h \lambda_p^j - v_j v_h u_p + (v_h \lambda_p^j)'\} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge e_p \wedge \cdots \wedge e_r.
\end{aligned} \tag{141}
$$

Comparing the coefficients of the vectors in (138) and (141), we obtain the following four equations:

$$\frac{w'_{11}}{w_{11}} v'_h = v''_h + (\sum_{j=2}^{r} v_j^2)u_h - \sum_{j=2}^{r} v_j u_j v_h + \sum_{j=2}^{r}\sum_{p=r+1}^{m-1} v_p \lambda_p^j \lambda_h^j - \sum_{j=2}^{r}\sum_{p=r+1}^{m-1} (\lambda_p^j)^2 v_h, \tag{142}$$

$$-\frac{w'_{11}}{w_{11}}(\sum_{j=2}^{r} v_j^2) = \sum_{p=r+1}^{m-1} v_p v'_p - 2\sum_{j=2}^{r} v_j v'_j + \sum_{j=2}^{r}\sum_{p=r+1}^{m-1} v_j \lambda_p^j v_p, \tag{143}$$

$$v'_h v_p - \sum_{j=2}^{r} v_j v_h \lambda_p^j - v'_p v_h + \sum_{j=2}^{r} v_j v_p \lambda_h^j = 0, \tag{144}$$

$$\frac{w'_{11}}{w_{11}} v_j v_h = v_j v'_h + (\sum_{k=2}^{r} v_k^2)\lambda_h^j + (v_j v_h)' - (\sum_{p=r+1}^{m-1} v_p^2)\lambda_h^j + v_h \sum_{p=r+1}^{m-1} v_p \lambda_p^j. \tag{145}$$

Substituting (139) into (143), we get

$$-\frac{w'_{11}}{w_{11}}(\sum_{j=2}^{r} v_j^2) = \sum_{p=r+1}^{m-1} v_p v'_p - \sum_{j=2}^{r} v_j v'_j + \frac{w_{11}}{v_h}\sum_{k=2}^{r} v_k \lambda_h^k \tag{146}$$

for some h with $v_h \neq 0$. Putting (146) into (144) gives

$$v_p v'_h = v_h v'_p,$$

which implies

$$v_h(s) = c_p^h v_p(s)$$

for some constant c_p^h. Therefore, we can put

$$v_h(s) = c_h v_{r+1}(s) \tag{147}$$

for some constants c_h and $h = r+1, \ldots, m-1$.

Recall that Equation (139) is valid for all $h = r+1, \ldots, m-1$. By replacing h with $r+1, \ldots, m-1$, respectively, and comparing equations obtained in such a way, with the help of (147) we can get

$$\lambda_h^j(s) = c_h \lambda_{r+1}^j(s) \tag{148}$$

for all $j = 2, \ldots, r$ and for all $h = r+1, \ldots, m-1$. By virtue of (147) and (148), Equation (142) is simplified as

$$\frac{w_{11}'}{w_{11}} v_h' = v_h'' + (\sum_{j=2}^{r} v_j^2) u_h - \sum_{j=2}^{r} v_j u_j v_h \tag{149}$$

for all $h = r+1, \ldots, m-1$. Putting (147) into (149) and repeating the method to get (148), we have

$$u_h(s) = c_h u_{r+1}(s) \tag{150}$$

for all $h = r+1, \ldots, m-1$. If we put (147) and (148) into (139), then we obtain

$$v_j' v_{r+1} = -(\sum_{k=2}^{r} v_k^2) \lambda_{r+1}^j \tag{151}$$

because of $w_{11} = \sum_j v_j^2 + \sum_h v_h^2$. Substituting (139) into (145) provides

$$\frac{w_{11}'}{w_{11}} v_j v_{r+1} = 2\{(v_j v_{r+1})' + (\sum_{k=2}^{r} v_k^2) \lambda_{r+1}^j\},$$

which yields

$$v_j (\frac{w_{11}'}{w_{11}} v_{r+1} - 2v_{r+1}') = 0 \tag{152}$$

with the help of (151).

If $v_j = 0$, (138) implies that

$$w_{11} \mathbf{C} = \sum_{h=r+1}^{m-1} v_h' e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \tag{153}$$

and hence

$$w_{11}' \mathbf{C} = \sum_{h=r+1}^{m-1} v_h'' e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r + \sum_{h=r+1}^{m-1} v_h v_h' A \wedge B \wedge e_2 \wedge \cdots \wedge e_r. \tag{154}$$

Combining (153) and (154), we have

$$\sum_{h=r+1}^{m-1} v_h v_h' = 0,$$

which means that the function $w_{11} = \sum_h v_h^2$ is constant. Since $w_{11} = v_{r+1}^2 \sum_h c_h^2$, the function v_{r+1} is constant, so are v_h for all $h = r+1, \ldots, m-1$. In (153), we can see that $\mathbf{C}$ is a zero vector because of $w_{11} \neq 0$, a contradiction. Therefore, from (152), we conclude that

$$\frac{w_{11}'}{w_{11}} v_{r+1} = 2v_{r+1}',$$

or, equivalently,

$$w_{11} = d_{r+1} v_{r+1}^2 \tag{155}$$

for some positive constant d_{r+1}. Since $w_{11} = \sum_j v_j^2 + \sum_h v_h^2 = \sum_j v_j^2 + v_{r+1}^2 \sum_h c_h^2$, we see that

$$\sum_{j=2}^{r} v_j^2 = (d_{r+1} - \sum_{h=r+1}^{m-1} c_h^2) v_{r+1}^2. \tag{156}$$

We now introduce another kind of generalized null scroll as follows:

For a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$, we consider a null frame $\{A(s), B(s) = e_1(s), e_2(s), \ldots, e_{m-1}(s)\}$ along $\tilde{\alpha}(s)$ satisfying

$$\begin{aligned} \langle A(s), A(s)\rangle = \langle B(s), B(s)\rangle = \langle A(s), e_i(s)\rangle = \langle B(s), e_i(s)\rangle = 0, \\ \langle A(s), B(s)\rangle = -1, \ \ \langle e_i(s), e_j(s)\rangle = \delta_{ij}, \ \ \tilde{\alpha}'(s) = A(s) \end{aligned}$$

for $i, j = 2, 3, \ldots, m-1$.

Let $X(s)$ be the matrix $(A(s)\ B(s)\ e_2(s) \cdots e_{m-1}(s))$ consisting of column vectors of $A(s)$, $B(s)$, $e_2(s)$, …, $e_{m-1}(s)$ with respect to the standard coordinate system in $\mathbb{L}^m$. Then we have

$$X^t(s) E X(s) = T,$$

where $X^t(s)$ denotes the transpose of $X(s)$, $E =$diag$(-1,\ 1, \ldots,\ 1)$ and

$$T = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Consider a system of ordinary differential equations

$$X'(s) = X(s) M(s), \tag{157}$$

where

$$M(s) = \begin{pmatrix} 0 & 0 & v_2 & \cdots & v_r & v_{r+1} & c_{r+2}v_{r+1} & \cdots & c_{m-1}v_{r+1} \\ 0 & 0 & -u_2 & \cdots & -u_r & -u_{r+1} & -c_{r+2}u_{r+1} & \cdots & -c_{m-1}u_{r+1} \\ -u_2 & v_2 & 0 & \cdots & 0 & -\lambda^2_{r+1} & -c_{r+2}\lambda^2_{r+1} & \cdots & -c_{m-1}\lambda^2_{r+1} \\ -u_3 & v_3 & 0 & \cdots & 0 & -\lambda^3_{r+1} & -c_{r+2}\lambda^3_{r+1} & \cdots & -c_{m-1}\lambda^3_{r+1} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -u_r & v_r & 0 & \cdots & 0 & -\lambda^r_{r+1} & -c_{r+2}\lambda^r_{r+1} & \cdots & -c_{m-1}\lambda^r_{r+1} \\ -u_{r+1} & v_{r+1} & \lambda^2_{r+1} & \cdots & \lambda^r_{r+1} & 0 & 0 & \cdots & 0 \\ -c_{r+2}u_{r+1} & c_{r+2}v_{r+1} & c_{r+2}\lambda^2_{r+1} & \cdots & c_{r+2}\lambda^r_{r+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -c_{m-1}u_{r+1} & c_{m-1}v_{r+1} & c_{m-1}\lambda^2_{r+1} & \cdots & c_{m-1}\lambda^r_{r+1} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where v_i $(2 \le i \le r+1)$, u_j $(2 \le j \le r+1)$ and λ^k_{r+1} $(2 \le k \le r,\ \ r+1 \le b \le m-1)$ are some smooth functions of s and c_h $(r+1 \le h \le m-1)$ are constant satisfying

$$\begin{aligned} \sum_{j=2}^{r} v_j^2 = d v_{r+1}^2, \quad v_j' = -d v_{r+1}\lambda^j_{r+1} \quad \text{and} \\ \frac{w_{11}'}{w_{11}} v_{r+1}' = v_{r+1}'' + (\sum_{j=2}^{r} v_j^2) u_{r+1} - \sum_{j=2}^{r} v_j u_j v_{r+1} \end{aligned} \tag{158}$$

for some positive constant d.

For a given initial condition $X(0) = (A(0)\ B(0)\ e_2(0) \cdots e_{m-1}(0))$ satisfying $X^t(0)EX(0) = T$, there exists a unique solution to $X'(s) = X(s)M(s)$ on the whole domain I of $\tilde{\alpha}(s)$ containing 0. Since T is symmetric and MT is skew-symmetric, $\frac{d}{ds}(X^t(s)EX(s)) = 0$ and hence we have

$$X^t(s)EX(s) = T$$

for all $s \in I$. Therefore, $A(s), B(s), e_2(s), \ldots, e_{m-1}(s)$ form a null frame along a null curve $\tilde{\alpha}(s)$ in $\mathbb{L}^m$ on I. Let $\alpha(s) = \int_0^s A(u)du$.

We now give the following definition.

Definition 2. *A generalized null scroll satisfying (157) parameterized by*

$$x(s, t_1, t_2, \cdots, t_r) = \alpha(s) + t_1 B(s) + \sum_{i=2}^{r} t_i e_i(s). \tag{159}$$

is called the generalized B-scroll kind.

Remark 2. *In the case of $m = 3$ with $v_2 \in \mathbb{R}$, a generalized B-scroll kind is a so-called B-scroll.*

Therefore, we can see that the parametrization of a generalized null scroll M with a pointwise 1-type Gauss map of the second kind can be given by (159). Furthermore, by combining the first two equations of (158), we can see that these ruled submanifolds satisfy

$$v'_{r+1} = -\sum_{j=2}^{r} v_j \lambda^j_{r+1}.$$

Conversely, for a generalized B-scroll kind M parameterized by (159), by computation, ΔG can be expressed as

$$\Delta G = \frac{2w_{11}}{Q^2}(G + \mathbf{C}),$$

where $\mathbf{C}$ is the constant vector given by

$$\begin{aligned} \mathbf{C} = \frac{1}{w_{11}} \{ &\sum_{h=r+1}^{m-1} c_h v'_{r+1} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r - \sum_{j=2}^{r} v_j^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\ &+ \sum_{j=2}^{r} \sum_{h=r+1}^{m-1} v_j c_h v_{r+1} e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge A \wedge e_{j+1} \wedge \cdots \wedge e_r \}. \end{aligned}$$

It means that the Gauss map G of M is of pointwise 1-type of the second kind.

Case 6. $f_{t_1} \neq 0$.

In this case, the open subset $W = \{p \in M | f_{t_1}(p) \neq 0\}$ is non-empty. Comparing the vectors composing the constant vector $\mathbf{C}$ of (133) and (135), by the orthogonality of them, we get

$$v_j v_h = 0 \quad \text{and} \quad v_h \lambda^j_p - v_p \lambda^j_h = 0$$

on W for all $j = 2, \ldots, r$ and $h, p = r+1, \ldots, m-1$.

If $v_h = 0$ for all $h = r+1, \ldots, m-1$, we obtain the result that the open subset W of M is part of a Lorentzian $(r+1)$-plane by Lemma 8 and Theorem 3.

If $v_h \neq 0$ for some $h \in \{r+1, \ldots, m-1\}$, then $v_j = 0$ and $Q = 1$ for all $j = 2, \ldots, r$. Then, Equation (133) is simplified as

$$
\begin{aligned}
&2\sum_{h=r+1}^{m-1}\{(\sum_{i=1}^{r}\langle B', e_i'\rangle v_h t_i + v_h')\}e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ 2\sum_{h=r+1}^{m-1} v_h^2 A \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
=& f A \wedge B \wedge e_2 \wedge \cdots \wedge e_r + f\sum_{h=r+1}^{m-1}(t_1 v_h + \sum_{j=2}^{r}\lambda_h^j t_j)e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r + f\mathbf{C},
\end{aligned} \tag{160}
$$

or,

$$
\begin{aligned}
\mathbf{C} = \frac{1}{f}\{2&\sum_{h=r+1}^{m-1}\{(\sum_{i=1}^{r}\langle B', e_i'\rangle v_h t_i + v_h')\}e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r \\
&+ (2\sum_{h=r+1}^{m-1} v_h^2 - f)A \wedge B \wedge e_2 \wedge \cdots \wedge e_r\} \\
&- \sum_{h=r+1}^{m-1}(t_1 v_h + \sum_{j=2}^{r}\lambda_h^j t_j)e_h \wedge B \wedge e_2 \wedge \cdots \wedge e_r
\end{aligned} \tag{161}
$$

on W. By differentiating (160) with respect to t_1 and using (161), we can obtain

$$
\frac{2f_{t_1}}{f}\sum_{h=r+1}^{m-1} v_h^2 = 0
$$

as the coefficients of $A \wedge B \wedge e_2 \wedge \cdots \wedge e_r$. Since $f_{t_1} \neq 0$ on W, we have $\sum_h v_h^2 = 0$, a contradiction to $v_h \neq 0$ for some h.

Therefore, we can conclude that if the open set W is non-empty, then the functions v_h are identically zero on W for all $h = r+1, \ldots, m-1$, and hence we see that W is an open part of a Lorentzian plane in $\mathbb{L}^m$. By continuity, M is a Lorentzian $(r+1)$-plane.

Therefore, we have

Theorem 4. *Let M be a generalized null scroll in the Minkowski m-space $\mathbb{L}^m$. Then, M has a pointwise 1-type Gauss map of the second kind if and only if M is part of a Lorentzian $(r+1)$-plane in $\mathbb{L}^m$ or a generalized B-scroll kind.*

In particular, by straightforward computation, we have

Corollary 1. *Let M be a null scroll in the Minkowski 3-space $\mathbb{L}^3$. Then, M has a pointwise 1-type Gauss map of the second kind if and only if M is part of a time-like plane or a flat B-scroll.*

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