symmetry

Article

Symmetric Representation of Ternary Forms Associated to Some Toeplitz Matrices †

Mao-Ting Chien [1,*] and Hiroshi Nakazato [2]

1 Department of Mathematics, Soochow University, Taipei 11102, Taiwan
2 Faculty of Science and Technology, Hirosaki University, Hirosaki 036-8561, Japan; nakahr@hirosaki-u.ac.jp
* Correspondence: mtchien@scu.edu.tw; Tel.: +886-2-28819471

† Portions of this article were presented at The First International Conference on Symmetry, Barcelona, 16–18 October 2017.

Received: 5 January 2018; Accepted: 22 February 2018 ; Published: 27 February 2018

Abstract: Let A be an $n \times n$ complex matrix. Assume the determinantal curve $\mathbb{V}_A = \{[(x, y, z)] \in \mathbb{CP}^2 : F_A(x, y, z) = \det(x\Re(A) + y\Im(A) + zI_n) = 0\}$ is a rational curve. The Fiedler formula provides a complex symmetric matrix S satisfying $F_S(x, y, z) = F_A(x, y, z)$. It is also known that every Toeplitz matrix is unitarily similar to a symmetric matrix. In this paper, we investigate the unitary similarity of the symmetric matrix S and the matrix A in the Fiedler theorem for a specific parametrized family of 4×4 nilpotent Toeplitz matrices A. We show that there are either one or at least three unitarily inequivalent symmetric matrices which admit the determinantal representation of the ternary from $F_A(x, y, z)$ associated to the specific 4×4 nilpotent Toeplitz matrices.

Keywords: determinantal representation; hyperbolic ternary forms; rational curves; toeplitz matrices; numerical range

1. Introduction

Let A be an $n \times n$ complex matrix. The numerical range of A is defined as the set

$$W(A) = \{\xi^* A \xi : \xi \in \mathbb{C}^n, \xi^* \xi = 1\}.$$

The set $W(A)$ is a convex set due to the Toeplitz-Hausdorff theorem [1]. The determinantal ternary form of A is defined by

$$F_A(x, y, z) = \det(x\Re(A) + y\Im(A) + zI_n),$$

where $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$. Kippenhahn [2] showed that $W(A)$ is the convex hull of the real affine part of the dual projective curve of $F_A(x, y, z) = 0$. A real ternary form $F(x, y, z)$ is *hyperbolic* with respect to $(0, 0, 1)$ if the equation $F(x_0, y_0, z) = 0$ has only real roots in z for any $(x_0, y_0) \in \mathbb{R}^2$, and $F(0, 0, 1) = 1$. Obviously, the ternary form $F_A(x, y, z)$ is hyperbolic. An irreducible plane algebraic curve $F(x, y, z) = 0$ is called rational if it has a parametric expression

$$x = u(s), \quad y = v(s), \quad z = w(s)$$

by three polynomials $u(s), v(s), w(s)$ in one variable s. Fiedler [3] made the following conjecture:

Fiedler conjecture: Let $F(x, y, z)$ be an n degree hyperbolic ternary form with respect to $(0, 0, 1)$ and $F(0, 0, 1) = 1$. Then there exists an $n \times n$ matrix A satisfying $F(x, y, z) = F_A(x, y, z)$.

Later, Fielder [4] reformulated his conjecture in a stronger sense: there exists an $n \times n$ complex symmetric matrix S satisfying $F(x,y,z) = F_S(x,y,z)$, and proved that the conjecture is true in the case $F(x,y,z) = 0$ is a rational curve. Historically, Fielder's stronger conjecture was already raised by Lax [5], namely,

Lax conjecture: Let $F(x,y,z)$ be an n degree hyperbolic ternary form with respect to $(0,0,1)$ and $F(0,0,1) = 1$. Then there exists an $n \times n$ symmetric matrix A satisfying $F(x,y,z) = F_A(x,y,z)$.

Recently, Helton and Vinnikov [6], see also [7], confirmed the Lax conjecture is true. Therefore, the hyperbolicity of ternary forms completely characterizes the boundary of the numerical ranges of matrices based on the duality of plane algebraic curves. Plaumann et al. [8] mentioned the number of unitarily inequivalent classes of real symmetric matrices (S_1, S_2) satisfying $F_{S_1+iS_2}(x,y,z) = F(x,y,z)$ is 2^g if the curve $F(x,y,z) = 0$ has no singular points, where g is the genus of the curve. For certain irreducible curve $F(x,y,z) = 0$ of degree 4 having singular points with genus 1, it is shown in [9], see also [10], that there are infinitely many inequivalent classes of S satisfying $F_S(x,y,z) = F(x,y,z)$. Typical hyperbolic ternary forms may admit determinantal representation by special matrices. For instance, it is proved in [11] that hyperbolic ternary forms satisfying weak symmetry admit determinantal representations via cyclic weighted shift matrices for lower degrees. Lentzos and Pasley [12] solved the problem for general degrees.

Let A be an $n \times n$ Toeplitz matrix. It is known that A is unitarily similar to a complex symmetric matrix (cf. [13]). Let S be a symmetric matrix which admits the determinantal representation of the ternary $F_A(x,y,z)$, i.e., $F_S(x,y,z) = F_A(x,y,z)$. In this paper, we investigate the unitary similarity of S and A, and examine the number of unitarily inequivalent classes of the symmetric matrices for certain Toeplitz matrices.

2. Symmetric Representation

Let $\beta_1, \beta_2, \ldots, \beta_m$ be complex numbers. The $n \times n$ upper triangular nilpotent Toeplitz matrix $T(\beta_1, \beta_2, \ldots, \beta_m)$ is the one whose first row is the ordered entries $0, \beta_1, \beta_2, \ldots, \beta_{m-1}, \beta_m, \bar{\beta}_{m-1}, \ldots, \bar{\beta}_1$ for $n = 2m$, and $0, \beta_1, \beta_2, \ldots, \beta_{m-1}, \beta_m, \beta_m, \bar{\beta}_{m-1}, \ldots, \bar{\beta}_1$ for $n = 2m+1$. The c-numerical range of this class of matrices is discussed in [14], and it is also shown that the ternary form $F_T(x,y,z) = 0$ is an irreducible rational curve if the corresponding graph of T is connected. In this case, the parametrization is given by

$$x = -\cos(m\theta), y = -\sin(m\theta), z = \frac{1}{2}\beta_m + \Re\Big(\sum_{k=1}^{m-1} \beta_{m-k} \exp(-i\,k\theta)\Big)$$

if $n = 2m$, and

$$x = -\cos(m\theta), y = -\sin(m\theta), z = \Re\Big(\sum_{k=1}^{m} \beta_{m+1-k} \exp(-i\,(2k-1)\theta)\Big)$$

if $n = 2m+1$. Changing the variable $s = \tan(\theta/2)$, the above parametrization coordinates x, y, z can be represented by rational functions in s (cf. [14], Theorem 3.1).

Let α, β be two real numbers. Assume $\beta \neq 0$. The 4×4 upper triangular nilpotent Toeplitz matrix

$$T(\beta, \alpha) = \begin{pmatrix} 0 & \beta & \alpha & \beta \\ 0 & 0 & \beta & \alpha \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{\beta}{\sqrt{2}} T(\sqrt{2}, \frac{\sqrt{2}}{\beta}\alpha),$$

hence we may assume for computation simplicity, a standard form of 4×4 upper triangular nilpotent Toeplitz matrices $T(\beta, \alpha)$ is the form $T(\sqrt{2}, a)$ for some a. The following preliminary lemma is essential to the study of the ternary from associated to certain Toeplitz matrices.

Lemma 1. *Let $A = T(\sqrt{2}, a)$ be a 4×4 upper triangular nilpotent Toeplitz matrix, a is a real number and $a \neq -2, 0, 2$. Then*

(i) The ternary form $F_A(x, y, z)$ is irreducible and the algebraic curve $F_A(x, y, z) = 0$ is rational;
(ii) The roots of $F_A(0, -1, z) = 0$ are $(a+2)/2, (a-2)/2, -(a-2)/2, -(a+2)/2$;
(iii) If a real symmetric matrix $C = (c_{jk})$ satisfies $F_{C/2+iB/2}(x, y, z) = F_A(x, y, z)$ then

(a) $c_{11} = 1, c_{22} = -1, c_{33} = -1, c_{44} = 1$;
(b) The entries c_{jk}, $1 \leq j < k \leq 4$ are solutions of the system of equations:

$$
\begin{aligned}
P_1 &= c_{12}^2 + c_{34}^2 + c_{13}^2 + c_{24}^2 + c_{14}^2 + c_{23}^2 - (2a^2 + 6) = 0, \\
P_2 &= a(c_{12}^2 - c_{34}^2) + 2(c_{13}^2 - c_{24}^2) = 0, \\
P_3 &= (4 - a^2)(c_{12}^2 + c_{34}^2) + (a^2 - 4)(c_{13}^2 + c_{24}^2) + (a-2)^2 c_{14}^2 \\
&\quad + (a+2)^2 c_{23}^2 - (2a^4 - 14a^2 + 24) = 0, \\
P_4 &= c_{12}(c_{13}c_{23} + c_{14}c_{24}) + c_{34}(c_{13}c_{14} + c_{23}c_{24}) + c_{14}^2 - c_{23}^2 - 8a = 0, \\
P_5 &= c_{12}\Big(-(a+2)c_{13}c_{23} + (2-a)c_{14}c_{24} \Big) + c_{34}\Big((a-2)c_{13}c_{14} \\
&\quad + (2+a)c_{23}c_{24} \Big) - 2c_{12}^2 + 2c_{34}^2 - ac_{13}^2 + ac_{24}^2 = 0, \\
P_6 &= c_{12}^2 c_{34}^2 + c_{13}^2 c_{24}^2 + c_{14}^2 c_{23}^2 - 2c_{12}c_{34}(c_{14}c_{23} + c_{13}c_{24}) \\
&\quad - 2c_{13}c_{24}c_{14}c_{23} + 2c_{12}(c_{13}c_{23} - c_{14}c_{24}) + 2c_{34}(c_{23}c_{24} - c_{13}c_{14}) \\
&\quad + c_{12}^2 + c_{34}^2 - c_{14}^2 - c_{23}^2 + c_{13}^2 + c_{24}^2 - a^4 + 8a^2 + 1 = 0,
\end{aligned}
$$

and $B = \mathrm{diag}(a+2, a-2, -a+2, -a-2)$.

Proof of Lemma 1. It is clear that the graph of A is connected. Then, by ([14], Theorem 3.2), the form $F_A(x, y, z)$ is irreducible and the algebraic curve $F_A(x, y, z) = 0$ is rational.

Observe that

$$
-\mathrm{diag}(1, -1, 1, -1)T(\sqrt{2}, a)\mathrm{diag}(1, -1, 1, -1) = T(\sqrt{2}, -a),
$$

the matrix $T(\sqrt{2}, -a)$ is unitarily similar to $-T(\sqrt{2}, a)$. Hence, we may assume that $0 \leq a < \infty$. We compute and find that

$$
\begin{aligned}
16F_A(x, y, z) = 16z^4 - 8(a^2 + 4)(x^2 + y^2)z^2 + 32ax(x^2 + y^2)z - a^2(8 - a^2)x^4 \\
+ (2a^4 - 16a^2 + 16)x^2y^2 + (a^4 - 8a^2 + 16)y^4. \quad (1)
\end{aligned}
$$

Then the equation $F_A(0, -1, z) = 16^4 - 8(a^2 + 4)z^2 + a^4 - 8a^2 + 16 = 0$ has four real roots $(a+2)/2, (a-2)/2, -(a-2)/2, -(a+2)/2$ which are mutually distinct if $0 \leq a \neq 2 < \infty$. Denote $\beta_1 = a + 2, \beta_2 = a - 2, \beta_3 = -a + 2, \beta_4 = -a - 2$, and $B = \mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$.

Suppose real symmetric matrix $C = (c_{jk})$ satisfies

$$
F_{C/2+iB/2}(x, y, z) = F_A(x, y, z). \quad (2)
$$

Then, by Fiedler formula [4],

$$
\frac{1}{2}c_{jj} = \beta_j \frac{\frac{\partial}{\partial x}F_A(0, -1, \beta_j/2)}{\frac{\partial}{\partial y}F_A(0, -1, \beta_j/2)},
$$

we obtain that the diagonal entries (c_{jj}) of C are given by

$$
c_{11} = c_{44} = 1, \quad c_{22} = c_{33} = -1
$$

which are independent of the parameter a. Comparing both sides of Equation (2), the off-diagonal entries c_{12}, c_{23}, c_{34}, c_{13}, c_{24}, c_{14} of C satisfy the system of equations $P_1 = P_2 = \cdots = P_6 = 0$ in (b) of (iii). $\square$

We conjecture: Let $T(b,a)$ be a 4×4 upper triangular nilpotent Toeplitz matrix, and let $n(T(b,a))$ be the number of unitarily inequivalent complex symmetric matrices S satisfying $F_S(x,y,z) = F_{T(b,a)}(x,y,z)$. Then $n(T(b,a)) = 1$ if $a \leq \sqrt{2}b$ and $n(T(b,a)) = 3$ if $\sqrt{2}b < a$.

For computation simplicity, we may assume $b = \sqrt{2}$ in the conjecture. The conjecture becomes $n(T(\sqrt{2},a)) = 1$ if $a \leq 2$ and $n(T(\sqrt{2},a)) = 3$ if $2 < a$. We verify $n(T(\sqrt{2},a)) = 1$ when $a = 1,2$. This means that the symmetric determinantal representation matrix S for the ternary for $F_{T(\sqrt{2},1)}(x,y,z)$ is unique up to unitary equivalence.

Theorem 1. *Let $A = T(\sqrt{2},1)$ be a 4×4 upper triangular nilpotent Toeplitz matrix, and let $B = \mathrm{diag}(\beta_1,\beta_2,\beta_3,\beta_4)$ be the diagonal matrix with diagonal entries consisting of eigenvalues of $\Im(A)$. Then the real symmetric matrix $C = (c_{ij})$ satisfying $F_{C+iB}(x,y,z) = F_A(x,y,z)$ is unique up to diagonal unitary similarity via $\mathrm{diag}(1,\epsilon_2,\epsilon_3,\epsilon_4)$, where $\epsilon_2,\epsilon_3,\epsilon_4 \in \{+1,-1\}$.*

Proof of Theorem 2. Assume $a = 1$ in $T(\sqrt{2},a)$. The ternary (1) becomes

$$16F_A(x,y,z) = 16z^4 - 40(x^2+y^2)z^2 + 32x(x^2+y^2)z - 7x^4 + 2x^2y^2 + 9y^4,$$

and the four roots of $F_A(0,-1,z) = 0$ are $3/2, 1/2, -1/2, -3/2$. Let

$$B = \mathrm{diag}(3/2, 1/2, -1/2, -3/2).$$

Assume that a real symmetric matrix $C = (c_{jk})$ admits the determinantal representation of the ternary $F_A(x,y,z)$, i.e.,

$$\det(xC + yB + zI_4) = F_A(x,y,z).$$

Then

$$c_{11} = c_{44} = 1/2, \quad c_{22} = c_{33} = -1/2.$$

Further, the off-diagonal entries c_{jk}, $1 \leq j,k \leq 4$ satisfy 6 simultaneous equations $P_1 = P_2 = \cdots = P_6 = 0$ in (b) of (iii).

One real solution of these six simultaneous equations is given by

$$c_{13} = c_{24} = \frac{1}{2}, \quad c_{12} = c_{23} = c_{34} = \frac{1}{2\sqrt{2}}, \quad c_{14} = \frac{3}{2\sqrt{2}}.$$

We claim that all real solutions of the system $P_1 = P_2 = \cdots = P_6 = 0$ are given by

$$(c_{12}, c_{23}, c_{34}, c_{13}, c_{24}, c_{14}) = (\frac{\epsilon_2}{2\sqrt{2}}, \frac{\epsilon_2\epsilon_3}{2\sqrt{2}}, \frac{\epsilon_3\epsilon_4}{2\sqrt{2}}, \frac{\epsilon_3}{2}, \frac{\epsilon_2\epsilon_4}{2}, \frac{3\epsilon_4}{2\sqrt{2}}), \tag{3}$$

for some $(\epsilon_2,\epsilon_3,\epsilon_4) \in \{+1,-1\}$.

To express the real solutions by rational numbers, we change the variables:

$$c_{13} = \frac{1}{2}C_{13}, c_{24} = \frac{1}{2}C_{24}, c_{12} = \frac{1}{2\sqrt{2}}C_{12}, c_{23} = \frac{1}{2\sqrt{2}}C_{23},$$

$$c_{34} = \frac{1}{2\sqrt{2}}C_{34}, c_{14} = \frac{3}{2\sqrt{2}}C_{14}.$$

Then the equations $P_j = 0$ are rewritten as

$$
\begin{aligned}
G_1 &= C_{12}^2 + C_{23}^2 + C_{34}^2 + 2C_{13}^2 + 2C_{24}^2 + 9C_{14}^2 - 16 = 0, \\
G_2 &= C_{12}^2 - C_{34}^2 + C_{13}^2 - C_{24}^2 = 0, \\
G_3 &= 3C_{14}^2 + 3C_{23}^2 + 2C_{13}^2 + 2C_{24}^2 - C_{12}^2 - C_{34}\,2 - 8 = 0, \\
G_4 &= -4C_{13}^2 + 4C_{24}^2 - C_{12}^2 + C_{34}^2 - 3C_{12}C_{13}C_{23} - 3C_{12}C_{14}C_{24} \\
&\quad + 3C_{13}C_{14}C_{34} + 3C_{23}C_{24}C_{34} = 0, \\
G_5 &= 9C_{14}^2 - C_{23}^2 + 3C_{12}C_{14}C_{24} + 3C_{13}C_{14}C_{34} + C_{12}C_{13}C_{23} \\
&\quad + C_{23}C_{24}C_{34} - 16 = 0, \\
G_6 &= -18C_{14}^2 + 4C_{13}^2 + 4C_{24}^2 + 2C_{12}^2 - 2C_{23}^2 + 2C_{24}^2 + 2C_{34}^2 + 4C_{12}C_{13}C_{23} \\
&\quad + 4C_{23}C_{24}C_{34} - 12C_{12}C_{14}C_{24} - 12C_{13}C_{14}C_{34} + 9C_{14}^2C_{23}^2 + 4C_{13}^2C_{24}^2 \\
&\quad + C_{12}^2C_{34}^2 - 12C_{13}C_{14}C_{23}C_{24} - 6C_{12}C_{14}C_{23}C_{34} \\
&\quad - 4C_{12}C_{13}C_{24}C_{34} + 32 = 0.
\end{aligned}
$$

Here, we apply Gröbner basis method for solving system of polynomial equations. The Mathematica function GroebnerBasis efficiently calculates the Gröbner basis for a list of polynomials. (For reference on the applications of Gröbner basis to solve systems of polynomial equations, see, for instance, [15].) Using Gröbner basis computation for the 6 polynomials $G_1, G_2, \ldots, G_6$, we eliminate the variables $C_{13}, C_{14}, C_{23}, C_{24}, C_{34}$, and obtain an equation $P_{12}(C_{12}) = 0$ in C_{12}. We factorize P_{12} in the polynomial ring $\mathbb{Z}[C_{13}]$ and abandon the factors which have only imaginary roots. We replace P_{12} by its factor related to real roots. Next, we eliminate $C_{12}, C_{14}, C_{23}, C_{24}, C_{34}$ from the seven polynomials $G_1, G_2, \ldots, G_6, P_{12}$ and get an equation $P_{13}(C_{13}) = 0$.

Again, we abandon the factors related to only imaginary roots. We continue this process to arrive at the step at which the above process does not have imaginary roots. At the final step, we have that

$$
\begin{aligned}
&P_{12} = (C_{12} - 1)(C_{12} + 1)(3C_{12}^4 - 4) = 0, \\
&P_{13} = (C_{13} - 1)(C_{13} + 1)(3C_{13}^4 - 6C_{13}^2 - 1) = 0, \\
&P_{14} = (C_{14} - 1)(C_{14} + 1)(3C_{14}^2 - 4) = 0, \\
&P_{23} = C_{23}(C_{23} - 1)(C_{23} + 1) = 0, \\
&P_{24} = (C_{24} - 1)(C_{24} + 1)(3C_{24}^4 - 6C_{24}^2 - 1) = 0, \\
&P_{34} = (C_{34} - 1)(C_{34} + 1)(3C_{34}^4 - 4) = 0.
\end{aligned}
$$

By eliminating $C_{14}, C_{23}, C_{24}, C_{34}$, we produce the Gröbner basis for the elimination ideal of $\{G_1, G_2, \ldots, G_6, \;\; P_{12}, \ldots, P_{14}\}$ with respect to C_{13}, C_{12}. It consists of

$$\{(C_{12} - 1)(C_{12} + 1)(3C_{12}^4 - 4), C_{13}^2 + 3C_{12}^4 + C_{12}^2 - 5\}.$$

Thus, if $(C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34})$ is a real solution of the equations $G_1 = G_2 = \cdots = G_6 = 0$, then

$$(C_{12} - 1)(C_{12} + 1)(3C_{12}^4 - 4) = 0 \text{ and } C_{13}^2 + 3C_{12}^4 + C_{12}^2 - 5 = 0.$$

Suppose $3C_{12}^4 - 4 = 0$. Then $C_{12}^2 = 2/\sqrt{3}$, and hence $3C_{13}^2 + 2\sqrt{3} - 3 = 0$, which is impossible for a real number C_{13}. This implies that $C_{12}^2 = 1$. We set

$$\tilde{P}_{12} = (C_{12} - 1)(C_{12} + 1).$$

We eliminate $C_{12}, C_{14}, C_{23}, C_{24}$, and produce the Gröbner basis for $\{G_1, G_2, \ldots, G_6, \tilde{P}_{12}, P_{13}, \ldots, P_{34}\}$. The basis consists of

$$\{(C_{34}-1)(C_{34}+1), (C_{13}-1)(C_{13}+1)\}.$$

Thus, any real solution $(C_{12}, C_{13}, \ldots)$ of $G_1 = G_2 = \cdots = G_6 = 0$ satisfies $C_{34}^2 = 1$ and $C_{13}^2 = 1$. Continuing similar arguments, we conclude that any real solution $(C_{12}, \ldots, C_{34})$ satisfies

$$C_{12}^2 = C_{13}^2 = C_{14}^2 = C_{23}^2 = C_{24}^2 = C_{34}^2 = 1.$$

This proves that the real vectors $(C_{12}, \ldots, C_{34})$ satisfying $G_1 = G_2 = \cdots = G_6 = 0$ are necessarily of the form

$$(C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34}) = (\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7)$$

for some $\epsilon_2, \ldots, \epsilon_7 \in \{+1, -1\}$. There are 64 possible vectors of the form. By direct computations, we find among them the solutions of the equations $G_1 = G_2 = \cdots = G_6 = 0$ are the following eight vectors:

$$(C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, C_{34}) = (\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_2\epsilon_3, \epsilon_2\epsilon_4, \epsilon_3\epsilon_4),$$

where $\epsilon_2, \epsilon_3, \epsilon_4 \in \{+1, -1\}$. $\square$

For any $0 \le a < \infty$, there exists a particular solution for the system of equations $P_1 = P_2 = \cdots = P_6 = 0$ in , namely,

$$c_{12} = c_{34} = 1, \quad c_{13} = c_{24} = \frac{a}{\sqrt{2}}, \quad c_{14} = \frac{a+2}{\sqrt{2}}, \quad c_{23} = \frac{-a+2}{\sqrt{2}}. \tag{4}$$

To find other real solutions for the system of equations $P_1 = \cdots = P_6 = 0$, $2 \le a < \infty$, we introduce an analytic function $c_{14} = c_{14}(a)$ on the interval $2 \le a < \infty$ by defining

$$c_{14} = -\frac{1}{\sqrt{2}a}\sqrt{a^4 - 2a^2 + 16a + 24 - 4\sqrt{(a+2)^3(a-2)}\sqrt{a^2-2}} \tag{5}$$

for $2 \le a \le \alpha$, and

$$c_{14} = \frac{1}{\sqrt{2}a}\sqrt{a^4 - 2a^2 + 16a + 24 - 4\sqrt{(a+2)^3(a-2)}\sqrt{a^2-2}}$$

for $\alpha \le a < \infty$, where the constant α is defined by

$$\alpha = \frac{2}{3} + \frac{1}{3}\left((116 + 6\sqrt{78})^{1/3} + (116 - 6\sqrt{78})^{1/3}\right)$$

which is numerically approximated by 3.83598. We find that $c_{14}(2) = c_{14}(\alpha) = 0$. Then, for $2 \le a < \infty$, the solutions of the remaining entries of the system of equations are given by

$$c_{12} = c_{34} = -\frac{1}{a}\sqrt{5a^2 - 12 + 4\sqrt{a^2-2}\sqrt{a^2-4}}, \tag{6}$$

$$c_{13} = c_{24} = \frac{1}{\sqrt{2}}\sqrt{a^2-2}, \tag{7}$$

$$c_{23} = -\frac{1}{\sqrt{2}a}\sqrt{a^4 - 2a^2 - 16a + 24 + 4\sqrt{(a-2)^3(a+2)}\sqrt{a^2-2}}, \tag{8}$$

and the real conjugates

$$c_{12} = c_{34} = -\frac{1}{a}\sqrt{5a^2 - 12 - 4\sqrt{a^2-2}\sqrt{a^2-4}}, \tag{9}$$

$$c_{13} = c_{24} = \frac{1}{\sqrt{2}}\sqrt{a^2-2}, \tag{10}$$

$$c_{14} = -\frac{1}{\sqrt{2}a}\sqrt{a^4 - 2a^2 + 16a + 24 + 4\sqrt{(a+2)^3(a-2)}\sqrt{a^2-2}}, \tag{11}$$

$$c_{23} = \frac{1}{\sqrt{2}a}\sqrt{a^4 - 2a^2 - 16a + 24 - 4\sqrt{(a-2)^3(a+2)}\sqrt{a^2-2}}. \tag{12}$$

It is not so hard to find the analytic functions given by (5)–(8) or the analytic functions given by (9)–(12) satisfying six simultaneous equations $P_j = 0$ in Lemma 1. The presentation of these functions here is rather a priori. In the proof of Theorem 4, we outline the process to determine some particular solutions for the system of equations $P_1 = P_2 = \cdots = P_6 = 0$ in Lemma 1.

In the case $a = 2$. One particular solution (4) is given by

$$c_{11} = c_{44} = 1, c_{12} = c_{34} = 1, c_{13} = c_{24} = \sqrt{2}, c_{22} = c_{33} = -1, c_{14} = 2\sqrt{2}. \tag{13}$$

The other solutions (5)–(8) and (9)–(11) are all the same as

$$c_{11} = c_{44} = 1, c_{12} = c_{34} = -\sqrt{2}, c_{13} = c_{24} = 1, c_{22} = c_{33} = -1, c_{14} = -2\sqrt{2}. \tag{14}$$

Thus, the matrix $S_1 = C + iB$ corresponding to the solution (13) is given by

$$S_1 = \begin{pmatrix} 1+4i & 1 & \sqrt{2} & 2\sqrt{2} \\ 1 & -1 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & -1 & 1 \\ 2\sqrt{2} & \sqrt{2} & 1 & 1-4i \end{pmatrix},$$

which is permutationally similar to

$$L = \begin{pmatrix} 1+4i & \sqrt{2} & 1 & 2\sqrt{2} \\ \sqrt{2} & -1 & 0 & 1 \\ 1 & 0 & -1 & \sqrt{2} \\ 2\sqrt{2} & 1 & \sqrt{2} & 1-4i \end{pmatrix}.$$

The matrix $S_2 = C + iB$ corresponding to the solution (14) is given by

$$S_2 = \begin{pmatrix} 1+4i & -\sqrt{2} & 1 & -2\sqrt{2} \\ -\sqrt{2} & -1 & 0 & 1 \\ 1 & 0 & -1 & -\sqrt{2} \\ -2\sqrt{2} & 1 & -\sqrt{2} & 1-4i \end{pmatrix},$$

which satisfies

$$\mathrm{diag}(1,-1,1,-1)S_2\mathrm{diag}(1,-1,1,-1) = L.$$

Hence, the two complex symmetric matrices S for which $F_S(x,y,z) = F_{T(\sqrt{2},1)}(x,y,z)$ are unitarily similar. The following result can be obtained by following the argument similar to that used in Theorem 2.

Theorem 2. *Let $A = T(\sqrt{2}, 2)$ be a 4×4 upper triangular nilpotent Toeplitz matrix. Then the complex symmetric matrix S satisfying $F_S(x, y, z) = F_A(x, y, z)$ is unique up to the diagonal unitary similarity.*

Next, we deal with the case $a = 3$ in the 4×4 upper triangular nilpotent Toeplitz matrix $A = T(\sqrt{2}, a)$. In this situation, the complex symmetric matrices admitting the ternary form $F_A(x, y, z)$ are not unique up to unitary equivalence. Indeed, we show $n(T(\sqrt{2}, 3)) \geq 3$.

Theorem 3. *Let $A = T(\sqrt{2}, 3)$ be a 4×4 upper triangular nilpotent Toeplitz matrix. Then there exist at least three unitarily inequivalent complex symmetric matrices S such that $F_S(x, y, z) = F_A(x, y, z)$.*

Proof of Theorem 4. Let $C = (c_{ij})$ be a real symmetric matrix and $B = \mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$ be the diagonal matrix with diagonal entries consisting of eigenvalues of $\Im(A)$ satisfying

$$\det(xC + yB + zI_4) = \det(x\Re(A) + y\Im(A) + zI_4).$$

Suppose that $c_{12} = c_{34}$ and $c_{13} = c_{24}$. Then the equations $P_2 = 0$ and $P_5 = 0$ in Lemma 1 hold.

To find the first solution of the system of equations in (iii) of Lemma 1, we assume that $c_{12} = c_{34} = 1$. By changing the variables

$$c_{13} = \frac{3}{\sqrt{2}}v_{13}, c_{23} = -\frac{1}{\sqrt{2}}v_{23}, c_{14} = \frac{5}{\sqrt{2}}v_{14},$$

the equations $P_1 = 0, \ldots, P_6 = 0$ are expressed as

$$\begin{aligned}
\tilde{P}_1 &= 18v_{13}^2 + 25v_{14}^2 + v_{23}^2 - 44 = 0, \\
\tilde{P}_3 &= 18v_{13}^2 + 5v_{14}^2 + 5v_{23}^2 - 28 = 0, \\
\tilde{P}_4 &= 25v_{14}^2 - v_{23}^2 + 30v_{13}v_{14} - 6v_{13}v_{23} - 48 = 0, \\
\tilde{P}_6 &= 81v_{13}^4 + 90v_{13}^2v_{14}v_{23} + 25v_{14}^2v_{23}^2 - 50v_{14}^2 - 2v_{23}^2 - 120v_{13}v_{14} \\
&\quad - 24v_{13}v_{23} + 20v_{14}v_{23} - 20 = 0.
\end{aligned}$$

It is easy to see that the condition $v_{13} = v_{14} = v_{23} = 1$ satisfies the above four equations. Hence, we obtain the first real solution

$$(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}) = (1, \frac{3}{\sqrt{2}}, \frac{5}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 1). \tag{15}$$

Next, we find the second and third real solutions of the four equations $P_1 = P_3 = P_4 = P_6 = 0$ under the assumption that $c_{34} = c_{12}$ and $c_{24} = c_{13}$. We choose $c_{13} = \sqrt{7/2}$, and change the variable $c_{12} = \sqrt{14}d_{12}$. The four equations are rewritten as

$$\begin{aligned}
P_1 &= 28d_{12}^2 + c_{14}^2 + c_{23}^2 - 17 = 0, \\
P_3 &= -140d_{12}^2 + c_{14}^2 + 25c_{23}^2 - 25 = 0, \\
P_4 &= 14d_{12}c_{14} + 14d_{12}c_{23} + c_{14}^2 - c_{23}^2 - 24 = 0, \\
P_6 &= 784d_{12}^4 - 112d_{12}^2c_{14}c_{23} + 4c_{14}^2c_{23}^2 - 280d_{12}^2 - 112d_{12}c_{14} + 112d_{12}c_{23} \\
&\quad - 28c_{14}c_{23} - 4c_{14}^2 - 4c_{23}^2 + 45 = 0.
\end{aligned}$$

Using Gröbner basis computation, we eliminate c_{14}, c_{23} from the equations $P_1 = P_3 = P_4 = P_6 = 0$, and obtain that

$$P_7 = 15876d_{12}^4 - 8316d_{12}^2 + 529 = 0.$$

Similarly, we get

$$P_8 = 324c_{14}^4 - 4860c_{14}^2 + 4225 = 0,$$

and

$$P_9 = 324c_{23}^4 - 1404c_{23}^2 + 961 = 0.$$

Now, we compute the Gröbner basis for the polynomials $P_1, P_3, P_4, P_6, P_7, P_8, P_9$ with respect to some order of the variables d_{12}, c_{14}, c_{23}. The basis is given by P_7 and

$$P_{10} = 23c_{14} + 756d_{12}^3 - 373d_{12}, \quad P_{11} = 23c_{23} - 378d_{12}^3 + 106d_{12}.$$

Substituting

$$c_{14} = \frac{1}{23}(-756d_{12}^3 + 373d_{12}), \tag{16}$$

$$c_{23} = \frac{2}{23}(189d_{12}^3 - 53d_{12}), \tag{17}$$

into the equations $P_1 = 0, P_3 = 0, P_4 = 0, P_6 = 0$, these four equations are rewritten as

$$\begin{aligned}
P_1 &= (15876d_{12}^4 - 8316d_{12}^2 + 529)(45d_{12}^2 - 17) = 0,\\
P_3 &= (15876d_{12}^4 - 8316d_{12}^2 + 529)(261d_{12}^2 - 25) = 0,\\
P_4 &= (15876d_{12}^4 - 8316d_{12}^2 + 529)(9d_{12}^2 - 8) = 0,\\
P_6 &= (15876d_{12}^4 - 8316d_{12}^2 + 529)(9d_{12}^2 - 8)(2286144d_{12}^8 - 2340576d_{12}^6\\
&\quad + 817812d_{12}^4 - 55800d_{12} + 2645) = 0.
\end{aligned}$$

Observe that these four equations have common real solutions

$$d_{12} = \pm 1/(3\sqrt{14})\sqrt{33 + 4\sqrt{35}}, \quad d_{12} = \pm 1/(3\sqrt{14})\sqrt{33 - 4\sqrt{35}},$$

which correspond to

$$c_{12} = \pm\sqrt{\frac{11}{3} + \frac{4}{9}\sqrt{35}}, \quad c_{12} = \pm\sqrt{\frac{11}{3} - \frac{4}{9}\sqrt{35}}.$$

Together with (16), (17), we obtain respectively two real solutions (c_{jk}) for the six equations $P_1 = P_2 = \cdots = P_6 = 0$ which are given by

$$(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34})$$

$$= (-\sqrt{\frac{11}{3} + \frac{4}{9}\sqrt{35}}, \sqrt{7/2}, -\sqrt{\frac{15}{2} - \frac{10}{9}\sqrt{35}}, -\sqrt{\frac{13}{6} + \frac{2}{9}\sqrt{35}}, \sqrt{7/2}, -\sqrt{\frac{11}{3} + \frac{4}{9}\sqrt{35}}) \tag{18}$$

and

$$(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34})$$

$$= (-\sqrt{\frac{11}{3} - \frac{4}{9}\sqrt{35}}, \sqrt{7/2}, -\sqrt{\frac{15}{2} + \frac{10}{9}\sqrt{35}}, \sqrt{7/2}, \sqrt{\frac{13}{6} - \frac{2}{9}\sqrt{35}}, -\sqrt{\frac{11}{3} - \frac{4}{9}\sqrt{35}}). \tag{19}$$

Numerically, we have respectively

$$(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}) \sim (-2.509, 1.871, -0.963, -1.866, 1.871, -2.509),$$

and

$$(c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}) \sim (-1.018, 1.871, -3.751, 0.923, 1.871, -1.018).$$

The three constructed real symmetric matrices C with entries (15), (18), (19) are not diagonally unitarily similar each other. Therefore, the number of unitarily inequivalent complex symmetric matrices $S + C + iB$ satisfying $F_S(x, y, z) = F_A(x, y, z)$ is at least three. $\square$

So far, the authors of this paper are not able to prove that the 6 simultaneous equations $P_j = 0$ have no inequivalent solutions other than the solutions satisfying $c_{12} = c_{34}, c_{13} = c_{24}$. In the case, $c_{12} = c_{34}$, $c_{13} = c_{24}$, the proof of Theorem 4 asserts that there are only three inequivalent real solutions.

3. Unitary Similarity

It is known that every Toeplitz matrix is unitarily similar to a complex symmetric matrix (cf [13]). Let A be an $n \times n$ Toeplitz matrix. Consider the hyperbolic ternary form $F_A(x, y, z)$, the affirmation of the Lax conjecture asserts that there exists an $n \times n$ complex symmetric matrix S such that $F_S(x, y, z) = F_A(x, y, z)$. It is interesting to ask if A and S are unitarily similar.

As an immediate consequence of Theorems 2 and 3, we have the following positive answer for some Toeplitz matrices.

Theorem 4. *Let $A = T(\sqrt{2}, a)$ be a 4×4 upper triangular nilpotent Toeplitz matrix, and let $B = \mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$ be the diagonal matrix with diagonal entries consisting of eigenvalues of $\Im(A)$. Then, for $a = 1, 2$, A is unitarily similar to the symmetric matrix $S = C + iB$, where C is a real symmetric matrix satisfying $\det(xC + yB + zI_4) = \det(x\Re(A) + y\Im(A) + zI_4)$.*

Proof of Theorem 5. It is proved in [13] that every Toeplitz matrix is unitarily similar to a complex symmetric matrix. The ternary form $F_A(x, y, z)$ is invariant under unitary similarity. The uniqueness of the symmetric matrix in Theorems 2 and 3 for the ternary $F_A(x, y, z)$ asserts the conclusion. □

Let $F(x, y, z)$ be hyperbolic ternary form. Suppose that the form $F(x, y, z)$ is irreducible and the curve $F(x, y, z) = 0$ is rational. Fiedler [4] constructed a symmetric matrix S which admits the determinantal representation of $F_A(x, y, z)$. We formulate the result of Fiedler construction.

Theorem 5. *(cf. [4]) Let $F(x, y, z)$ be a degree n real ternary form hyperbolic with respect to $(0, 0, 1)$ and $F(0, 0, 1) = 1$. Suppose that the form $F(x, y, z)$ is irreducible in the polynomial ring $\mathbb{C}[x, y, z]$, and defines a rational curve $F(x, y, z) = 0$ parametrized by three real polynomials $x = u(s), y = v(s), z = w(s)$ in one variable s. Further, assume that the curve $F(x, y, z) = 0$ and the line $x = 0$ intersect at distinct n real points $P_j = (x, y, z) = (0, -1, \beta_j)$, with $\beta_j \neq 0$, $j = 1, 2, \ldots, n$. Then a complex symmetric matrix $S = C + i\,\mathrm{diag}(\beta_1, \ldots, \beta_n)$ satisfying the relation $F_S(x, y, z) = F(x, y, z)$ is given by a real symmetric matrix $C = (c_{jk})$ determined by*

$$c_{jj} = \beta_j \frac{F_x(0, -1, \beta_j)}{F_y(0, -1, \beta_j)},$$

and

$$c_{jk} = -\epsilon \frac{1}{Q_j - Q_k} \left(\frac{w(Q_j)}{v(Q_j)} - \frac{w(Q_k)}{v(Q_k)} \right) \sqrt{\frac{v(Q_j)v(Q_k)}{u'(Q_j)u'(Q_k)}},$$

$j = 1, 2, \ldots, n$, $j \neq k$, where $s = Q_j$ is the point in the parameter s-space corresponding to P_j, that is,

$$u(Q_j) = 0, \quad \frac{w(Q_j)}{v(Q_j)} = -\beta_j,$$

the polynomial $u'(s)$ is the derivative of $u(s)$ with respect to s, and $\epsilon \in \{+1, -1\}$ satisfies $\epsilon u'(Q_j)v(Q_j) > 0$ for all j.

Let $T(\sqrt{2}, a)$ be a 4×4 upper triangular nilpotent Toeplitz matrix. For $a = 0$, the Hermitian matrix $\Im(T(\sqrt{2}, 0))$ has multiple eigenvalues. We modify the Toeplitz matrix and consider

$$A = (1 + i)\, T(\sqrt{2}, 0) = (1 + i)\sqrt{2}\, T(1, 0).$$

In the following, we show that the symmetric matrix S for the ternary form $F_A(x,,y,z)$ constructed by the Fiedler formula, Theorem 6, is unitarily similar to A.

Theorem 6. *Let $A = (1+i)\sqrt{2}\,T(1,0)$ be a 4×4 upper triangular nilpotent Toeplitz matrix. Then the symmetric representation S for the ternary form $F_A(x,y,z)$ constructed by the Fiedler formula is unitarily similar to A.*

Proof of Theorem 6. Observe that the form F_A satisfies

$$\begin{aligned}&F_A(-\cos(\theta+\pi/4), -\sin(\theta+\pi/4), z)\\ &= (z-2\cos(\theta/2))(z+2\cos(\theta/2))(z+2\sin(\theta/2))(z-2\sin(\theta/2))\end{aligned}$$

for any angle $0 \le \theta \le 2\pi$. By introducing the parameter $s = \tan(\theta/4)$, the curve $F_A(x,y,z) = 0$ is parametrized by

$$\begin{aligned}x &= u(s) = s^4 + 4s^3 - 6s^2 - 4s + 1 = 0,\\ y &= v(s) = u(-s) = s^4 - 4s^3 - 6s^2 + 4s + 1,\\ z &= w(s) = 2\sqrt{2}(s^4 - 1).\end{aligned}$$

The intersection points of the curve $F_A(x,y,z) = 0$ and the line $x = 0$ are $P_j = (0,-1,\beta_j)$, $j = 1,2,3,4$, where

$$\beta_1 = -\sqrt{2+\sqrt{2}} < \beta_2 = -\sqrt{2-\sqrt{2}} < \beta_3 = \sqrt{2-\sqrt{2}} < \beta_4 = \sqrt{2+\sqrt{2}}.$$

The corresponding Q_j in Theorem 6 of the intersection points P_j are computed by

$$Q_1 = -1-\sqrt{2}-\sqrt{2}\sqrt{2+\sqrt{2}}, \quad Q_2 = -1+\sqrt{2}-\sqrt{2}\sqrt{2-\sqrt{2}},$$

$$Q_3 = -1+\sqrt{2}+\sqrt{2}\sqrt{2-\sqrt{2}}, \quad Q_4 = -1-\sqrt{2}+\sqrt{2}\sqrt{2+\sqrt{2}}.$$

Now, applying the Fiedler formula of Theorem 6, the real symmetric matrix C are given by

$$c_{11} = -\frac{1}{\sqrt{4+2\sqrt{2}}}, \quad c_{22} = \frac{1}{\sqrt{4-2\sqrt{2}}}$$

$$c_{33} = -\frac{1}{\sqrt{4-2\sqrt{2}}}, \quad c_{44} = \frac{1}{\sqrt{4+2\sqrt{2}}},$$

and

$$c_{12} = c_{23} = -c_{34} = -\frac{1}{2}\sqrt{2-\sqrt{2}},$$

$$c_{13} = c_{14} = c_{24} = -\frac{1}{2}\sqrt{2+\sqrt{2}}.$$

Define matrix $K = \frac{-2}{\sqrt{2+\sqrt{2}}}(C + i\mathrm{diag}(\beta_1,\beta_2,\beta_3,\beta_4))$ which is given by

$$K = \begin{pmatrix} \sqrt{2}-1+2i & \sqrt{2}-1 & 1 & 1\\ \sqrt{2}-1 & -1+(2\sqrt{2}-2)i & \sqrt{2}-1 & 1\\ 1 & \sqrt{2}-1 & 1-(2\sqrt{2}-2)i & -\sqrt{2}+1\\ 1 & 1 & -\sqrt{2}+1 & -\sqrt{2}+1-2i \end{pmatrix}.$$

By direct computations, we find that

$$\mathrm{Ker}(K) = \mathbb{C}f_1, \quad \mathrm{Ker}(K^2) = \mathbb{C}e_1 + \mathbb{C}e_2, \mathrm{Ker}(K^3) = \mathbb{C}f_1 + \mathbb{C}f_2 + \mathbb{C}f_3,$$

where $\{f_1, f_2, f_3, f_4\}$ is an orthonormal basis for $\mathbb{C}^4$ given by

$$f_1 = [\frac{i}{2}, \frac{1+i}{2\sqrt{2}}, \frac{1-i}{2\sqrt{2}}, \frac{1}{2}]^T, \quad f_2 = [\frac{-1}{2}, \frac{1+i}{2\sqrt{2}}, \frac{-1+i}{2\sqrt{2}}, \frac{-i}{2}]^T,$$

$$f_3 = [\frac{i}{2}, \frac{-1-i}{2\sqrt{2}}, \frac{-1+i}{2\sqrt{2}}, \frac{1}{2}]^T, \quad f_4 = [\frac{1}{2}, \frac{1+i}{2\sqrt{2}}, \frac{-1+i}{2\sqrt{2}}, \frac{i}{2}]^T.$$

Consider the unitary matrix

$$V = [f_1, f_2, f_3, f_4].$$

Then we have the unitary equivalence

$$V^*KV = \begin{pmatrix} 0 & -2+2(\sqrt{2}-1)i & 0 & 2+2(\sqrt{2}-1)i \\ 0 & 0 & 2-2(\sqrt{2}-1)i & 0 \\ 0 & 0 & 0 & 2-2(\sqrt{2}-1)i \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Choose the diagonal unitary matrix $W = \mathrm{diag}(1, \eta_2, \eta_3, \eta_4)$, where

$$\eta_3 = \frac{1-i}{\sqrt{2}}, \quad -\eta_4 = \eta_2 = \frac{1+(\sqrt{2}-1)i}{\sqrt{4-2\sqrt{2}}}.$$

Then

$$W^*(V^*KV)W = \frac{-2}{\sqrt{2+\sqrt{2}}}(1+i)\sqrt{2}T(1,0).$$

Hence, the matrix $S = C + i\mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$ is unitarily similar to A. □

Acknowledgments: The authors thank the anonymous referees for their careful reading of the text and helpful suggestions which led to improvements. The first author was partially supported by Taiwan Ministry of Science and Technology under MOST 106-2115-M-031-002, and the J.T. Tai & Co Foundation Visiting Research Program. The second author was supported in part by Japan Society for the Promotion of Science, KAKENHI No. 15K04890.

Author Contributions: The two authors made an equal contribution to this work.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

References

1. Hausdorff, F. Das Wertevorrat einer Bilinearform. *Math. Z.* **1919**, *3*, 314–316.
2. Kippenhahn, R. Über den wertevorrat einer Matrix. *Math. Nachr.* **1951**, *6*, 193–228.
3. Fiedler, M. Geometry of the numerical range of matrices. *Linear Algebra Appl.* **1981**, *37*, 81–96.
4. Fiedler, M. Pencils of real symmetric matrices and real algebraic curves. *Linear Algebra Appl.* **1990**, *141*, 53–60.
5. Lax, P.D. Differential equations, difference equations and matrix theory. *Comm. Pure Appl. Math.* **1958**, *11*, 175–194.

6. Helton, J.W.; Vinnikov, V. Linear matrix inequality representations of sets. *Commun. Pure Appl. Math.* **2007**, *60*, 654–674.
7. Helton, J.W.; Spitkovsky, I.M. The possible shapes of numerical ranges. *Oper. Matrices* **2012**, *6*, 607–611.
8. Plaumann, D.; Sturmfels, B.; Vinzant, C. Computing linear matrix representations of Helton-Vinnikov curves. In *Mathematical Methods in Systems, Optimization and Controls*; Birkhauser: Basel, Switzerland, 2012; Volume 222, pp. 259–277.
9. Chien, M.T.; Nakazato, H. Unitary similarity of the determinantal representation of unitary bordering matrices. *Linear Algebra Appl.* **2018**, *541*, 13–35.
10. Chien, M.T.; Nakazato, H. Computation of Riemann matrices for the hyperbolic curves of determinantal polynomials. *Ann. Funct. Anal.* **2017**, *8*, 152–167.
11. Chien, M.T.; Nakazato, H. Determinantal representations of hyperbolic forms via weighted shift matrices. *Appl. Math. Comput.* **2015**, *258*, 172–181.
12. Lentzos, K.; Pasley, L. Determinantal representations of invariant hyperbolic plane curves. *arXiv* **2017**, arXiv:1707.07724v1.
13. Chien, M.T.; Liu, J.Z.; Nakazato, H.; Tam, T.Y. Toeplitz matrices are unitarily similar to symmetric matrices. *Linear Multilinear Algebra* **2017**, *65*, 2131–2144.
14. Chien, M.T.; Nakazato, H. Boundary generating curves of *c*-numerical range. *Linear Algebra Appl.* **1999**, *294*, 67–84.
15. Fröeberg, R. *An Introduction to Gröbner Bases*; John Wiley & Sons: New York, NY, USA, 1997.