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Noether's Theorem and Symmetry

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Received: 20 November 2018; Accepted: 9 December 2018; Published: 12 December 2018

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Abstract: In Noether's original presentation of her celebrated theorem of 1918, allowance was made for the dependence of the coefficient functions of the differential operator, which generated the infinitesimal transformation of the action integral upon the derivatives of the dependent variable(s), the so-called generalized, or dynamical, symmetries. A similar allowance is to be found in the variables of the boundary function, often termed a gauge function by those who have not read the original paper. This generality was lost after texts such as those of Courant and Hilbert or Lovelock and Rund confined attention to point transformations only. In recent decades, this diminution of the power of Noether's theorem has been partly countered, in particular in the review of Sarlet and Cantrijn. In this Special Issue, we emphasize the generality of Noether's theorem in its original form and explore the applicability of even more general coefficient functions by allowing for nonlocal terms. We also look for the application of these more general symmetries to problems in which parameters or parametric functions have a more general dependence on the independent variables.

Keywords: Noether's theorem; action integral; generalized symmetry; first integral; invariant; nonlocal transformation; boundary term; conservation laws; analytic mechanics

1. Introduction

Noether's theorem [1] treats the invariance of the functional of the calculus of variations—the action integral in mechanics—under an infinitesimal transformation. This transformation can be considered as being generated by a differential operator, which in this case is termed a Noether symmetry. The theorem was not developed ab initio by Noether. Not only is it steeped in the philosophy of Lie's approach, but also, it is based on earlier work of more immediate relevance by a number of writers. Hamel [2,3] and Herglotz [4] had already applied the ideas developed in her paper to some specific finite groups. Fokker [5] did the same for specific infinite groups. A then recently-published paper by Kneser [6] discussed the finding of invariants by a similar method. She also acknowledged the contemporary work of Klein [7]. Considering that the paper was presented to the Festschrift in honor of the fiftieth anniversary of Klein's doctorate, this final attribution must have been almost obligatory.

For reasons obscure Noether's theorem has been subsequently subject to downsizing by many authors of textbooks [8–10], which has then given other writers (cf. [11]) the opportunity to 'generalize' the theorem or to demonstrate the superiority of some other method [12,13] to obtain more general results [14–16]. This is possibly due to the simplified form presented in Courant and Hilbert [8]. As Hilbert was present at the presentation by Noether of her theorem to the Festschrift in honor of the fiftieth anniversary of Felix Klein's doctorate, it could be assumed that his description would be accurate. However, Hilbert's sole contribution to the text was his name.

This particularizing tendency has not been uniform, e.g., the review by Sarlet and Cantrijn [17]. According to Noether [1] (pp. 236–237), "In den Transformationen können auch die Ableitungen der u nach den x, also $\partial u/\partial x$, $\partial^2 u/\partial x^2, \ldots$ auftreten", so that the introduction of generalized transformations is made before the statement of the theorem [1] (p. 238). On page 240, after the statement of the theorem, Noether does mention particular results if one restricts the class of transformations admitted and this may be the source of the usage of the restricted treatments mentioned above.

We permit the coefficient functions of the generator of the infinitesimal transformation to be of unspecified dependence subject to any requirement of differentiability.

For the purposes of the clarity of exposition, we develop the theory of the theorem in terms of a first-order Lagrangian in one dependent and one independent variable. The expressions for more complicated situations are given below in a convenient summary format.

2. Noether Symmetries

We consider the action integral:

$$A = \int_{t_0}^{t_1} L\left(t, q, \dot{q}\right) \mathrm{d}t. \tag{1}$$

Under the infinitesimal transformation:

$$\bar{t} = t + \varepsilon\tau, \qquad \bar{q} = q + \varepsilon\eta \tag{2}$$

generated by the differential operator:

$$\Gamma = \tau\partial_t + \eta\partial_q,$$

the action integral (1) becomes:

$$\bar{A} = \int_{\bar{t}_0}^{\bar{t}_1} L\left(\bar{t}, \bar{q}, \dot{\bar{q}}\right) \mathrm{d}\bar{t}$$

($\dot{\bar{q}}$ is $\mathrm{d}\bar{q}/\mathrm{d}\bar{t}$ in a slight abuse of standard notation), which to the first order in the infinitesimal, ε, is:

$$\begin{aligned} \bar{A} \;\; &= \;\; \int_{t_0}^{t_1} \left[L + \varepsilon\left(\tau\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial q} + \zeta\frac{\partial L}{\partial \dot{q}} + \dot{\tau}L\right)\right] \mathrm{d}t \\ &\quad + \varepsilon\left[\tau t_1 L(t_1, q_1, \dot{q}_1) - \tau t_0 L(t_0, q_0, \dot{q}_0)\right], \end{aligned} \tag{3}$$

where $\zeta = \dot{\eta} - \dot{q}\dot{\tau}$ and $L(t_0, q_0, \dot{q}_0)$ and $L(t_1, q_1, \dot{q}_1)$ are the values of L at the endpoints t_0 and t_1, respectively.

We demonstrate the origin of the terms outside of the integral with the upper limit. The lower limit is treated analogously.

$$\begin{aligned} \int^{\bar{t}_1} &= \int^{t_1 + \varepsilon\tau(t_1)} \\ &= \int^{t_1} + \int_{t_1}^{t_1 + \varepsilon\tau(t_1)} \\ &= \varepsilon \int^{t_1} + \varepsilon\tau(t_1) L(t_1, q_1, \dot{q}_1) \end{aligned}$$

to the first order in ε. We may rewrite (3) as:

$$\bar{A} = A + \varepsilon \int_{t_0}^{t_1} \left(\tau\frac{\partial L}{\partial t} + \eta\frac{\partial L}{\partial q} + \zeta\frac{\partial L}{\partial \dot{q}} + \dot{\tau}L\right) \mathrm{d}t + \varepsilon F,$$

where the number, F, is the value of the second term in bracketsin (3). As F depends only on the endpoints, we may write it as:

$$F = -\int_{t_0}^{t_1} \dot{f}\,\mathrm{d}t,$$

where the sign is chosen as a matter of later convenience.

The generator, Γ, of the infinitesimal transformation, (2), is a Noether symmetry of (1) if:

$$\bar{A} = A,$$

i.e.,

$$\int_{t_0}^{t_1} \left(\tau \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \dot{\tau} L - \dot{f} \right) \mathrm{d}t = 0$$

from which it follows that:

$$\dot{f} = \tau \frac{\partial L}{\partial t} + \eta \frac{\partial L}{\partial q} + \zeta \frac{\partial L}{\partial \dot{q}} + \dot{\tau} L. \tag{4}$$

Remark 1. *The symmetry is the generator of an infinitesimal transformation, which leaves the action integral invariant, and the existence of the symmetry has nothing to do with the Euler–Lagrange equation of the calculus of variations. The Euler–Lagrange equation follows from the application of Hamilton's principle in which q is given a zero endpoint variation. There is no such restriction on the infinitesimal transformations introduced by Noether.*

3. Noether's Theorem

We now invoke Hamilton's principle for the action integral (1). We observe that the zero-endpoint variation of (1) imposed by Hamilton's principle requires that (1) take a stationary value; not necessarily a minimum! The principle of least action enunciated by Fermat in 1662 as "Nature always acts in the shortest ways" was raised to an even more metaphysical status by Maupertuis [18] (p. 254, p. 267). That the principle applies in classical (Newtonian) mechanics is an accident of the metric! We can only wonder that the quasi-mystical principle has persisted for over two centuries in what are supposed to be rational circles. In the case of a first-order Lagrangian with a positive definite Hessian with respect to $\dot{q}$, Hamilton's principle gives a minimum. This is not necessarily the case otherwise.

The Euler–Lagrange equation:

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{5}$$

follows from the application of Hamilton's principle. We manipulate (4) as follows:

$$\begin{aligned}
0 &= \dot{f} - \tau \frac{\partial L}{\partial t} - \dot{\tau} L - \eta \frac{\partial L}{\partial q} - (\dot{\eta} - \dot{q}\dot{\tau}) \frac{\partial L}{\partial \dot{q}} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(f - \tau L) + \tau \left(\dot{q} \frac{\partial L}{\partial q} + \ddot{q} \frac{\partial L}{\partial \dot{q}} \right) + \dot{\tau} \left(\dot{q} \frac{\partial L}{\partial \dot{q}} \right) \\
&\quad - \eta \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}} \right) - \dot{\eta} \frac{\partial L}{\partial \dot{q}} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[f - \tau L - (\eta - \tau \dot{q}) \frac{\partial L}{\partial \dot{q}} \right]
\end{aligned}$$

in the second line of which we have used the Euler–Lagrange Equation (5), to change the coefficient of η. Hence, we have a first integral:

$$I = f - \left[\tau L + (\eta - \dot{q}\tau) \frac{\partial L}{\partial \dot{q}} \right] \tag{6}$$

and an initial statement of Noether's Theorem.

Noether's theorem: If the action integral of a first-order Lagrangian, namely:

$$A = \int_{t_0}^{t_1} L\,(t, q, \dot{q})\,\mathrm{d}t$$

is invariant under the infinitesimal transformation generated by the differential operator:

$$\Gamma = \tau\partial_t + \eta_i\partial_{q_i},$$

there exists a function f such that:

$$\dot{f} = \tau\frac{\partial L}{\partial t} + \eta_i\frac{\partial L}{\partial q_i} + \zeta_i\frac{\partial L}{\partial \dot{q}_i} + \dot{\tau}L, \tag{7}$$

where $\zeta_i = \dot{\eta}_i - \dot{q}_i\dot{\tau}$, and a first integral given by:

$$I = f - \left[\tau L + (\eta_i - \dot{q}_i\tau)\,\frac{\partial L}{\partial \dot{q}_i}\right].$$

Γ is called a Noether symmetry of L and I a Noetherian first integral. The symmetry Γ exists independently of the requirement that the variation of the functional be zero. When the extra condition is added, the first integral exists.

We note that there is not a one-to-one correspondence between a Noether symmetry and a Noetherian integral. Once the symmetry is determined, the integral follows with minimal effort. The converse is not so simple because, given the Lagrangian and the integral, the symmetry is the solution of a differential equation with an additional dependent variable, the function f arising from the boundary terms. There can be an infinite number of coefficient functions for a given first integral. The restriction of the symmetry to a point symmetry may reduce the number of symmetries, too effectively, to zero. The ease of determination of a Noetherian integral once the Noether symmetry is known is in contrast to the situation for the determination of first integrals in the case of Lie symmetries of differential equations. The computation of the first integrals associated with a Lie symmetry can be a highly nontrivial matter.

4. Nonlocal Integrals

We recall that the variable dependences of the coefficient functions τ and η were not specified and do not enter into the derivation of the formulae for the coefficient functions or the first integral. Consequently, not only can we have the generalized symmetries of Noether's paper, but we can also have more general forms of symmetry such as nonlocal symmetries [19,20] without a single change in the formalism. Of course, as has been noted for the calculation of first integrals [21] and symmetries in general [22], the realities of computational complexity may force one to impose some constraints on this generality. Once the Euler–Lagrange equation is invoked, there is an automatic constraint on the degree of derivatives in any generalized symmetry.

If one has a standard Lagrangian such as (1), a nonlocal Noether's symmetry will usually produce a nonlocal integral through (6). In that the total time derivative of this function is zero when the Euler–Lagrange equation, (5), is taken into account, it is formally a first integral. However, the utility of such a first integral is at best questionable. Here, Lie and Noether have generically differing outcomes. An exponential nonlocal Lie symmetry can be expected to lead to a local first integral, whereas one could scarcely envisage the same for an exponential nonlocal Noether symmetry.

On the other hand, if the Lagrangian was nonlocal, the combination of nonlocal symmetry and nonlocal Lagrangian could lead to a local first integral. However, we have not constructed a formalism to deal with nonlocal Lagrangians—as opposed to nonlocal symmetries—and so, we cannot simply apply what we have developed above.

The introduction of a nonlocal term into the Lagrangian effectively increases the order of the Lagrangian by one (in the case of a simple integral) and the order of the associated Euler–Lagrange equation by two so that for a Lagrangian regular in $\dot{q}$ instead of a second-order differential equation, we would have a fourth order differential equation in q. To avoid that the Lagrangian would have to be degenerate, i.e., linear, in $\dot{q}$, this cannot, as is well-known, lead to a second-order differential equation. It would appear that nonlocal symmetries in the context of Noether's theorem do not have the same potential as nonlocal Lie symmetries of differential equations.

There is often some confusion of identity between Lie symmetries and Noether symmetries. Although every Noether symmetry is a Lie symmetry of the corresponding Euler–Lagrange equation, we stress that they have different provenances. There is a difference that is more obvious in systems of higher dimension. A Noether symmetry can only give rise to a single first integral because of (3). In an n-dimensional system of second-order ordinary differential equations, a single Lie symmetry gives rise to $(2n-1)$ first integrals [23–26].

5. Extensions: One Independent Variable

The derivation given above applies to a one-dimensional discrete system. The theorem can be extended to continuous systems and systems of higher order. The principle is the same. The mathematics becomes more complicated. We simply quote the relevant results.

For a first-order Lagrangian with n dependent variables:

$$G = \tau\partial_t + \eta_i\partial_{q_i} \tag{8}$$

is a Noether symmetry of the Lagrangian, $L(t, q_i, \dot{q}_i)$, if there exists a function f such that:

$$\dot{f} = \dot{\tau}L + \tau\frac{\partial L}{\partial t} + \eta_i\frac{\partial L}{\partial q_i} + (\dot{\eta}_i - \dot{q}_i\dot{\tau})\frac{\partial L}{\partial \dot{q}_i} \tag{9}$$

and the corresponding Noetherian first integral is:

$$I = f - \left[\tau L + (\eta_i - \dot{q}_i\tau)\frac{\partial L}{\partial \dot{q}_i}\right] \tag{10}$$

which are the obvious generalizations of (4) and (6), respectively.

In the case of an n^{th}-order Lagrangian in one dependent variable and one independent variable, $L(t, q, \dot{q}, \ldots, q^{(n)})$ with $q^{(n)} = \mathrm{d}^n q/\mathrm{d}t^n$, the Euler–Lagrange equation is:

$$\sum_{j=0}^{n}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{\partial L}{\partial q^{(j)}}\right). \tag{11}$$

$\Gamma = \tau\partial_t + \eta\partial_q$ is a Noether symmetry if there exists a function f such that:

$$\dot{f} = \dot{\tau}L + \tau\frac{\partial L}{\partial t} + \sum_{j=0}^{n}(-1)^j\zeta^j\left(\frac{\partial L}{\partial q^{(j)}}\right), \tag{12}$$

where:

$$\zeta^j = \eta^{(j)} - \sum_{k=1}^{j}\begin{pmatrix} j \\ k \end{pmatrix} q^{(j+1-k)}\tau^{(k)}. \tag{13}$$

The expression for the first integral is:

$$I = f - \left[\tau L + \sum_{i=0}^{n-1}\sum_{j-0}^{n-1-i}(-1)^j\left(\eta - \dot{q}\tau\right)^{(i)}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(\frac{\partial L}{\partial q^{(i+j+1)}}\right)\right]. \tag{14}$$

In the case of an n^{th}-order Lagrangian in m dependent variables and one independent variable, $L(t, q_k, \dot{q}_k, \ldots, q_k^{(n)})$ with $q_k^{(n)} = \mathrm{d}^n q_k/\mathrm{d}t^n$, $k = 1, m$, the Euler–Lagrange equation is:

$$\sum_{j=0}^{n} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \left(\frac{\partial L}{\partial q_k^{(j)}} \right), \; k = 1, m. \tag{15}$$

$\Gamma = \tau \partial_t + \sum_{k=1}^{m} \eta_k \partial_{q_k}$ is a Noether symmetry if there exists a function f such that:

$$\dot{f} = \dot{\tau} L + \tau \frac{\partial L}{\partial t} + \sum_{k=1}^{m} \sum_{j=0}^{n} (-1)^j \zeta_k^j \left(\frac{\partial L}{\partial q_k^{(j)}} \right), \tag{16}$$

where:

$$\zeta_k^j = \eta_k^{(j)} - \sum_{k=1}^{m} \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} q_k^{(j+1-i)} \tau^{(i)}. \tag{17}$$

The expression for the first integral is:

$$I = f - \left[\tau L + \sum_{k=1}^{m} \sum_{i=0}^{n-1} \sum_{j-0}^{n-1-i} (-1)^j \left(\eta_k - \dot{q}_k \tau \right)^{(i)} \frac{\mathrm{d}^j}{\mathrm{d}t^j} \left(\frac{\partial L}{\partial q_k^{(i+j+1)}} \right) \right]. \tag{18}$$

The expressions in (14) and (18), although complex enough, conceals an even much greater complexity because each derivative with respect to time is a total derivative and so affects all terms in the Lagrangian and its partial derivatives.

6. Observations

In the case of a first-order Lagrangian with one independent variable, it is well-known [17] that one can achieve a simplification in the calculations of the Noether symmetry in the case that the Lagrangian has a regular Hessian with respect to the $\dot{q}_i$. We suppose that we admit generalized symmetries in which the maximum order of the derivatives present in τ and the η_i is one, i.e., equal to the order of the Lagrangian. Then, the coefficient of each $\ddot{q}_j$ in (9) is separately zero since the Euler–Lagrange equation has not yet been invoked. Thus, we have:

$$\frac{\partial f}{\partial \dot{q}_j} = \frac{\partial \tau}{\partial \dot{q}_j} L + \left(\frac{\partial \eta_i}{\partial \dot{q}_j} - \dot{q}_i \frac{\partial \tau}{\partial \dot{q}_j} \right) \frac{\partial L}{\partial \dot{q}_i}. \tag{19}$$

We differentiate (10) with respect to $\dot{q}_j$ to obtain:

$$\frac{\partial I}{\partial \dot{q}_j} = \frac{\partial f}{\partial \dot{q}_j} - \left[\frac{\partial \tau}{\partial \dot{q}_j} L + \tau \frac{\partial L}{\partial \dot{q}_j} + \left(\frac{\partial \eta_i}{\partial \dot{q}_j} - \delta_{ij} \tau - \dot{q}_i \frac{\partial \tau}{\partial \dot{q}_j} \right) \frac{\partial L}{\partial \dot{q}_i} + \left(\eta_i - \dot{q}_i \tau \right) \frac{\partial^2 L}{\partial \dot{q}_i} \partial \dot{q}_j \right], \tag{20}$$

where δ_{ij} is the usual Kronecker delta, which, when we take (19) into account, gives:

$$\frac{\partial I}{\partial \dot{q}_j} = - \left(\eta_i - \dot{q}_i \tau \right) \frac{\partial^2 L}{\partial \dot{q}_i} \partial \dot{q}_j. \tag{21}$$

Consequently, if the Lagrangian is regular with respect to the $\dot{q}_i$, we have:

$$\left(\eta_i - \dot{q}_i \tau \right) = -g_{ij} \frac{\partial I}{\partial \dot{q}_j}, \tag{22}$$

where:

$$g_{ik}\frac{\partial^2 L}{\partial \dot{q}_k}\partial \dot{q}_j = \delta_{ij}.$$

The relations (21) and (22) reveal two useful pieces of information. The first is that the derivative dependence of the first integral is determined by the nature of the generalized symmetry (modulo the derivative dependence in the Lagrangian). The second is that there is a certain freedom of choice in the structure of the functions τ and η_i in the symmetry. Provided generalized symmetries are admitted, there is no loss of generality in putting one of the coefficient functions equal to zero. An attractive candidate is τ as it appears the most frequently. The choice should be made before the derivative dependence of the coefficient functions is assumed. We observe that in the case of a 'natural' Lagrangian, i.e., one quadratic in the derivatives, the first integrals can only be linear or quadratic in the derivatives if the symmetry is assumed to be point.

7. Examples

The free particle:

We consider the simple example of the free particle for which:

$$L = \tfrac{1}{2}y'^2.$$

Equation (7) is:

$$(\eta' - y'\xi')y' + \tfrac{1}{2}y'^2 = f'. \tag{23}$$

If we assume that Γ is a Noether point symmetry, (23) gives the following determining equations:

$$\begin{array}{lcl} y'^3 : & -\tfrac{1}{2}\dfrac{\partial \xi}{\partial y} = & 0 \\ y'^2 & : & \dfrac{\partial \eta}{y} - \tfrac{1}{2}\dfrac{\partial \xi}{\partial x} = 0 \\ y'^1 & : & \dfrac{\partial \eta}{\partial x} - \dfrac{\partial f}{\partial y} = 0 \\ y'^0 & : & \dfrac{\partial f}{\partial x} = 0 \end{array}$$

from which it is evident that:

$$\begin{aligned} \xi &= a(x) \\ \eta &= \tfrac{1}{2}a'y + b(x) \\ f &= \tfrac{1}{4}a''^2 + b'y + c(x) \\ 0 &= \tfrac{1}{4}a'''^2 + b''y + c'. \end{aligned}$$

Hence:

$$\begin{aligned} a &= A_0 + A_1 x + A_2 x^2 \\ b &= B_0 + B_1 x \\ c &= C_0. \end{aligned}$$

Because c is simply an additive constant, it is ignored. There are five Noether point symmetries, which is the maximum for a one-dimensional system [27]. They and their associated first integrals are:

$$\begin{aligned}
\Gamma_1 &= \partial_y & I_1 &= -y' \\
\Gamma_2 &= x\partial_y & I_2 &= y - xy' \\
\Gamma_3 &= \partial_x & I_3 &= \tfrac{1}{2}y'^2 \\
\Gamma_4 &= x\partial_x + \tfrac{1}{2}y\partial_y & I_4 &= -\tfrac{1}{2}y'(y - xy') \\
\Gamma_5 &= x^2\partial_x + xy\partial_y & I_5 &= \tfrac{1}{2}(y - xy'^2.
\end{aligned}$$

The corresponding Lie algebra is isomorphic to $A_{5,40}$ [28]. The algebra is structured as $2A_1 \oplus_s sl(2,R)$, which is a proper subalgebra of the Lie algebra for the differential equation for the free particle, namely $sl(3,R)$, which is structured as $2A_1 \oplus_s \{sl(2,R) \oplus_s A_1\} \oplus_s 2A_1$. The missing symmetries are the homogeneity symmetry and the two non-Cartan symmetries. The absence of the homogeneity symmetry emphasizes the distinction between the Lie and Noether symmetries.

Noether symmetries of a higher-order Lagrangian:

Suppose that $L = \frac{1}{2}y''^2$. The condition for a Noether point symmetry is that:

$$\zeta_2 \frac{\partial L}{\partial y''} + \xi' L = f', \tag{24}$$

where $\zeta_2 = \eta'' - 2y''\xi' - y'\xi''$, so that (24) becomes:

$$(\eta'' - 2y''\xi' - y'\xi'')y'' + \tfrac{1}{2}\xi' y''^2 = f'. \tag{25}$$

Assume a point transformation, i.e., $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$. Then:

$$\begin{aligned}
&\left[\frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x}\partial y + y'^2\frac{\partial^2\eta}{\partial y^2} + y''\frac{\partial\eta}{\partial y} - 2y''\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y}\right)\right. \\
&\left.-y'\left(\frac{\partial^2\xi}{\partial x^2} + 2y'\frac{\partial^2\xi}{\partial x}\partial y + y'^2\frac{\partial^2 xi}{\partial y^2} + y''\frac{\partial\xi}{\partial y}\right)\right]y'' + \tfrac{1}{2}\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y}\right)y''^2 \\
= \; & \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'}.
\end{aligned}$$

from the coefficient of $y'y''^2$, videlicet:

$$-\tfrac{5}{2}\frac{\partial\xi}{\partial y} = 0,$$

we obtain:

$$\xi = a(x).$$

The coefficient of y''^2,

$$\frac{\partial\eta}{\partial y} - \tfrac{3}{2}\frac{\partial\xi}{\partial x} = 0,$$

results in:

$$\eta = \tfrac{3}{2}a'y + b(x)$$

and the coefficient of y'',

$$\frac{\partial^2\eta}{\partial x^2} + 2y'\frac{\partial^2\eta}{\partial x}\partial y + y'^2\frac{\partial^2\eta}{\partial y^2} - y'\frac{\partial^2\xi}{\partial x^2} = \frac{\partial f}{\partial y'},$$

gives f as:

$$f = a''y'^2 + (\tfrac{3}{2}a'''y + b'')y' + c(x,y).$$

The remaining terms give:

$$y'\frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} = 0,$$

i.e.,

$$y'\left[\frac{3}{2}a'''y' + \frac{\partial c}{\partial y}\right] + a'''y'^2 + \left(\tfrac{3}{2}a^{iv}y + b'''\right)y' + \frac{\partial c}{\partial x} = 0.$$

The coefficient of y'^2 is $\frac{5}{2}a''' = 0$, from which it follows that:

$$a = A_0 + A_1 x + A_2 x^2.$$

The coefficient of y' is:

$$\frac{\partial c}{\partial y} + \tfrac{3}{2}a^{iv}y + b''' = 0$$

and so:

$$c = -b'''y + d(x).$$

The remaining terms give:

$$\frac{\partial c}{\partial x} = 0,$$

i.e.,

$$-b^{iv}y + d' = 0$$

from which d is a constant (and can therefore be ignored) and:

$$b = B_0 + B_1 x + B_2 x^2 + B_3 x^3.$$

There are seven Noetherian point symmetries for $L = \frac{1}{2}y''^2$. They and the associated "gauge functions" are:

$$\begin{array}{lll}
B_0: & \Gamma_1 = \partial_y & f_1 = 0 \\
B_1: & \Gamma_2 = x\partial_y & f_2 = 0 \\
B_2: & \Gamma_3 = x^2\partial_y & f_3 = 2xy' \\
B_3: & \Gamma_4 = x^3\partial_y & f_4 = 6xy' - 6y \\
A_0: & \Gamma_5 = \partial_x & f_5 = 0 \\
A_1: & \Gamma_6 = x\partial_x + \tfrac{3}{2}y\partial_y & f_6 = 0 \\
A_2: & \Gamma_7 = x^2\partial_x + 3xy\partial_y & f_7 = 2y'^2.
\end{array}$$

The Euler–Lagrange equation for $L = \frac{1}{2}y''^2$ is $y^{(iv)} = 0$, which has Lie point symmetries the same as the Noether point symmetries plus $\Gamma_8 = y\partial_y$. Note that there is a contrast here in comparison with the five Noether point symmetries of $L = \frac{1}{2}y'^2$ and the eight Lie point symmetries of $y'' = 0$. The additional Lie symmetries are $y\partial_y$ as above for $y^{iv} = 0$ and the two non-Cartan symmetries, $X_1 = y\partial_x$ and $X_2 = xy\partial_x + y^2\partial_y$.

For $L = \frac{1}{2}y''^2$, the associated first integrals have the structure:

$$I = f - \tfrac{1}{2}\xi y''^2 + (\eta - y'\xi)y''' - (\eta' - y''\xi - y'\xi')y''$$

and are:

$$\begin{array}{rcl}
I_1 & = & y''' \\
I_2 & = & xy''' - y'' \\
I_3 & = & x^2y''' - 2xy'' + 2xy' \\
I_4 & = & x^3y'''^2y'' + 6xy' - 6y \\
I_5 & = & -y'y''' + \tfrac{1}{2}y''^2 \\
I_6 & = & -xy'y + \tfrac{1}{2}xy''^2 - \tfrac{1}{2}y'y'' + \tfrac{3}{2}yy''' \\
I_7 & = & x(3y - xy')y''' - (3y - xy' - \tfrac{1}{2}x^2y'')y'' + 2y'^2.
\end{array}$$

Note that I_1–I_4 associated with Γ_1–Γ_4, respectively, are also integrals obtained by the Lie method. However, each Noether symmetry produces just one first integral, whereas each Lie symmetry has three first integrals associated with it.

In this example, only point Noether symmetries have been considered. One may also determine symmetries that depend on derivatives, effectively up to the third order when one is calculating first integrals of the Euler–Lagrange equation.

Omission of the gauge function:

In some statements of Noether's theorem, the so-called gauge function, f, is taken to be zero. In the derivation given here, f comes from the contribution of the boundary terms produced by the infinitesimal transformation in t and so is not a gauge function in the usual meaning of the term. However, it does function as one since it is independent of the trajectory in the extended configuration space and depends only on the evaluation of functions at the boundary (end points in a one-degree-of-freedom case) and can conveniently be termed one especially in light of Boyer's theorem [29].

Consider the example $L = \frac{1}{2}y'^2$ without f. The equation for the symmetries,

$$f' = \xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial y} + (\eta' - y'\xi')\frac{\partial L}{\partial y'} + \xi' L,$$

becomes:

$$0 = \left(\frac{\partial \eta}{\partial x} + y'\frac{\partial \eta}{\partial y} - y'\frac{\partial \xi}{\partial x} - y'^2\frac{\partial \xi}{\partial y}\right)y' + \tfrac{1}{2}y'^2\left(\frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y}\right).$$

We solve this in the normal way: the coefficients of y'^3, y'^2 and of y' give in turn:

$$\begin{aligned} \xi &= a(x) \\ \eta &= \tfrac{1}{2}a'y + b(x) \\ \tfrac{1}{2}a''y + b' &= 0 \end{aligned}$$

which hold provided that:

$$a = A_0 + A_1 x \qquad\qquad b = B_0,$$

i.e., only three symmetries are obtained instead of the five when the gauge function is present.

It makes no sense to omit the gauge function when the infinitesimal transformation is restricted to be point and only in the dependent variables.

A higher-dimensional system:

We determine the Noether point symmetries and their associated first integrals for:

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2)$$

(which is the standard Lagrangian for the free particle in two dimensions). The determining equation is:

$$\begin{aligned} \frac{\partial f}{\partial t} + \dot{x}\frac{\partial f}{\partial x} + \dot{y}\frac{\partial f}{\partial y} &= \left(\frac{\partial \eta}{\partial t} + \dot{x}\frac{\partial \eta}{\partial x} + \dot{y}\frac{\partial \eta}{\partial y} - \dot{x}\left(\frac{\partial \xi}{\partial t} + \dot{x}\frac{\partial \xi}{\partial x} + \dot{y}\frac{\partial \xi}{\partial y}\right)\right)\dot{x} \\ &\quad + \left(\frac{\partial \zeta}{\partial t} + \dot{x}\frac{\partial \zeta}{\partial x} + \dot{y}\frac{\partial \zeta}{\partial y} - \dot{y}\left(\frac{\partial \xi}{\partial t} + \dot{x}\frac{\partial \xi}{\partial x} + \dot{y}\frac{\partial \xi}{\partial y}\right)\right)\dot{y}, \end{aligned}$$

where $\eta_1 = \eta$ and $\eta_2 = \zeta$.

We separate by powers of $\dot{x}$ and $\dot{y}$. Firstly taking the third-order terms, we have:

$$\begin{aligned} \dot{x}^3: \quad & -\frac{\partial \xi}{\partial x} &=& \; 0 \\ \dot{x}^2\dot{y}: \quad & -\frac{\partial \xi}{\partial y} &=& \; 0 \\ \dot{x}\dot{y}^2: \quad & -y\frac{\partial \xi}{\partial x} &=& \; 0 \\ \dot{y}^3: \quad & -\frac{\partial \xi}{\partial y} &=& \; 0 \end{aligned}$$

which implies $\xi = a(t)$. We now consider the second-order terms: the coefficient of $\dot{x}^2$ gives η as $\eta = \dot{a}x + b(y,t)$; that of $\dot{x}\dot{y}$ gives ζ as:

$$\zeta = -\frac{\partial b}{\partial y}x + c(y,t);$$

and that of $\dot{y}^2$ gives $c = \dot{a}y + d(t)$ and $b = e(t)y + g(t)$. Thus far, we have:

$$\xi = a(t) \qquad \eta = \dot{a}x + ey + g \qquad \zeta = -ex + \dot{a}y + d.$$

The coefficient of $\dot{x}$ gives f as:

$$f = \tfrac{1}{2}\ddot{a}x^2 + \dot{e}xy + \dot{g}x + K(y,t).$$

The coefficient of $\dot{y}$ requires that:

$$\dot{e}x + \frac{\partial K}{\partial y} = -\dot{e}x + \ddot{a}y + \dot{d}$$

which implies:

$$\dot{e} = 0 \qquad K = \tfrac{1}{2}\ddot{a}y^2 + \dot{d}y + h(t).$$

The remaining term requires that:

$$\tfrac{1}{2}\dddot{a}\, x^2 + \ddot{g}x + \tfrac{1}{2}\dddot{a}\, y^2 + \ddot{d}y + \dot{h} = 0$$

whence:

$$\begin{aligned} a &= A_0 + A_1 t + A_2 t^2 \\ g &= G_0 + G_1 t \\ d &= D_0 + D_1 t \\ h &= H_0 \end{aligned}$$

(we ignore H_0, as it is an additive constant to f).

The coefficient functions are:

$$\begin{aligned} \xi &= A_0 + A_1 t + A_2 t^2 \\ \eta &= (A_1 + 2A_2 t)x + E_0 y + G_0 + G_1 t \\ \zeta &= -E_0 x + (A_1 + 2A_2 t)y + D_0 + D_1 t \end{aligned}$$

and the gauge function is:

$$f = A_2 x^2 + G_1 x + A_2 y^2 + D_1 y.$$

We obtain three symmetries from a, namely:

$$\begin{aligned}\Gamma_1 &= \partial_t \\ \Gamma_2 &= t\partial_t + x\partial_x + y\partial_y \\ \Gamma_3 &= t^2\partial_t + 2t\left(x\partial_x + y\partial_y\right)\end{aligned}$$

which form $sl(2,R)$, one from e,

$$\Gamma_4 = y\partial_x - x\partial_y$$

which is $so(2)$, and four from g and d, namely:

$$\begin{aligned}\Gamma_5 &= \partial_x \\ \Gamma_6 &= t\partial_x \\ \Gamma_7 &= \partial_y \\ \Gamma_8 &= t\partial_y.\end{aligned}$$

The last four are the "solution" symmetries and form the Lie algebra $4A_1$.

8. More than One Independent Variable: Preliminaries

8.1. Euler–Lagrange Equation

Noether's original formulation of her theorem was in the context of Lagrangians for functions of several independent variables. We have deliberately separated the case of one independent variable from the general discussion to be able to present the essential ideas in as simple a form as possible. The discussion of the case of several independent variables is inherently more complex simply from a notational point of view, although there is no real increase in conceptual difficulty.

We commence with the simplest instance of a Lagrangian of this class, which is $L(t,x,u,u_t,u_x)$, i.e., one dependent variable, u, and two dependent variables, t and x. We recall the derivation of the Euler–Lagrange equation for $u(t,x)$ consequent upon the application of Hamilton's principle. In the action integral:

$$A = \int_\Omega L\left(t,x,u,u_t,u_x\right)\mathrm{d}x\mathrm{d}t \tag{26}$$

we introduce an infinitesimal variation of the dependent variable,

$$\bar{u} = u + \varepsilon v(t,x), \tag{27}$$

where ε is the infinitesimal parameter, $v(t,x)$ is continuously differentiable in both independent variables and is required to be zero on the boundary, $\partial\Omega$, of the domain of integration, Ω, which in this introductory case is some region in the (t,x) plane. Otherwise, v is an arbitrary function. We have:

$$\bar{A} = \int_\Omega L\left(t,x,\bar{u},\bar{u}_t,\bar{u}_x\right)\mathrm{d}x\mathrm{d}t \tag{28}$$

and we require the action integral to take a stationary value, i.e., $\delta A = \bar{A} - A$ be zero. Now:

$$
\begin{aligned}
\delta A &= \int_\Omega \left[L\left(t,x,\bar{u},\bar{u}_t,\bar{u}_x\right) - L\left(t,x,u,u_t,u_x\right)\right] \mathrm{d}x\mathrm{d}t \\
&= \varepsilon \int_\Omega \left[\frac{\partial L}{\partial u} v + \frac{\partial L}{\partial u_t}\frac{v}{t} + \frac{L}{u_x}\frac{v}{x}\right] \mathrm{d}x\mathrm{d}t + O(\varepsilon^2) \\
&= \varepsilon \int_\Omega \left[\frac{L}{u} - \frac{\partial}{\partial t}\left(\frac{L}{u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \bar{u}_x}\right)\right] v\mathrm{d}t\mathrm{d}x \\
&\quad + \int_{\partial\Omega} \left[\frac{\partial L}{\partial u_t}\mathrm{d}t + \frac{\partial L}{\partial u_x}\mathrm{d}x\right] v + O(\varepsilon^2).
\end{aligned} \tag{29}
$$

The second integral in (29) is the sum of four integrals, two along each of the intervals (t_1, t_2) and (x_1, x_2) with $x = x_1$ and $x = x_2$ for the two integrals with respect to t and $t = t_1$ and $t = t_2$ for the two integrals with respect to x. Because v is zero on the boundary, this term must be zero. As v is otherwise arbitrary, the expression within the bracketsin the first integral must be zero, and so, we have the Euler–Lagrange equation:

$$
\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \bar{u}_x}\right) = 0. \tag{30}
$$

A conservation law for (30) is a vector-valued function, $\mathbf{f}$, of t, x, u and the partial derivatives of u, which is divergence free, i.e.,

$$
\begin{aligned}
\mathrm{div}.\mathbf{f} &= \frac{\partial f^1}{\partial t} + \frac{\partial f^2}{\partial x} \\
&= 0.
\end{aligned} \tag{31}
$$

In (31), the operators ∂_t and ∂_x are operators of total differentiation with respect to t and x, respectively, and henceforth, we denote these operators by D_t and D_x. The standard symbol for partial differentiation, ∂_A, indicates differentiation solely with respect to A. In this notation, (30) and (31) become respectively:

$$
\frac{\partial L}{\partial u} - D_t\frac{\partial L}{\partial u_t} - D_x\frac{\partial L}{\partial u_x} = 0 \quad \text{and} \tag{32}
$$

$$
\mathrm{div}.\mathbf{f} = D_t f^1 + D_x f^2 = 0. \tag{33}
$$

Naturally, there is no distinction between D_t, $\partial u/\partial t$ and u_t,likewise for the derivatives with respect to x.

8.2. Noether's Theorem for $L(t, x, u, u_t, u_x)$

We introduce into the action integral an infinitesimal transformation,

$$
\bar{t} = t + \varepsilon\tau \quad \bar{x} = x + \varepsilon\xi \quad \bar{u} = u + \varepsilon\eta \tag{34}
$$

generated by the differential operator:

$$
\Gamma = \tau\partial_t + \xi\partial_x + \eta\partial_u, \tag{35}
$$

which, because the Lagrangian depends on u_t and u_x, we extend once to give:

$$
\Gamma^{[1]} = \Gamma + \left(D_t\eta - u_t D_t\tau - u_t D_t\xi\right)\partial_{u_t} + \left(D_x\eta - u_x D_t\tau - u_x D_x\xi\right)\partial_{u_x}. \tag{36}
$$

The coefficient functions τ, ξ and η may depend on the derivatives of u, as well as t, x, and u.

The change in the action due to the infinitesimal transformation is given by:

$$\begin{aligned} \delta A &= \bar{A} - A \\ &= \int_{\bar{\Omega}} L\left(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{\bar{t}}, \bar{u}_{\bar{x}} \mathrm{d}\bar{t}\right) \mathrm{d}\bar{x} - \int_{\Omega} L\left(t, x, u, u_t, u_x\right) \mathrm{d}t\mathrm{d}x, \end{aligned} \tag{37}$$

where $\bar{\Omega}$ is the transformed domain. We recall that Noether's theorem comes in two parts. In the first part, with which we presently deal, the discussion is about the action integral and not the variational principle. Consequently, there is no reason why the domain over which integration takes place should be the same before and after the transformation. Equally, there is no reason to require that the coefficient functions vanish on the boundary of Ω. To make progress in the analysis of (37), we must reconcile the variables and domains of integration. For the variables of integration, we have:

$$\begin{aligned} \mathrm{d}\bar{t}\mathrm{d}\bar{x} &= \frac{\partial\left(\bar{t}, \bar{x}\right)}{\partial\left(t, x\right)} \mathrm{d}t\mathrm{d}x \\ &= \begin{vmatrix} D_t\bar{t} & D_t\bar{x} \\ D_x\bar{t} & D_x\bar{x} \end{vmatrix} \mathrm{d}t\mathrm{d}x \\ &= \begin{vmatrix} 1+\varepsilon D_t\tau & \varepsilon D_t\xi \\ \varepsilon D_x\tau & 1+\varepsilon D_x\xi \end{vmatrix} \mathrm{d}t\mathrm{d}x \\ &= \left[1 + \varepsilon\left(D_t\tau + D_x\xi\right) + O\left(\varepsilon^2\right)\right] \mathrm{d}t\mathrm{d}x. \end{aligned} \tag{38}$$

For the domain, we have simply that:

$$\bar{\Omega} = \Omega + \delta\Omega \tag{39}$$

which, as the transformation is infinitesimal, in general means the evaluation of the surface integral and in this two-dimensional case the evaluation of the line integral along the boundary of the original domain. Although this domain is arbitrary, it is fixed for the variational principle we are using. We can use the divergence theorem to express this in terms of the volume integral over the original domain of the divergence of some vector-valued function. Combining these considerations with (37) and (38) and expanding the integrand of the first integral in (37) as a Taylor series, we can write the condition that the action integral be invariant under the infinitesimal transformation as:

$$\begin{aligned} 0 &= \int_{\Omega} \left\{ L + \varepsilon\left[\tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial u} + \left(D_t\eta - u_tD_t\tau - u_tD_t\xi\right)\frac{\partial L}{\partial u_t}\right.\right. \\ &\quad \left.\left. + \left(D_x\eta - u_xD_t\tau - u_xD_x\xi\right)\frac{\partial L}{\partial u_x}\right] - \varepsilon\mathrm{div}.\mathbf{F}\right\} \left[1 + \varepsilon\left(D_t\tau + D_x\xi\right)\right] \mathrm{d}t\mathrm{d}x \\ &\quad - \int_{\Omega} L\mathrm{d}t\mathrm{d}x + O\left(\varepsilon^2\right), \end{aligned} \tag{40}$$

where $\mathbf{F}$ represents the contribution from the boundary term. If we require that this be true for any domain in which the Lagrangian is validly defined, the first-order term in (40) gives the condition for the Lagrangian to possess a Noether symmetry, videlicet:

$$\begin{aligned} \mathrm{div}.\mathbf{F} &= \left(D_t\tau + D_x\xi\right)L + \tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial u} \\ &\quad + \left(D_t\eta - u_tD_t\tau - u_xD_t\xi\right)\frac{\partial L}{\partial u_t} + \left(D_x\eta - u_tD_x\tau - u_xD_x\xi\right)\frac{\partial L}{\partial u_x}. \end{aligned} \tag{41}$$

The rest is just a matter of computation! There does not appear to be code that enables one to solve (41) for a given Lagrangian even for point symmetries. One is advised [30] (p. 273) to calculate the (generalized) symmetries of the corresponding Euler–Lagrange equation and then test whether there exists an $\mathbf{F}$ such that each of these symmetries in turn satisfies (41).

There is a theorem in Olver [31] (p. 326) that the set of generalized symmetries of the Euler–Lagrange equation contains the set of generalized Noether symmetries of the Lagrangian. A purist could well prefer to be able to solve (41) directly. Alan Head, the distinguished Australian scientist, who wrote one of the more successful codes for differential equations in 1978, considered the effort involved to write the requisite code twenty years later excessive when the indirect route was available (A K Head, private communication, December, 1997).

A conservation law corresponding to a Noether symmetry "derived" from (41) is obtained when the Euler–Lagrange equation is taken into account. We rewrite the right side of (41) and have:

$$\begin{aligned}
\mathrm{div.}\mathbf{F} &= D_t\left[\tau L+\eta\frac{\partial L}{\partial u_t}\right]+D_x\left[\xi L+\eta\frac{\partial L}{\partial u_x}\right]+\tau\frac{\partial L}{\partial t}+\xi\frac{\partial L}{\partial x}+\eta\frac{\partial L}{\partial u}\\
&\quad -\left(u_tD_t\tau+u_xD_t\xi\right)\frac{\partial L}{\partial u_t}-\left(u_tD_x\tau+u_xD_x\xi\right)\frac{\partial L}{\partial u_x}\\
&\quad -\tau D_tL-\xi D_xL-\eta D_t\frac{\partial L}{\partial u_t}-\eta D_x\frac{\partial L}{\partial u_x}\\
&= D_t\left[\tau L+\eta\frac{\partial L}{\partial u_t}\right]+D_x\left[\xi L+\eta\frac{\partial L}{\partial u_x}\right]+\tau\frac{\partial L}{\partial t}+\xi\frac{\partial L}{\partial x}-\left(u_tD_t\tau+u_xD_t\xi\right)\frac{\partial L}{\partial u_t}\\
&\quad -\left(u_tD_x\tau+u_xD_x\xi\right)\frac{\partial L}{\partial u_x}-\tau D_tL-\xi D_xL\\
&= D_t\left[\tau L+\left(\eta-u_t\tau-u_x\xi\right)\frac{\partial L}{\partial u_t}\right]+D_x\left[\xi L+\left(\eta-u_t\tau-u_x\xi\right)\frac{\partial L}{\partial u_x}\right]\\
&\quad +\tau\left[\frac{\partial L}{\partial t}-D_tL+D_t\left(u_t\frac{\partial L}{\partial u_t}\right)+D_x\left(u_t\frac{\partial L}{\partial u_x}\right)\right]\\
&\quad +\xi\left[\frac{\partial L}{\partial x}-D_xL+D_t\left(u_x\frac{\partial L}{\partial u_t}\right)+D_x\left(u_x\frac{\partial L}{\partial u_x}\right)\right]\\
&= D_t\left[\tau L+\left(\eta-u_t\tau-u_x\xi\right)\frac{\partial L}{\partial u_t}\right]+D_x\left[\xi L+\left(\eta-u_t\tau-u_x\xi\right)\frac{\partial L}{\partial u_x}\right] \qquad (42)
\end{aligned}$$

when the Euler–Lagrange equation is taken into account. Hence, there is the vector of the conservation law:

$$\mathbf{I}=\mathbf{F}-\left[\tau L+\left(\eta-u_t\tau-u_x\xi\right)\frac{\partial L}{\partial u_t}\right]\mathbf{e}_t-\left[\xi L+\left(\eta-u_t\tau-u_x\xi\right)\frac{\partial L}{\partial u_x}\right]\mathbf{e}_x, \qquad (43)$$

where $\mathbf{e}_t$ and $\mathbf{e}_x$ are the unit vectors in the (t,x) plane.

We consider the simple example of the Lagrangian:

$$L=\tfrac{1}{12}\left(u_x\right)^4+\tfrac{1}{2}\left(u_t\right)^2.$$

The condition for the existence of a Noether symmetry, (41), becomes:

$$\begin{aligned}
\mathrm{div.}\mathbf{F} &= \left(D_t\tau+D_x\xi\right)\left(\tfrac{1}{12}\left(u_x\right)^4+\tfrac{1}{2}\left(u_t\right)^2\right)+\left(D_t\eta-u_tD_t\tau-u_xD_t\xi\right)u_t\\
&\quad +\left(D_x\eta-u_tD_x\tau-u_xD_x\xi\right)\tfrac{1}{3}u_x^3. \qquad (44)
\end{aligned}$$

The Lagrangian has the Euler–Lagrange equation:

$$u_x^2u_{xx}+u_{tt}=0. \qquad (45)$$

The Lie point symmetries of (45) are:

$$\begin{array}{ll} \Gamma_1 = \partial_t & \Gamma_4 = t\partial_u \\ \Gamma_2 = \partial_x & \Gamma_5 = t\partial_t - u\partial_u \\ \Gamma_3 = \partial_u & \Gamma_6 = x\partial_x + 2u\partial_u. \end{array} \tag{46}$$

The Lie point symmetries Γ_1–Γ_4 give a zero vector $\mathbf{F}$ except for Γ_4, which gives $(u, 0)$. The symmetries Γ_5 and Γ_6 give nonlocal vectors and so nonlocal conservation laws, which could be interpreted as meaning that they are not Noether symmetries for the given Lagrangian. The four local conservation laws are:

$$\begin{aligned} \mathbf{I}_1 &= \left(\tfrac{1}{2}u_t^2 - u_x^4, \tfrac{1}{12}u_t u_x^3\right) \\ \mathbf{I}_2 &= \left(u_t u_x, \tfrac{1}{4}u_x^4 - \tfrac{1}{2}u_t^2\right) \\ \mathbf{I}_3 &= \left(u_t, \tfrac{1}{3}u_x^3\right) \\ \mathbf{I}_4 &= \left(u - tu_t, -\tfrac{1}{3}tu_x^3\right). \end{aligned}$$

9. The General Euler–Lagrange Equation

In the case of a p^{th}-order Lagrangian in m dependent variables, u_i, $i = 1, m$, and n independent variables, x_j, $j = 1, n$, the Lagrangian, $L(x, u, u_1, \ldots, u_p)$, under an infinitesimal transformation:

$$\bar{u}^i(x) = u^i(x) + \varepsilon v^i(x),$$

where ε is the parameter of smallness and $v(x)$ is $k-1$-times differentiable and zero on the boundary $\partial\Omega$ of the domain of integration Ω of the action integral,

$$A = \int_\Omega L\left(x, u, u_1, \ldots, u_p\right) \mathrm{d}x, \tag{47}$$

becomes:

$$\begin{aligned} \bar{L} &= L\left(\bar{x}, \bar{u}, \bar{u}_1, \ldots, \bar{u}_p\right) \\ &= L\left(x, u, u_1, \ldots, u_p\right) + \varepsilon v^i_{j_1, j_2, \ldots, j_k} \frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} + O\left(\varepsilon^2\right) \end{aligned} \tag{48}$$

in which summation over repeated indices is implied and $i = 1, m$, $j = 1, n$, and $k = 0, p$. The variation in the action integral is:

$$\begin{aligned} \delta A &= \int_\Omega [\bar{L} - L]\, \mathrm{d}x \\ &= \varepsilon \int_\Omega v^i_{j_1, j_2, \ldots, j_k} \frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} \mathrm{d}x + O\left(\varepsilon^2\right). \end{aligned} \tag{49}$$

We consider one set of terms in (49) with summation only over j_k.

$$\begin{aligned} & \int_\Omega v^i_{j_1, j_2, \ldots, j_k} \frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} \mathrm{d}x \\ = & \int_\Omega \left\{ D_{j_k} \left[v^i_{j_1, j_2, \ldots, j_{k-1}} \frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} \right] - v^i_{j_1, j_2, \ldots, j_{k-1}} D_{j_k} \left[\frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} \right] \right\} \mathrm{d}x \\ = & \int_{\partial\Omega} D_{j_k} \left[v^i_{j_1, j_2, \ldots, j_{k-1}} \frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} \right] n_{j_k} \mathrm{d}\sigma - \int_\Omega v^i_{j_1, j_2, \ldots, j_{k-1}} D_{j_k} \left[\frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} \right] \mathrm{d}x \\ = & \int_\Omega v^i_{j_1, j_2, \ldots, j_{k-1}} D_{j_k} \left[\frac{\partial L}{\partial u^i_{j_1, j_2, \ldots, j_k}} \right] \mathrm{d}x, \end{aligned} \tag{50}$$

where on the right side in passing from the first line to the second line, we have made use of the divergence theorem and from the second to the third the requirement that v and its derivatives up to the $(p-1)^{\text{th}}$ be zero on the boundary. If we apply this stratagem repeatedly to (50), we eventually obtain that:

$$\int_{\Omega} v^i_{j_1,j_2,\ldots,j_k} \frac{\partial L}{\partial u^i_{j_1,j_2,\ldots,j_k}} \mathrm{d}x = (-1)^k \int_{\Omega} v^i D_{j_1} D_{j_2} \ldots D_{j_k} \frac{\partial L}{\partial u^i_{j_1,j_2,\ldots,j_k}} \mathrm{d}x. \tag{51}$$

We substitute (51) into (49) to give:

$$\delta A = \varepsilon(-1)^k \int_{\Omega} v^i D_{j_1} D_{j_2} \ldots D_{j_k} \frac{\partial L}{\partial u^i_{j_1,j_2,\ldots,j_k)}} \mathrm{d}x + O\left(\varepsilon^2\right). \tag{52}$$

Hamilton's principle requires that δA be zero for a zero-boundary variation. As the functions $v^i(x)$ are arbitrary subject to the differentiability condition, the integrand in (52) must be zero for each value of the index i, and so, we obtain the m Euler–Lagrange equations:

$$(-1)^k D_{j_1} D_{j_2} \ldots D_{j_k} \frac{\partial L}{\partial u^i_{j_1,j_2,\ldots,j_k}} = 0, \quad i = 1, m, \tag{53}$$

with the summation on j being from one to n and on k from zero to p.

10. Noether's Theorem: Original Formulation

Under the infinitesimal transformation:

$$\bar{x}^j = x^j + \varepsilon\xi^j \qquad \bar{u}^i = u^i + \varepsilon\eta^i \tag{54}$$

of both independent and dependent variables generated by the differential operator:

$$\Gamma = \xi_j \partial_{x_j} + \eta_i \partial_{u_i}, \tag{55}$$

in which summation on i and j from 1–m and from 1–n, respectively, is again implied, the action integral,

$$A = \int_{\Omega} L\left(x, u, u_1, \ldots, u_p\right) \mathrm{d}x, \tag{56}$$

becomes:

$$\begin{aligned} \bar{A} &= \int_{\bar{\Omega}} L\left(\bar{x}, \bar{u}, \bar{u}_1, \ldots, \bar{u}_p\right) \mathrm{d}\bar{x} \\ &= \int_{\Omega+\delta\Omega} L\left(x + \varepsilon\xi, u + \varepsilon\eta, u_1 + \varepsilon\eta^{(1)}, \ldots, u_p + \varepsilon\eta^{(p)}\right) J\left(\bar{x}, x\right) \mathrm{d}x. \end{aligned} \tag{57}$$

The notation $\delta\Omega$ indicates the infinitesimal change in the domain of integration Ω induced by the infinitesimal transformation of the independent variables.

The notation $\eta^{(j)}$ is a shorthand notation for the j^{th} extension of Γ. For the j_1^{th} derivative of u^i, we have specifically:

$$\eta^{i(1)}_{j_1} = D_{j_1}\eta^i - u^i_l D_{j_1}\xi^l \tag{58}$$

and for higher derivatives, we can use the recursive definition:

$$\eta^{i(k)}_{j_1 j_2 \cdots j_k} = D_k \eta^{i(k-1)}_{j_1 j_2 \cdots j_{k-1}} - u^i_{j_1 j_2 \cdots j_k l} D_{j_k}\xi^l \tag{59}$$

in which the terms in parentheses are not to be taken as summation indices.

The Jacobian of the transformation may be written as:

$$\begin{aligned} J(\bar{x}, x) &= \left|\frac{\partial \bar{x}^i}{\partial x^j}\right| \\ &= \left|\delta_{ij} + \varepsilon D_j \xi^i + O\left(\varepsilon^2\right)\right| \\ &= 1 + \varepsilon D_j \xi^j + O\left(\varepsilon^2\right). \end{aligned} \tag{60}$$

We now can write (57) as:

$$\bar{A} = \int_\Omega \left\{ L + \varepsilon \left[\varepsilon L D_j \xi^i + \xi^j \frac{\partial L}{\partial x_j} + \eta^{i(k)}_{j_1 j_2 \cdots j_k} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \right] \right\} \mathrm{d}x + \int_{\delta\Omega} L \mathrm{d}x + O\left(\varepsilon^2\right). \tag{61}$$

Because the transformation is infinitesimal, to the first order in the infinitesimal parameter, ε, the integral over $\delta\Omega$ can be written as:

$$\begin{aligned} \int_{\delta\Omega} L \mathrm{d}x &= \varepsilon \int_{\partial\Omega} L \mathrm{d}\sigma \\ &= -\int_\Omega D_j F^j \mathrm{d}x, \end{aligned} \tag{62}$$

where $\mathbf{F}$ is an as yet arbitrary function. The requirement that the action integral be invariant under the infinitesimal transformation now gives:

$$D_j F_j = L D_j \xi^j + \xi^j \frac{\partial L}{\partial x_j} + \eta^{i(k)}_{j_1 j_2 \cdots j_k} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}}. \tag{63}$$

This is the condition for the existence of a Noether symmetry for the Lagrangian. We recall that the variational principle was not used in the derivation of (63), and so, the Noether symmetry exists for all possible curves in the phase space and not only the trajectory for which the action integral takes a stationary value.

To obtain a conservation law corresponding to a given Noether symmetry, we manipulate (63) taking cognizance of the Euler–Lagrange equations. As:

$$\xi^i D_j L = \xi^j \left(\frac{\partial L}{\partial x_j} + D_j u^i_{j_1 j_2 \cdots j_k} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \right), \tag{64}$$

we may write (63) as:

$$\begin{aligned} D_j \left[F_j - L\xi^j \right] &= \left[\eta^{i(k)}_{j_1 j_2 \cdots j_k} - \xi^j D_j u^i_{j_1 j_2 \cdots j_k} \right] \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \\ &= \left(\eta^i - \xi^j D_j u^i \right) \frac{\partial L}{\partial u^i} + \sum_{k=1}^{p} \left[\eta^{i(k)}_{j_1 j_2 \cdots j_k} - \xi^j D_j u^i_{j_1 j_2 \cdots j_k} \right] \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \\ &= -\left(\eta^i - \xi^j D_j u^i \right) (-1)^k D_{j_1} D_{j_2} \ldots D_{j_k} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \\ &\quad + \sum_{k=1}^{p} \left[\eta^{i(k)}_{j_1 j_2 \cdots j_k} - \xi^j D_j u^i_{j_1 j_2 \cdots j_k} \right] \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \end{aligned} \tag{65}$$

in the second line of which we have separated the first term from the summation and used the Euler–Lagrange equation. We may rewrite the first term as:

$$D_{j_k}\left[\left(\eta^i - \xi^j D_j u^i\right)(-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-1}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}}\right]$$
$$-\left(D_{j_k}\eta^i - \left(D_{j_k}\xi^j\right) D_{j_k} u^i - \xi^j D_{jj_k} u^i\right)(-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-1}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}}.$$

The first term, being a divergence, can be moved to the left side (after replacing the repeated index j_k with j). We observe that the second term may be written as (58):

$$\begin{aligned}
& -\left(\eta^{i(1)}_{j_k} - \xi^j u^i_{jj_k}\right)(-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-1}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \\
\Leftrightarrow \quad & -D_{j_{k-1}}\left[\left(\eta^{i(1)}_{j_k} - \xi^j u^i_{jj_k}\right)(-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-2}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}}\right] \\
& + \left[D_{j_{k-1}}\left(\eta^{i(1)}_{j_k} - \xi^j u^i_{jj_k}\right)\right](-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-2}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}} \\
\Leftrightarrow \quad & -D_{j_{k-1}}\left[\left(\eta^{i(1)}_{j_k} - \xi^j u^i_{jj_k}\right)(-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-2}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}}\right] \\
& + \left(\eta^{i(2)}_{j_{k-1} j_k} - \xi^j u^i_{jj_{k-1} j_k}\right)(-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-2}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}}
\end{aligned} \tag{66}$$

in which we see the same process repeated. Eventually, all terms can be included with the divergence, and we have the conservation law:

$$D_j\left\{F_j - L\xi^j - \left(\eta^{i(l)}_{jk \ldots j_{k-l+1}} - \xi^m u^i_{mjk \ldots j_{k-l+1}}\right)(-1)^k D_{j_1} D_{j_2} \ldots D_{j_{k-l}} \frac{\partial L}{\partial u^i_{j_1 j_2 \cdots j_k}}\right\} = 0. \tag{67}$$

The relations (63) and (67) constitute Noether's theorem for Hamilton's principle.

11. Noether's Theorem: Simpler Form

The original statement of Noether's theorem was in terms of infinitesimal transformations depending on dependent and independent variables and the derivatives of the former. Thus, the theorem was stated in terms of generalized symmetries ab initio. The complexity of the calculations for even a system of a moderate number of variables and derivatives of only low order in the coefficient functions is difficult to comprehend and the thought of hand calculations depressing. We have already mentioned that one is advised to calculate generalized Lie symmetries for the corresponding Euler–Lagrange equation using some package and then to check whether there exists an **F** such that (63) is satisfied for the Lie symmetries obtained. Even this can be a nontrivial task. Fortunately, there exists a theoretical simplification, presented by Boyer in 1967 [29], which reduces the amount of computation considerably. The basic result is that under the set of generalized symmetries:

$$\Gamma = \xi^i \partial_{x^i} + \eta^i \partial_{u^i},$$

where the ξ^i and η^i are functions of u, x and the derivatives of u with respect to x, and:

$$\Gamma = \bar{\eta}^i \partial_{u^i}, \quad \bar{\eta}^i = \eta^i - u^i_j \xi^j$$

one obtains the same results [32,33].

This enables (63) and (67) to be written without the coefficient functions ξ^i. This is a direct generalization of the result for a first-order Lagrangian in one independent variable. One simply must ensure that generality is not lost by allowing for a sufficient generality in the dependence of the η^i upon the derivatives of the dependent variables. The only caveat one should bear in mind is that the physical or geometric interpretation of a symmetry may be impaired if the symmetry is given in a form that is not its natural form. This does raise the question of what is the "natural" form of a symmetry. It does not provide the beginnings of an answer. It would appear that the natural form is often determined in the eye of the beholder, cf. [34].

The proof of the existence of equivalence classes of generalized transformation depends on the fact that two transformations can produce the same effect on a function.

12. Conclusions

In this review article, we perform a detailed discussion on the formulation of Noether's theorems and on its various generalizations. More specifically, we discuss that in the original presentation of Noether's work [1], the dependence of the coefficient functions of the infinitesimal transformation can be upon the derivatives of the dependent variables. Consequently, a series of generalizations on Noether' theorem, like hidden symmetries, generalized symmetries, etc., are all included in the original work of Noether. That specific point and that the boundary function on the action integral can include higher-order derivatives of the dependent variables were the main subjects of discussion for this work. Our aim was to recover for the audience that generality that has been lost after texts, for instance Courant, Hilbert, Rund, and many others, where they identify as Noether symmetries only the point transformations. The discussion has been performed for ordinary and partial differential equations, while the corresponding conservation laws/flows are given in each case.

Author Contributions: The authors contributed equivalently.

Funding: APacknowledges the financial support of FONDECYT Grant No. 3160121. PGLL Thanks the Durban University of Technology, the University of KwaZulu-Natal, and the National Research Foundation of South Africa for support. AKH expresses grateful thanks to UGC(India), NFSC, Award No. F1-17.1/201718/RGNF-2017-18-SC-ORI-39488, for financial support.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data, in the writing of the manuscript; nor in the decision to publish the results.

References and Notes

1. Noether, E. Invariante Variationsprobleme. *Königlich Gesellschaft der Wissenschaften Göttingen Nachrichten Mathematik-Physik Klasse* **1918**, *2*, 235–267.
2. Hamel, G. Ueber die Grundlagen der Mechanik. *Math. Ann.* **1908**, *66*, 350–397. [CrossRef]
3. Hamel, G. Ueber ein Prinzig der Befreiung bei Lagrange. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **1917**, *25*, 60–65.
4. Herglotz, G. Über den vom Standpunkt des Relativitätsprinzips aus als "starr" zu bezeichnenden Körper. *Annalen der Physik* **1910**, *336*, 393–415. [CrossRef]
5. Fokker. *Verslag die Amsterdamer Akad*; Holland, 1917.
6. Kneser, A. Kleinste Wirkung und Galileische Relativität. *Math. Z.* **1918**, *2*, 326–349. [CrossRef]
7. Klein, F. *Königlich Gesellschaft der Wissenschaften Göttingen Nachrichten Mathematik-Physik Klasse*; Germany, 1918; p. 2.
8. Courant, R.; Hilbert, D. *Methods of Mathematical Physics*; Wiley Interscience: New York, NY, USA, 1953.
9. Lovelock, D.; Rund, H. *Tensors, Differential Forms and Variational Principles*; Wiley: New York, NY, USA, 1975.
10. Logan, J.D. *Invariant Variational Principles*; Academic: New York, NY, USA, 1977.
11. Stan, M. On the Noether's Theorem in Hamiltonian formalism. *Sci. Bull. Politech. Univ. Buchar. Ser. A Appl. Math. Phys.* **1994**, *56*, 93–98.

12. Anco, S.C. Generalization of Noether's Theorem in Modern Form to Non-variational Partial Differential Equations. In *Recent Progress and Modern Challenges in Applied Mathematics, Modeling and Computational Science*; Melnik, R., Makarov, R., Belair, J., Eds.; Fields Institute Communications; Springer: New York, NY, USA, 2017; Volume 79.
13. Hojman, S.A. A new conservation law constructed without using either Lagrangians or Hamiltonians. *J. Phys. A Math. Gen.* **1992**, *25*, L291–L295. [CrossRef]
14. González-Gascón, F. Geometric foundations of a new conservation law discovered by Hojman. *J. Phys. A Math. Gen.* **1994**, *27*, L59–L60. [CrossRef]
15. Crampin, M. Hidden symmetries and Killing tensors. *Rep. Math. Phys.* **1984**, *20*, 31–40. [CrossRef]
16. Crampin, M.; Mestdag, T. The Cartan form for constrained Lagrangian systems and the nonholonomic Noether theorem. *Int. J. Geom. Methods Mod. Phys.* **2011**, *8*, 897–923. [CrossRef]
17. Sarlet, W.; Cantrijn, F. Generalizations of Noether's Theorem in Classical Mechanics. *SIAM Rev.* **1981**, *23*, 467–494. [CrossRef]
18. Dugas, R. *A History of Mechanics*; Maddox, J.R., Translated; Dover: New York, NY, USA, 1988.
19. Govinder, K.S.; Leach, P.G.L. Paradigms of Ordinary Differential Equations. *S. Afr. J. Sci.* **1995**, *91*, 306–311.
20. Pillay, T.; Leach, P.G.L. Comment on a theorem of Hojman and its generalizations. *J. Phys. A Math. Gen.* **1996**, *29*, 6999–7002. [CrossRef]
21. Leach, P.G.L. Applications of the Lie theory of extended groups in Hamiltonian mechanics: The oscillator and the Kepler problem. *J. Aust. Math. Soc.* **1981**, *23*, 173–186. [CrossRef]
22. Govinder, K.S.; Leach, P.G.L.; Maharaj, S.D. Integrability analysis of a conformal equation arising in relativity. *Int. J. Theor. Phys.* **1995**, *34*, 625–639. [CrossRef]
23. Leach, P.G.L.; Govinder, K.S.; Abraham-Shrauner, B. Symmetries of First Integrals and Their Associated Differential Equations. *J. Math. Anal. Appl.* **1999**, *235*, 58–83. [CrossRef]
24. Leach, P.G.L.; Warne, R.R.; Caister, N.; Naicker, V.; Euler, N. Symmetries, integrals and solutions of ordinary differential equations of maximal symmetry. *Proc. Math. Sci.* **2010**, *120*, 123. [CrossRef]
25. Leach, P.G.L.; Mahomed, F.M. Maximal subalgebra associated with a first integral of a system possessing sl(3,R) algebra. *J. Math. Phys.* **1988**, *29*, 1807. [CrossRef]
26. Paliathanasis, A.; Leach, P.G.L. Nonlinear Ordinary Differential Equations: A discussion on Symmetries and Singularities. *Int. J. Geom. Meth. Mod. Phys.* **2016**, *13*, 1630009. [CrossRef]
27. Mahomed, F.M.; Kara, A.H.; Leach, P.G.L. Lie and Noether counting theorems for one-dimensional systems. *J. Math. Anal. Appl.* **1993**, *178*, 116–129. [CrossRef]
28. Patera, J.; Winternitz, P. Algebras of real three- and four-dimensional Lie algebras. *J. Math. Phys.* **1977**, *18*, 1449–1455. [CrossRef]
29. Boyer, T.H. Continuous symmetries and conserved currents. *Ann. Phys.* **1967**, *42*, 445–466. [CrossRef]
30. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.
31. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: Berlin, Germany, 1986.
32. Katzin, G.H.; Levine, J. Characteristic functional structure of infinitesimal symmetry mappings of classical dynamical systems. I. Velocity-dependent mappings of second-order differential equations. *J. Math. Phys.* **1985**, *26*, 3080. [CrossRef]
33. Katzin, G.H.; Levine, J. Characteristic functional structure of infinitesimal symmetry mappings of classical dynamical systems. II. Mappings of first-order differential equations. *J. Math. Phys.* **1985**, *26*, 3100. [CrossRef]
34. Moyo, S.; Leach, P.G.L. Exceptional properties of second and third order ordinary differential equations of maximal symmetry. *J. Math. Anal. Appl.* **2000**, *252*, 840–863. [CrossRef]